3. Given $\sqrt{x+7}+\sqrt{x-5}=6$.

SOLUTION.

Given,	$\sqrt{x+7}+\sqrt{x-5}=6,$
transposing,	$\sqrt{x+7}=6-\sqrt{x-5},$
squaring,	$x+7=36-12\sqrt{x-5}+x-5,$
transposing and reducing,	$\sqrt{x-5}=2,$
squaring,	$x-5=4,$
transposing,	$x=9.$

EXAMPLES.

4. Given $\sqrt{2x}+5=9$, to find x. *Ans.* $x=8$.
5. Given $\sqrt{(x-3)}+2=5$, to find x. *Ans.* $x=12$.
6. Given $\sqrt{(x+5)}=\sqrt{x}+1$, to find x. *Ans.* $x=4$.
7. Given $3+\sqrt{(2x+4)}=7$, to find x. *Ans.* $x=6$.
8. Given $8-\sqrt{x}=\sqrt{(x-16)}$, to find x. *Ans.* $x=25$.
9. Given $\sqrt{(6+\sqrt[3]{3x})}+5=8$, to find x. *Ans.* $x=9$.
10. Given $\sqrt{x}-2=\sqrt{(x-24)}$, to find x. *Ans.* $x=49$
11. Given $\sqrt{(x+2)}=\dfrac{5}{\sqrt{(x+2)}}$, to find x. *Ans.* $x=3$.
12. Given $\sqrt{(x-9)}+\sqrt{(x+11)}=10$, to find x. *Ans.* $x=25$.
13. Given $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{2}\sqrt{a}$, to find x. *Ans.* $x=\dfrac{25a}{16}$.
14. Given $\dfrac{x-2}{\sqrt{x}}=\dfrac{2\sqrt{x}}{3}$, to find x. *Ans.* $x=6$.
15. Given $\sqrt{(x+4ab)}=2a-\sqrt{x}$, to find x. *Ans.* $x=(a-b)^2$.
16. Given $x+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a-x)}}$, to find x. *Ans.* $x=a-1$.
17. Given $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}$, to find x. *Ans.* $x=a$.
18. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$, to find x. *Ans.* $x=\dfrac{1}{1-a}$.

SECTION VII

QUADRATIC EQUATIONS.

274. A **Quadratic Equation** is one in which the second power is the highest power of the unknown quantity; as, $x^2=4$, and $2x^2+3x=5$.

275. There are *two classes* of quadratic equations—*Pure* or *Incomplete* quadratics, and *Affected* or *Complete* quadratics.

276. The term which does not contain the unknown quantity is called the *absolute term*, or the term *independent of the unknown quantity.*

NOTES.—1. A quadratic equation is also called an equation of the *second degree*. An equation of the fourth degree is called a *bi-quadratic equation.*

2. When there are two unknown quantities, a quadratic is defined as an equation in which the greatest sum of the exponents in any term is two.

PURE QUADRATICS.

277. A **Pure Quadratic Equation** is one which contains the second power only of the unknown quantity; as $x^2=16$.

278. The **General Form** of a pure quadratic equation, is $ax^2=b$.

1. Given $2x^2+6=24$, to find x.

SOLUTION.

Given,	$2x^2+6=24,$
transposing and reducing,	$x^2=9,$
extracting the square root,	$x=\pm 3.$

2. Given $ax^2+n=cx^2+m$, to find x.

SOLUTION.

Given,	$ax^2+n=cx^2+m,$
transposing,	$ax^2-cx^2=m-n,$
factoring,	$(a-c)x^2=m-n,$
reducing,	$x^2=\dfrac{m-n}{a-c},$
extracting square root,	$x=\pm\sqrt{\dfrac{m-n}{a-c}}.$

Rule.—I. *Reduce the equation to the form* $ax^2=b$.

II. *Divide by the coefficient of* x^2, *and extract the square root of both members.*

EXAMPLES.

