INVOLUTION OF RADICALS.

262. Involution of Radicals is the process of raising radical quantities to any required power.

1. Find the cube of $2\sqrt{a}$.

Solution. Expressing the radical with a fractional exponent, we have $(2a^{\frac{1}{2}})^3$; raising it to the third power, we have $2^3\times a^{\frac{3}{2}}$; which reduced gives $8a\sqrt{a}$.

OPERATION.

$$(2\sqrt{a})^3=(2\times a^{\frac{1}{2}})^3$$
$$=2^3\times a^{\frac{3}{2}}=8\sqrt{a^3}$$
$$=8a\sqrt{a}$$

Rule.—*Raise the rational part to the required power, and multiply the fractional exponent by the index of the power;* or,

Raise the rational and radical parts to the required power, and reduce the result to its simplest form.

Note.—*Dividing the index of the radical by any number raises the radical to a power denoted by the number.* Thus, the square of $\sqrt[6]{a}$ is $\sqrt[3]{a}$.

EXAMPLES.

Find the—

2. Square of $3\sqrt[3]{(ac^2)}$. *Ans.* $9c\sqrt[3]{(a^2c)}$.
3. Cube of $2\sqrt{(ax)}$. *Ans.* $8ax\sqrt{(ax)}$.
4. Cube of $3\sqrt{(2x)}$. *Ans.* $54x\sqrt{(2x)}$.
5. Square of $\frac{a}{2}\sqrt{(2a)}$. *Ans.* $\frac{a^3}{2}$.
6. Cube of $4\sqrt[4]{\frac{ax^3}{4}}$. *Ans.* $16x^2\sqrt[4]{(4a^3x)}$.
7. Fourth power of $3\sqrt{\frac{a}{3}}$. *Ans.* $9a^2$.
8. Fourth power of $2a\sqrt[3]{\frac{x}{a}}$. *Ans.* $16a^2x\sqrt[3]{(a^2x)}$.
9. nth power of $a\sqrt[3]{x}$. *Ans.* $a^n\sqrt[3]{x^n}$.
10. Third power of $\sqrt[3]{2a^2}\times\sqrt[6]{(ax^3)}$. *Ans.* $2a^2x\sqrt{(ax)}$.
11. Square of $\sqrt{a}-\sqrt{x}$. *Ans.* $a-2\sqrt{(ax)}+x$.
12. Square of $\sqrt{2}+a\sqrt{2}$. *Ans.* $2+4a+2a^2$.

EVOLUTION OF RADICALS.

263. Evolution of Radicals is the process of extracting any required root of radical quantities.

1. Find the cube root of $a^3\sqrt[4]{c^3}$.

Solution. Reducing the radical to the form of a fractional exponent, we have $a^3c^{\frac{3}{4}}$; extracting the cube root by dividing the exponents by 3, we have $ac^{\frac{1}{4}}$, which equals $a\sqrt[4]{c}$.

OPERATION.

$$\sqrt[3]{a^3\sqrt[4]{c^3}}=\sqrt[3]{(a^3c^{\frac{3}{4}})}=$$
$$ac^{\frac{1}{4}}=a\sqrt[4]{c}$$

Rule.—*Extract the required root of the rational part, and divide the fractional exponent by the index of the root;* or,

Extract the required root of the rational and radical parts, and reduce the result to its simplest form.

Note.—*Multiplying the index of the radical by a number extracts a root of the radical denoted by the number.* Thus, the square root of $\sqrt[3]{a}$ is $\sqrt[6]{a}$.

EXAMPLES.