3. Given $x^2+5=3x^2-13$, to find x. *Ans.* $x=\pm 3$.
4. Given $(x-6)(x+6)=-11$, to find x. *Ans.* $x=\pm 5$.
5. Given $x^2+ab=5x^2$, to find x. *Ans.* $x=\pm\frac{1}{2}\sqrt{(ab)}$.
6. Given $2x^2-3=\dfrac{x^2}{3}+12$, to find x. *Ans.* $x=\pm 3$.
7. Given $x^2+a^2+b^2=2ab+2x^2$, to find x. *Ans.* $x=\pm(a-b)$.
8. Given $4x+8=(x+2)^2$, to find x. *Ans.* $x=\pm 2$.
9. Given $\dfrac{1}{1-x}+\dfrac{1}{1+x}=3$, to find x. *Ans.* $x=\pm\frac{1}{3}\sqrt{3}$.
10. Given $\dfrac{4}{x-3}-\dfrac{4}{x+3}=\dfrac{1}{3}$, to find x. *Ans.* $x=\pm 9$.
11. Given $2x+5x^{-1}=3x-11x^{-1}$, to find x. *Ans.* $x=\pm 4$.
12. Given $\dfrac{x+3}{x-3}+\dfrac{x-3}{x+3}=3\frac{1}{3}$, to find x. *Ans.* $x=\pm 6$.
13. Given $\dfrac{x}{x+1}+\dfrac{x}{x+4}=1$, to find x. *Ans.* $x=\pm 2$.
14. Given $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{x}{b}+\dfrac{b}{x}$, to find x. *Ans.* $x=\pm\sqrt{(ab)}$.
15. Given $(n-x)^3+(n+x)^3=3n^3$, to find x. *Ans.* $x=\pm\frac{n}{6}\sqrt{6}$.

279. Radical Equations sometimes become pure quadratics when cleared of radicals.

1. Given $\sqrt{(x^2+9)}=\sqrt{(2x^2-7)}$, to find x. *Ans.* $x=\pm 4$.
2. Given $\sqrt{(x^2-4)}=2\sqrt{(a-1)}$, to find x. *Ans.* $x=\pm 2\sqrt{a}$.
3. Given $\sqrt{\left(\dfrac{10x^2-2}{2x}\right)}=\sqrt{x}$, to find x. *Ans.* $x=\pm\frac{1}{2}$.
4. Given $(x+a)^{\frac{1}{2}}=\dfrac{a}{(x-a)^{\frac{1}{2}}}$, to find x. *Ans.* $x=\pm a\sqrt{2}$.
5. Given $\sqrt{(x+m)}=\sqrt{\{x+\sqrt{(n^2+x^2)}\}}$, to find x. *Ans.* $x=\sqrt{(m^2-n^2)}$.
6. Given $\sqrt{(x+a)}=\dfrac{b}{\sqrt{(x-a)}}$, to find x. *Ans.* $x=\pm\sqrt{(a^2+b^2)}$.
7. Given $\sqrt{\{x^2+2ax+\sqrt{(x^2-4)}\}}=a+x$, to find x. *Ans.* $x=\pm\sqrt{(a^4+4)}$.
8. Given $\sqrt{\{x^2+\sqrt{(x^4-n^4)}\}}=n$, to find x. *Ans.* $x=\pm n$.
9. Given $\sqrt{(x+m)}=\sqrt[4]{(x^2+n^2)}$, to find x. *Ans.* $x=\dfrac{n^2-m^2}{2m}$.
10. Given $x+\sqrt{(a^2+x^2)}=\dfrac{2a^2}{\sqrt{(a^2+x^2)}}$, to find x. *Ans.* $x=\pm\frac{1}{3}a\sqrt{3}$.

PRINCIPLES OF PURE QUADRATICS.

280. The **Principles** of Pure Quadratics relate to the form of the equation and the relative value of its roots.

PRINCIPLES.

1. *Every pure quadratic equation may be reduced to the form* $ax^2=b$.

For, it is evident we can reduce all the terms containing x^2 to one term, as ax^2, and all the known terms to one term, as b; hence the form will become $ax^2=b$.

2. *Every pure quadratic equation has two roots, equal in numerical value, but of opposite signs.*

DEM. 1ST. The general form of a pure quadratic is $ax^2=b$; dividing by a, we have $x^2=b$ divided by a; representing the quotient by m^2, we have $x^2=m^2$; extracting the square root, we have $x=\pm m$. Therefore, etc.

OPERATION.

$$ax^2=b$$
$$x^2=\frac{b}{a}=m^2$$
$$x=\pm m$$

DEM. 2D. From the equation $x^2=m^2$, by transposition, we have $x^2-m^2=0$; factoring, we have $(x+m)(x-m)=0$. This equation can be satisfied by making $x-m$ equal 0, or by making $x+m$ equal 0, and in no other way (Art. 192, Prin. 1). Making $x-m=0$, we have $x=+m$; making $x+m=0$, we have $x=-m$. Therefore, etc.

$$x^2=m^2$$
$$x^2-m^2=0$$
$$(x+m)(x-m)=0$$
$$x-m=0$$
$$x=+m$$
$$x+m=0$$
$$x=-m$$

PROBLEMS

PRODUCING PURE QUADRATICS.