Find the—

2. Square root of $4\sqrt[3]{a^2}$. *Ans.* $\pm 2\sqrt[3]{a}$.
3. Square root of $9\sqrt[3]{(4x^2)}$. *Ans.* $\pm 3\sqrt[3]{(2x)}$.
4. Square root of $16\sqrt[3]{(3x)}$. *Ans.* $\pm 4\sqrt[6]{(3x)}$.
5. Cube root of $2\sqrt{(ax)}$. *Ans.* $\sqrt[6]{(4ax)}$.
6. Cube root of $2a\sqrt{(2a)}$. *Ans.* $\sqrt{(2a)}$.
7. Fourth root of $3a\sqrt[3]{(3a)}$. *Ans.* $\sqrt[3]{(3a)}$.
8. Fourth root of $\frac{1}{4}\sqrt{(2a)}$. *Ans.* $\sqrt[8]{(\frac{1}{2}a)}$.
9. Fifth root of $4c^2\sqrt{(2c)}$. *Ans.* $\sqrt{(2c)}$.
10. Fifth root of $4x\sqrt[4]{(4x)}$. *Ans.* $\sqrt[4]{(4x)}$.
11. Cube root of $\frac{a}{2}\sqrt{\frac{a}{2}}$. *Ans.* $\frac{1}{2}\sqrt{(2a)}$.
12. Square root of $\frac{a}{3}\sqrt[3]{\frac{a}{3}}$. *Ans.* $\frac{1}{3}\sqrt[3]{(3a^2)}$.
13. Fourth root of $\frac{a}{4}\sqrt[3]{\frac{a}{4}}$. *Ans.* $\frac{1}{2}\sqrt[3]{(2a)}$.

RATIONALIZATION.

264. Rationalization is the process of removing the radical sign from a quantity.

CASE I.

265. To rationalize any monomial surd.

1. Rationalize $\sqrt{a}$, also $a^{\frac{1}{3}}$.

SOLUTION. Multiplying $\sqrt{a}$ by $\sqrt{a}$, we have a, a rational quantity.

OPERATIONS.
$\sqrt{a}\times\sqrt{a}=a$

SOLUTION. Multiplying $a^{\frac{1}{3}}$ by $a^{\frac{2}{3}}$, we have a, a rational quantity.

$a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$

Rule.—*Multiply the surd by the same quantity with a fractional exponent which added to the given exponent shall equal unity.*

EXAMPLES.

2. What factor will rationalize $a^{\frac{3}{5}}$? *Ans.* $a^{\frac{2}{5}}$.
3. What factor will rationalize $\sqrt[3]{(a^2c)}$? *Ans.* $\sqrt[3]{ac^2}$.
4. What factor will rationalize $2\sqrt[4]{(ab)}$? *Ans.* $\sqrt[4]{(a^3b^3)}$.
5. What factor will rationalize $3\sqrt[3]{(a^2b)}$? *Ans.* $\sqrt[3]{(ab^2)}$.

CASE II.

266. To rationalize a binomial surd of the second degree.

1. Rationalize $\sqrt{a}-\sqrt{b}$.

SOLUTION. Since the product of the sum and difference of two quantities equals the difference of their squares, if we multiply $\sqrt{a}-\sqrt{b}$ by $\sqrt{a}+\sqrt{b}$, we obtain $a-b$, a rational quantity.

OPERATION.

$$\begin{array}{r}\sqrt{a}-\sqrt{b}\\ \sqrt{a}+\sqrt{b}\\ \hline a-b\end{array}$$

Rule.—*Multiply the given binomial by the same binomial, with the signs of one of the terms changed.*

EXAMPLES.

What factor will—

2. Rationalize $\sqrt{a}+\sqrt{x}$? *Ans.* $\sqrt{a}-\sqrt{x}$.
3. Rationalize $\sqrt{2}-\sqrt{3}$? *Ans.* $\sqrt{2}+\sqrt{3}$.
4. Rationalize $2\sqrt{a}+\sqrt{5}$? *Ans.* $2\sqrt{a}-\sqrt{5}$.
5. Rationalize $a+\sqrt{b}$? *Ans.* $a-\sqrt{b}$.

CASE III.

267. To rationalize either of the terms of a fractional surd.

1. Rationalize the denominator of $\frac{a}{\sqrt{x}}$.

SOLUTION. Multiplying both terms of the fraction by $\sqrt{x}$, we have $\frac{a\sqrt{x}}{x}$, in which the denominator is rational, and the value of the fraction is not changed.

OPERATION.

$$\frac{a}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}=\frac{a\sqrt{x}}{x}$$

Rule.—*Multiply both terms of the fraction by a factor that will render either term rational which may be required.*

EXAMPLES.

2. Rationalize the denominator of $\frac{1}{\sqrt{2}}$. *Ans.* $\frac{1}{2}\sqrt{2}$.
3. Rationalize the denominator of $\frac{a}{2\sqrt{3}}$. *Ans.* $\frac{a\sqrt{3}}{6}$.
4. Rationalize the denominator of $\frac{1}{1+\sqrt{3}}$. *Ans.* $\frac{\sqrt{3}-1}{2}$.
5. Rationalize the denominator of $\frac{\sqrt{3}}{3-\sqrt{3}}$. *Ans.* $\frac{\sqrt{3}+1}{2}$.
6. Rationalize the denominator of $\frac{a}{\sqrt{x}-\sqrt{y}}$.
   Ans. $\frac{a(\sqrt{x}+\sqrt{y})}{x-y}$.
7. Rationalize the denominator of $\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}$.
   Ans. $\frac{(\sqrt{a}+\sqrt{c})^2}{a-c}$.

IMAGINARY QUANTITIES.

268. An **Imaginary Quantity** is an indicated even root of a negative quantity.

269. Imaginary quantities, though they represent impossible operations, are of use in some departments of mathematics.

PRINCIPLE.—*Every imaginary quantity may be reduced to the form of* $a\sqrt{-1}$; or $a\sqrt[2n]{-1}$.

Thus, $\sqrt{-a^2}=\sqrt{a^2\times-1}=\sqrt{a^2}\times\sqrt{-1}=\pm a\sqrt{-1}$;

Also, $\sqrt{-n}=\sqrt{n\times-1}=\pm\sqrt{n}\times\sqrt{-1}$; or, putting $a=\sqrt{n}$, we have $\pm a\sqrt{-1}$.

In all higher powers the form will be $\pm a\sqrt[2n]{-1}$.

REDUCTION OF IMAGINARY QUANTITIES.

Reduce to simplest form—

1. $\sqrt{-n^2}$. *Ans.* $\pm n\sqrt{-1}$.
2. $\sqrt{-4}$. *Ans.* $\pm 2\sqrt{-1}$.
3. $\sqrt{-9a^4}$. *Ans.* $\pm 3a^2\sqrt{-1}$.
4. $\sqrt{-16a^6}$. *Ans.* $\pm 4a^3\sqrt{-1}$.
5. $\sqrt{-n^3}$. *Ans.* $\pm n^{\frac{3}{2}}\sqrt{-1}$.
6. $\sqrt{-4a^2c^4}$. *Ans.* $\pm 2ac^2\sqrt{-1}$.
7. $\sqrt[4]{-16a^2b^4}$. *Ans.* $\pm 2a^{\frac{1}{2}}b\sqrt{-1}$.
8. $\sqrt{-8a^2c^3}$. *Ans.* $\pm 2ac^{\frac{3}{2}}\sqrt{-2}$.

ADDITION AND SUBTRACTION OF IMAGINARY QUANTITIES.

1. Add $\sqrt{-a^2}$ and $\sqrt{-b^2}$.

SOLUTION. $\sqrt{-a^2}=a\sqrt{-1}$; $\sqrt{-b^2}=b\sqrt{-1}$; adding these two quantities, we have $(a+b)\sqrt{-1}$.

OPERATION.

$$\begin{array}{r} \sqrt{-a^2}=a\sqrt{-1} \\ \sqrt{-b^2}=b\sqrt{-1} \\ \hline (a+b)\sqrt{-1} \end{array}$$

EXAMPLES.