1. The product of two numbers is 48, and the greater is 3 times the smaller; required the numbers.

SOLUTION.

Let $x=$ the smaller number,
and $3x=$ the larger number;
then $3x\times x=3x^2=48$;
whence, $x^2=16$;
then $x=\pm 4$,
and $3x=\pm 12$.

2. The sum of two numbers is 10, and their product is 24; required the numbers.

SOLUTION. Let $5+x$ represent the greater number, and $5-x$ the smaller number; their sum will be 10, which satisfies the first condition of the problem. By the second condition, $(5+x)(5-x)=24$, or $25-x^2=24$. Reducing, we have $x=\pm 1$. Using the positive value, we have 6 or 4; using the negative value, we have 4 or 6.

OPERATION.

$5+x=$ the greater number;
$5-x=$ the less number.

$$(5+x)(5-x)=24$$
$$25-x^2=24$$
$$x^2=1$$
$$x=\pm 1$$
$$5+x=6 \text{ or } 4$$
$$5-x=4 \text{ or } 6$$

3. The sum of two numbers is 15, and their product is 54; required the numbers. *Ans.* 9; 6.

4. The sum of two numbers is 12, and the sum of their squares is 74; required the numbers. *Ans.* 7; 5.

5. The difference of two numbers is 5, and their product is 84; required the numbers. *Ans.* 12; 7.

6. Divide the number 24 into two such parts that their product shall be 140. *Ans.* 14; 10.

7. The difference of two numbers is 4 and the sum of their squares is 208; what are the numbers? *Ans.* 12; 8.

8. Required a number whose square is 432 more than the square of $\frac{1}{2}$ the number. *Ans.* 24.

9. What number is that which, if multiplied by $\frac{1}{2}$ of itself, is 72 greater than the square of its $\frac{2}{3}$? *Ans.* 36.

10. What two numbers are to each other as 4 to 5, and the difference of whose squares is 81? *Ans.* 12; 15.

11. Required two numbers whose product is 48, and the quotient of the greater divided by the less, 3. *Ans* 12; 4.

12. There is a rectangular field containing 4 acres whose length is to its breadth as 5 to 2; required its dimensions.
Ans. Length, 40 rods; breadth, 16 rods.

13. There is a number whose third part squared and subtracted from 170 leaves a remainder of 26; what is the number?
Ans. 36.

14. $\frac{3}{4}$ of the square of twice a number is equal to $\frac{5}{4}$ of the square of $\frac{4}{3}$ of the number, increased by 28; what is the number?
Ans. 6.

15. A man has two cubical bundles of hay, one of which contains 999 cubic feet more than the other; what are the dimensions of each, if the smaller is $\frac{3}{4}$ as long as the larger?
Ans. 12 ft.; 9 ft.

16. A merchant bought a piece of cloth for $216, and the number of dollars he paid for a yard was to the number of yards as 2 to 3; required the price per yard and the number of yards. *Ans.* 18 yards at $12 a yard.

17. A man bought a field whose length was to its breadth as 5 to 4; the price per acre was equal to the number of rods in the length of the field, and 5 times the distance around the

field equaled the number of dollars that it cost; required the length and breadth of the field.

Ans. Length, 60 rods; breadth, 48 rods.

18. From a cask containing 81 gallons of wine a vintner draws off a certain quantity, and then, filling the cask with water, draws off the same quantity again, and then there remains only 36 gallons of pure wine; how much wine did he draw off each time? *Ans.* First, 27 gals.; second, 18 gals.

AFFECTED QUADRATICS.

281. An **Affected Quadratic Equation** is one that contains both the second and first powers of the unknown quantity; as, $x^2+4x=12$.

282. The **General Forms** of an affected quadratic are $ax^2+bx=c$, and $x^2+2px=q$.

1. Given $x^2+6x=16$, to find the value of x.

SOLUTION. If the first member of this equation were a perfect square, we could extract the square root of both members, and find the value of x from the resulting equation. Let us, therefore, see if we can make the first member a perfect square.