2. Add $\sqrt{-a^2}$ and $\sqrt{-c^2}$. *Ans.* $(a+c)\sqrt{-1}$.
3. Add $\sqrt{-4}$ and $\sqrt{-9}$. *Ans.* $5\sqrt{-1}$.

4. Add $\sqrt{-8}$ and $\sqrt{-18}$. *Ans.* $5\sqrt{-2}$.
5. Subtract $\sqrt{-4}$ from $\sqrt{-16}$. *Ans.* $2\sqrt{-1}$.
6. Subtract $\sqrt{-2m^2}$ from $\sqrt{-8m^2}$. *Ans.* $m\sqrt{-2}$.

MULTIPLICATION OF IMAGINARY QUANTITIES.

1. Multiply $3\sqrt{-2}$ by $\sqrt{-3}$.

SOLUTION. $3\sqrt{-2}=3\sqrt{2}\times\sqrt{-1}$, and $\sqrt{-3}=\sqrt{3}\times\sqrt{-1}$; multiplying, we have $3\sqrt{6}\times(\sqrt{-1})^2$, which equals $3\sqrt{6}\times-1$, or $-3\sqrt{6}$.

OPERATION.

$$\begin{array}{r} 3\sqrt{-2}=3\sqrt{2}\times\sqrt{-1} \\ \sqrt{-3}=\ \sqrt{3}\times\sqrt{-1} \\ \hline 3\sqrt{6}\times(\sqrt{-1})^2=-3\sqrt{6} \end{array}$$

NOTE.—Had we multiplied the quantities under the radical sign at first, we could not have determined the sign of the product.

EXAMPLES.

2. Multiply $\sqrt{-3}$ by $2\sqrt{-2}$. *Ans.* $-2\sqrt{6}$.
3. Multiply $a\sqrt{-b^2}$ by $2\sqrt{-b^2}$. *Ans.* $-2ab^2$.
4. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* 2.
5. Multiply $1+\sqrt{-1}$ by $1+\sqrt{-1}$. *Ans.* $2\sqrt{-1}$.

DIVISION OF IMAGINARY QUANTITIES.

1. Divide $4\sqrt{-6}$ by $2\sqrt{-3}$.

SOLUTION. $4\sqrt{-6}=4\sqrt{6}\times\sqrt{-1}$, and $2\sqrt{-3}=2\sqrt{3}\times\sqrt{-1}$; dividing the dividend by the divisor and canceling the common factors, we have $2\sqrt{2}$.

OPERATION.

$$\frac{4\sqrt{-6}=4\sqrt{6}\times\sqrt{-1}}{2\sqrt{-3}=2\sqrt{3}\times\sqrt{-1}}=2\sqrt{2}$$

NOTE.—In dividing, it is not necessary to reduce to the general form, though it is sometimes convenient.

EXAMPLES.

2. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.
3. Divide $4\sqrt{-a^2}$ by $a\sqrt{-2}$. *Ans.* $2\sqrt{2}$.
4. Divide $a\sqrt{-6c}$ by $\sqrt{-2ac}$. *Ans.* $\sqrt{3a}$.
5. Divide $2\sqrt{-1}$ by $1+\sqrt{-1}$. *Ans.* $1+\sqrt{-1}$.

PRINCIPLES.

1. *The* PRODUCT *of two imaginary quantities is real, with the sign before the radical the* REVERSE *of that given by the common rule.*

Thus, $\sqrt{-a^2}\times\sqrt{-b^2}=-ab$; and $-\sqrt{-a^2}\times-\sqrt{-b^2}=-ab$; but $-\sqrt{-a^2}\times\sqrt{-b^2}=+ab$, etc.

2. *The* QUOTIENT *of two imaginary quantities is real, with the sign before the radical the* SAME *as that given by the common rule.*

Thus, $\sqrt{-a^2}\div\sqrt{-c^2}=+\frac{a}{c}$; $-\sqrt{-a^2}\div-\sqrt{-c^2}=+\frac{a}{c}$; also $-\sqrt{-a^2}\div+\sqrt{-c^2}=-\frac{a}{c}$, etc.