The square of a binomial is equal to the square of the first term, plus twice the product of the first term into the second, plus the square of the second. If now we consider x^2+6x as the first two terms of the *square of a binomial*, the first term of this binomial will be the square root of x^2 or x; and $6x$ will be twice the first term of the binomial into the second; hence, if we divide $6x$ by twice the first term, $2x$, the quotient, 3, must be the second term of the binomial, and its square, 9, added to the first member of the equation, will render it a perfect square.

OPERATION.

$$x^2+6x=16$$
$$x^2+6x+9=25$$
$$x+3=\pm 5$$
$$x=-3\pm 5$$
$$x=+2$$
$$x=-8$$

Adding 9 to the first member to complete the square, and to the second member to preserve the equality, we have $x^2+6x+9=25$. Extracting the square root, we have $x+3=\pm 5$; from which, using the positive value of 5, we have $x=2$; and using the negative value, we have $x=-8$.

Rule.—I. *Reduce the given equation to the general form* $x^2+2px=q$.

II. *Add to both members the square of one-half the coefficient of* x.

III. *Extract the square root of both members, and solve the resulting simple equation.*

2. Given $x^2-8x=-7$, to find x.

SOLUTION. Completing the square by adding the square of half the coefficient of x to both members, we have $x^2-8x+16=9$. Extracting the square root of both members, we have $x-4=\pm 3$. Using the positive value of 3, we have $x=7$; using the negative value, we have $x=1$.

OPERATION.

$$x^2-8x=-7 \quad (1)$$
$$x^2-8x+16=9 \quad (2)$$
$$x-4=\pm 3 \quad (3)$$
$$x=7 \quad (4)$$
$$x=1 \quad (5)$$

VERIFICATION.

$$7^2-8\times 7=-7;\ \text{or}\ 49-56=-7;$$
also, $$1^2-8\times 1=-7;\ \text{or}\ 1-8=-7.$$

3. Given $x^2+4x=32$, to find x.

SOLUTION.

Given the equation,	$x^2+4x=32$,
completing the square,	$x^2+4x+4=36$,
extracting the root,	$x+2=\pm 6$
whence,	$x=-2+6=+4$,
and	$x=-2-6=-8$.

Both of which will verify the equation.

4. Given $x^2-12x=-20$, to find x.

SOLUTION.

Given the equation,	$x^2-12x=-20$,
completing the square,	$x^2-12x+36=16$,
extracting the root,	$x-6=\pm 4$;
whence,	$x=6+4=10$,
and	$x=6-4=2$.

5. Given $x^2-5x=6$, to find x.

SOLUTION.

Given the equation,	$x^2-5x=6$,
completing the square,	$x^2-5x+(\frac{5}{2})^2=6+(\frac{5}{2})^2$,
which reduced, equals	$x^2-5x+(\frac{5}{2})^2=6+\frac{25}{4}=\frac{49}{4}$,
extracting the root,	$x-\frac{5}{2}=\pm\frac{7}{2}$;
whence,	$x=\frac{5}{2}+\frac{7}{2}=+6$,
and	$x=\frac{5}{2}-\frac{7}{2}=-1$.

6. Given $3x^2 - 7x = 66$, to find x.

SOLUTION.

Given the equation,	$3x^2 - 7x = 66$,
dividing by the coefficient of x^2,	$x^2 - \frac{7x}{3} = 22$,
completing the square,	$x^2 - \frac{7}{3}x + (\frac{7}{6})^2 = 22 + (\frac{7}{6})^2$,
which reduced, equals	$x^2 - \frac{7}{3}x + (\frac{7}{6})^2 = \frac{841}{36}$,
extracting the root,	$x - \frac{7}{6} = \pm\frac{29}{6}$;
whence,	$x = \frac{7}{6} + \frac{29}{6} = +6$,
and	$x = \frac{7}{6} - \frac{29}{6} = -\frac{11}{3}$.

Reduce the following equations:

7. $x^2 + 2x = 24$. *Ans.* $x = 4$, or -6.
8. $x^2 + 2x = 35$. *Ans.* $x = 5$, or -7.
9. $x^2 + 4x = 32$. *Ans.* $x = 4$, or -8.
10. $x^2 - 6x = 16$. *Ans.* $x = 8$, or -2.
11. $x^2 - 3x = 18$. *Ans.* $x = 6$, or -3.
12. $x^2 - 5x = 84$. *Ans.* $x = 12$, or -7.
13. $x^2 - 11x = 60$. *Ans.* $x = 15$, or -4.
14. $x^2 - 13x = 140$. *Ans.* $x = 20$, or -7.
15. $x^2 - 14x = -24$. *Ans.* $x = 12$, or 2.
16. $x^2 - 4x = 1$. *Ans.* $x = 2 \pm \sqrt{5}$
17. $3x^2 - 4x = 39$. *Ans.* $x = 4\frac{1}{3}$, or -3.
18. $(x-1)(x-2) = 20$. *Ans.* $x = 6$, or -3.
19. $x^2 - 6x = -13$. *Ans.* $x = 3 \pm \sqrt{-4}$.
20. $4(x^2 - 1) = 4x - 1$ *Ans.* $x = \frac{3}{2}$, or $-\frac{1}{2}$.
21. $(2x-3)^2 = 8x$. *Ans.* $x = 4\frac{1}{2}$, or $\frac{1}{2}$.
22. $x^2 - 3 = \frac{x-3}{6}$. *Ans.* $x = 1\frac{2}{3}$, or $-1\frac{1}{2}$.
23. $x^2 + 2px = q$. *Ans.* $x = -p \pm \sqrt{(p^2+q)}$.
24. $x^2 - ax = b$. *Ans.* $x = \frac{1}{2}(a \pm \sqrt{a^2+4b})$.
25. $x^2 - 2nx = m^2 - n^2$. *Ans.* $x = n \pm m$.
26. $x^2 - ax - bx = -ab$. *Ans.* $x = a$, or b.

27. $x + \frac{1}{x-3} = 5$. *Ans.* $x = 4$.
28. $x = 2 + \frac{5}{4x}$ *Ans.* $x = 2\frac{1}{2}$, or $-\frac{1}{2}$.
29. $\frac{x-1}{x-3} + 2x = 12$. *Ans.* $x = 5$, or $3\frac{1}{2}$.
30. $\frac{2}{x+3} + \frac{x+3}{2} = \frac{10}{3}$. *Ans.* 3, or $-2\frac{1}{3}$.
31. $\frac{2x}{x+2} + \frac{x+2}{2x} = 2$. *Ans.* $x = 2$.
32. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}$. *Ans.* $x = 2$, or -3.
33. $\frac{x+2}{x+1} + \frac{x+1}{x+2} = \frac{13}{6}$. *Ans.* $x = 1$, or -4.
34. $\frac{x+1}{x-1} - \frac{x-2}{x+2} = \frac{9}{5}$ *Ans.* $x = 3$, or $-\frac{2}{3}$.
35. $x^2 + 2ax = a^2$. *Ans.* $x = a(-1 \pm \sqrt{2})$.
36. $3a^2x^{-1} - x = -2a$. *Ans.* $x = 3a$, or $-a$.
37. $x^2 - 2ax = b^2 - a^2$. *Ans.* $x = a + b$, or $a - b$.
38. $x^2 - (a - b + c)x = (b - a)c$. *Ans.* $x = a - b$, or $+c$.

283. The roots of a complete quadratic may be directly written by the following rule:

Rule.—I. *Reduce the quadratic to the form* $x^2 + 2px = q$.

II. *Take, with a contrary sign, one-half of the coefficient of* x, *plus or minus the square root of the sum obtained by adding the square of half the coefficient of* x *to the second member.*

The reason for this rule may be shown as follows: Solving the general quadratic, $x^2 + 2px = q$, we find that $x = -p \pm \sqrt{(q + p^2)}$; comparing the root with the general equation, we see that $-p$ is $\frac{1}{2}$ of $2p$ with the sign changed, and $\sqrt{(q+p^2)}$ is the square root of the second member plus the square of half the coefficient of x.

NOTE.—Pupils should be required to solve some of the previous examples by this rule

SECOND METHOD OF COMPLETING THE SQUARE.

284. A SECOND METHOD of completing the square enables us to avoid fractions in the solutions of all forms of quadratics.

285. The method already given involves fractions when the coefficient of x is an odd number or a fraction.

NOTE.—This method is sometimes called the *Hindoo Method* of solving quadratics.

1. Given $ax^2+bx=c$, to find the value of x.

SOLUTION. To complete the square without using fractions, it is evident that the first term must be a perfect square and the second term must be divisible by 2.