SQUARE ROOT OF BINOMIAL SURDS.

270. Some binomials containing a radical quantity are squares of binomials, and will thus admit of the extraction of the square root.

Thus, $(2+\sqrt{3})^2=4+4\sqrt{3}+3$, or $7+4\sqrt{3}$; hence the square root of $7+4\sqrt{3}$ is $2+\sqrt{3}$.

271. Since the second term in the square of a binomial is *twice* the *product* of the other two terms, if we reduce the binomial surd to the form $a+2\sqrt{b}$, a will be the *sum* and b the *product* of two numbers.

1. Find the square root of $11+6\sqrt{2}$.

SOLUTION.—$11+6\sqrt{2}=11+2\sqrt{18}$; now find two numbers whose sum is 11 and product 18. These numbers, we see by inspection, are 9 and 2; then $11+2\sqrt{18}=9+2\sqrt{18}+2$, the square root of which is $3+\sqrt{2}$.

NOTE.—The numbers 9 and 2 can be obtained by letting $x+y=11$ and $xy=18$, and finding x and y.

EXAMPLES.

Find the

2. Square root of $14+6\sqrt{5}$. *Ans.* $3+\sqrt{5}$.
3. Square root of $28+10\sqrt{3}$. *Ans.* $5+\sqrt{3}$.
4. Square root of $11-4\sqrt{7}$. *Ans.* $2-\sqrt{7}$.
5. Square root of $15+6\sqrt{6}$. *Ans.* $3+\sqrt{6}$.
6. Square root of $5+2\sqrt{6}$. *Ans.* $\sqrt{2}+\sqrt{3}$.

7. Square root of $8-2\sqrt{15}$. *Ans.* $\sqrt{3}-\sqrt{5}$.
8. Square root of $9+2\sqrt{14}$. *Ans.* $\sqrt{2}+\sqrt{7}$.
9. Square root of $14-4\sqrt{6}$. *Ans.* $2\sqrt{3}-\sqrt{2}$.
10. Square root of $30+12\sqrt{6}$. *Ans.* $2\sqrt{3}+3\sqrt{2}$.
11. Square root of $3a-2a\sqrt{2}$. *Ans.* $\sqrt{a}-\sqrt{2a}$.
12. Square root of $2a+2\sqrt{a^2-b^2}$. *Ans.* $\sqrt{a+b}+\sqrt{a-b}$.
13. Square root of $m^2-2n\sqrt{m^2-n^2}$. *Ans.* $\sqrt{m^2-n^2}-n$.
14. Square root of $7+30\sqrt{-2}$. *Ans.* $5+3\sqrt{-2}$.
15. Square root of $12\sqrt{-2}-14$. *Ans.* $2+3\sqrt{-2}$.
16. Square root of $24\sqrt{-2}-23$. *Ans.* $3+4\sqrt{-2}$.
17. Square root of $40\sqrt{-3}-23$. *Ans.* $5+4\sqrt{-3}$.
18. Square root of $\sqrt{18}-4$. *Ans.* $\sqrt[4]{2}(\sqrt{2}-1)$.
19. Square root of $4\sqrt{3}-6$. *Ans.* $\sqrt[4]{3}(\sqrt{3}-1)$.
20. Square root of $6\sqrt{5}-10$. *Ans.* $\sqrt[4]{5}(\sqrt{5}-1)$.

NOTE.—In the 18th, let $x+y=\sqrt{18}$ and $xy=4$; in 19th, let $x+y=4\sqrt{3}$, and, since $6=2\sqrt{9}$, let $xy=9$, etc.

RADICAL EQUATIONS.

SOLVED AS SIMPLE EQUATIONS.

272. Radical Equations are those which contain the unknown quantity in the form of a radical.

273. Some radical equations may be solved by the principles of simple equations, examples of which will now be presented.

1. Given $\sqrt{x}-3=2$, to find x.

SOLUTION.—Given, $\sqrt{x}-3=2$,
transposing, $\sqrt{x}=5$,
squaring, $x=25$. *Ans.*

2. Given $\sqrt{(5+2\sqrt[3]{x})}=3$, to find x.

SOLUTION.—Given, $\sqrt{(5+2\sqrt[3]{x})}=3$,
squaring, $5+2\sqrt[3]{x}=9$,
transposing and reducing, $2\sqrt[3]{x}=4$,
dividing and cubing, $x=8$.

3. Given $\sqrt{x+7}+\sqrt{x-5}=6$.

SOLUTION.

Given,	$\sqrt{x+7}+\sqrt{x-5}=6$,
transposing,	$\sqrt{x+7}=6-\sqrt{x-5}$,
squaring,	$x+7=36-12\sqrt{x-5}+x-5$,
transposing and reducing,	$\sqrt{x-5}=2$,
squaring,	$x-5=4$,
transposing,	$x=9$.

EXAMPLES.

4. Given $\sqrt{2x}+5=9$, to find x. *Ans.* $x=8$.
5. Given $\sqrt{(x-3)}+2=5$, to find x. *Ans.* $x=12$.
6. Given $\sqrt{(x+5)}=\sqrt{x}+1$, to find x. *Ans.* $x=4$.
7. Given $3+\sqrt{(2x+4)}=7$, to find x. *Ans.* $x=6$.
8. Given $8-\sqrt{x}=\sqrt{(x-16)}$, to find x. *Ans.* $x=25$.
9. Given $\sqrt{(6+\sqrt[3]{3x})}+5=8$, to find x. *Ans.* $x=9$.
10. Given $\sqrt{x}-2=\sqrt{(x-24)}$, to find x. *Ans.* $x=49$
11. Given $\sqrt{(x+2)}=\dfrac{5}{\sqrt{(x+2)}}$, to find x. *Ans.* $x=3$.
12. Given $\sqrt{(x-9)}+\sqrt{(x+11)}=10$, to find x. *Ans.* $x=25$.
13. Given $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{2}\sqrt{a}$, to find x. *Ans.* $x=\dfrac{25a}{16}$.
14. Given $\dfrac{x-2}{\sqrt{x}}=\dfrac{2\sqrt{x}}{3}$, to find x. *Ans.* $x=6$.
15. Given $\sqrt{(x+4ab)}=2a-\sqrt{x}$, to find x. *Ans.* $x=(a-b)^2$.
16. Given $x+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a-x)}}$, to find x. *Ans.* $x=a-1$.
17. Given $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}$, to find x. *Ans.* $x=a$.
18. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$, to find x. *Ans.* $x=\dfrac{1}{1-a}$.

SECTION VII

QUADRATIC EQUATIONS.

274. A **Quadratic Equation** is one in which the second power is the highest power of the unknown quantity; as, $x^2=4$, and $2x^2+3x=5$.

275. There are *two classes* of quadratic equations—*Pure* or *Incomplete* quadratics, and *Affected* or *Complete* quadratics.

276. The term which does not contain the unknown quantity is called the *absolute term*, or the term *independent of the unknown quantity*.

NOTES.—1. A quadratic equation is also called an equation of the *second degree*. An equation of the fourth degree is called a *bi-quadratic equation*.

2. When there are two unknown quantities, a quadratic is defined as an equation in which the greatest sum of the exponents in any term is two.

PURE QUADRATICS.

277. A **Pure Quadratic Equation** is one which contains the second power only of the unknown quantity; as $x^2=16$.

278. The **General Form** of a pure quadratic equation, is $ax^2=b$.

1. Given $2x^2+6=24$, to find x.

SOLUTION.

Given,	$2x^2+6=24$,
transposing and reducing,	$x^2=9$,
extracting the square root,	$x=\pm 3$.