OPERATION.

$$ax^2+bx=c \qquad (1)$$
$$4a^2x^2+4abx=4ac \qquad (2)$$
$$4a^2x^2+4abx+b^2=4ac+b^2 \qquad (3)$$
$$2ax+b=\pm\sqrt{(4ac+b^2)} \qquad (4)$$
$$2ax=-b\pm\sqrt{(4ac+b^2)} \qquad (5)$$
$$x=\frac{-b\pm\sqrt{(4ac+b^2)}}{2a} \qquad (6)$$

To make the first term a perfect square and the second term divisible by 2, we multiply both members by $4a$, which gives eq. (2). If now we consider $4a^2x^2+4abx$ as the first two terms of the square of a binomial, the first term of this binomial will be the square root of $4a^2x^2$, or $2ax$; $4abx$ will be twice the product of the first term by the second: hence, if we divide $4abx$ by two times $2ax$, or $4ax$, the quotient b will be the second term of the binomial, and its square, b^2, added to both members of the equation, will complete the square.

Extracting the square root and reducing, we have the values of x expressed in equation (6).

Rule.—I. *Reduce the equation to the form $ax^2+bx=c$, in which the three terms are integral and prime to each other.*

II. *Multiply the equation by 4 times the coefficient of* x²*; add the square of the coefficient of* x *to both members.*

III. *Extract the square root, and find the value of* x *in the resulting simple equation.*

NOTES.—1. A quadratic may also be solved by multiplying by the coefficient of x^2, and adding the square of one-half the coefficient of x.

2. Let pupils make a rule from the formula which expresses the root of the general quadratic, $ax^2+bx=c$.

2. Reduce $3x^2+5x=22$.

SOLUTION.

Given the equation,	$3x^2+5x=22$,
multiplying by 4×3 or 12,	$36x^2+60x=264$,
completing the square,	$36x^2+60x+25=264+25$,
extracting the root,	$6x+5=\pm 17$,
transposing and reducing,	$6x=-5\pm 17$;
whence,	$x=+2$,
and	$x=-3\frac{2}{3}$.

EXAMPLES.

3. Reduce $2x^2-3x=9$. *Ans.* $x=3$, or $-1\frac{1}{2}$.
4. Reduce $3x^2+8x=28$. *Ans.* $x=2$, or $-4\frac{2}{3}$.
5. Reduce $4x^2+7x=11$. *Ans.* $x=1$, or $-2\frac{3}{4}$.
6. Reduce $5x^2+2x=88$. *Ans.* $x=4$, or $-4\frac{2}{5}$.
7. Reduce $5x^2-4x=156$. *Ans.* $x=6$, or $-5\frac{1}{5}$.
8. Reduce $4x^2-45x=36$. *Ans.* $x=12$, or $-\frac{3}{4}$.
9. Reduce $8x^2-7x=165$. *Ans.* $x=5$, or $-4\frac{1}{8}$.
10. Reduce $9x^2-7x=116$. *Ans.* $x=4$, or $-3\frac{2}{9}$.
11. Reduce $2x^2+ax=b$. *Ans.* $x=\frac{1}{4}\{-a\pm\sqrt{(a^2+8b)}\}$.
12. Reduce $\frac{x-3a}{b}=\frac{9(b-a)}{x}$. *Ans.* $x=3b$, or $3(a-b)$.
13. Reduce $x^2+3x=5$. *Ans.* $1.1925+$, or $-4.1925+$.
14. Reduce $x^2+2x=5$. *Ans.* $1.449+$, or $-3.449+$.
15. Reduce $x^2-8x=-8$. *Ans.* $6.828+$, or $1.172+$.
16. Reduce $5x^2-4x=2$. *Ans.* 1.148, or -0.348.

THIRD METHOD OF COMPLETING THE SQUARE.

286. A THIRD METHOD of completing the square is stated in the following rule:

Rule.—I. *Make the coefficient of the first term of the quadratic a positive square.*

II. *Divide the second term by twice the square root of the first, and add the square of the quotient to both members.*

EQUATIONS IN THE QUADRATIC FORM.

287. Any equation which is in, or may be put in, the quadratic form, may be solved by the following methods.

288. An equation is in the quadratic form when it contains but two powers of an unknown quantity, and the index of one power is twice that of the other as, $x^{2n}+ax^n=b$, or $ax^{2n}+bx^n=c$.

CASE I.

289. When the unknown term of the quadratic is a monomial.

1. Given $x^4-6x^2=-8$, to find x.

SOLUTION.

Given the equation,	$x^4-6x^2=-8$,
completing the square,	$x^4-6x^2+9=1$,
extracting the square root,	$x^2-3=\pm 1$,
transposing and reducing,	$x^2=4$, or 2,
extracting the square root,	$x=\pm 2$, or $\pm\sqrt{2}$.

2. Given $x^6-4x^3=32$, to find x.

SOLUTION.

Given the equation,	$x^6-4x^3=32$,
completing the square,	$x^6-4x^3+4=36$,
extracting the square root,	$x^3-2=\pm 6$,
transposing and reducing,	$x^3=8$, or -4,
extracting the cube root,	$x=2$,
and	$x=\sqrt[3]{(-4)}$.

NOTE.—Since the *odd roots* of a negative quantity are real, by extracting the cube root of 4 and prefixing the minus sign we find the approximate value of the second root of the above equation.

EXAMPLES.

3. Given $x^4+4x^2=32$, to find x. *Ans.* $x=\pm 2$, or $\pm 2\sqrt{(-2)}$.

4. Given $x^4-5x^2=36$, to find x. *Ans.* $x=\pm 3$, or $\pm 2\sqrt{(-1)}$.

5. Given $x^6-7x^3=8$, to find x. *Ans.* $x=2$, or -1.

6. Given $x^2+4x^{-2}=5$, to find x. *Ans.* $x=\pm 2$, or ± 1.

7. Given $x+3\sqrt{x}=18$, to find x. *Ans.* $x=9$, or 36.

8. Given $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=5$, to find x. *Ans.* $x=1$, or -125.

9. Given $x^{2n}-ax^n=b$, to find x. *Ans.* $x=(\frac{1}{2}a\pm\sqrt{b+\frac{1}{4}a^2})^{\frac{1}{n}}$.

10. Given $x^{2n}+4x^n=12$, to find x. *Ans.* $x=\sqrt[n]{2}$, or $\sqrt[n]{-6}$.

11. Given $x^3+7x^{\frac{3}{2}}=3\frac{3}{4}$, to find x. *Ans.* $x=\frac{1}{2}\sqrt[3]{2}$, or $\frac{1}{2}\sqrt[3]{450}$.

12. Given $\sqrt[3]{x}+2\sqrt[3]{x^2}=\frac{3}{8}$, to find x. *Ans.* $x=\frac{1}{64}$, or $-\frac{27}{64}$.

13. Given $x^n-ax^{\frac{n}{2}}=b$, to find x. *Ans.* $x=(\frac{1}{2}a\pm\sqrt{b+\frac{1}{4}a^2})^{\frac{2}{n}}$.

14. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4+\sqrt{x}}{\sqrt{x}}$, to find x. *Ans.* $x=64$, or 4.

CASE II.

290. When the unknown term of the quadratic is a polynomial.

291. When a polynomial becomes the basis of the quadratic form, we may consider it as a single quantity, and proceed as in the previous case.

1. Given $(x^2+2x)^2+4(x^2+2x)=96$, to find x.

SOLUTION.

Given the equation,	$(x^2+2x)^2+4(x^2+2x)=96$;
completing the square,	$(x^2+2x)^2+4(x^2+2x)+4=100$;
extracting the sq. root,	$x^2+2x+2=\pm 10$;
transposing,	$x^2+2x=8$, or -12;
completing the square,	$x^2+2x+1=9$, or -11;
extracting the sq. root,	$x+1=\pm 3$, or $\pm\sqrt{(-11)}$;
whence,	$x=2$, or -4,
and	$x=-1\pm\sqrt{(-11)}$.

NOTE.—This problem may also be solved by placing $x^2+2x=y$, giving the equation, $y^2+4y=96$, which may be solved and the value of y thus found placed equal to x^2+2x, from which the value of x can readily be found.

2. Given $(x^2-2)^2+2x^2=67$, to find x.

SOLUTION.

Given the equation,	$(x^2-2)^2+2x^2=67$;
subtracting 4,	$(x^2-2)^2+2x^2-4=63$;
factoring,	$(x^2-2)^2+2(x^2-2)=63$;
completing the square,	$(x^2-2)^2+2(x^2-2)+1=64$;
extracting the sq. root,	$(x^2-2)+1=\pm 8$;
whence,	$x^2=9$, or -7,
and	$x=\pm 3$, or $\pm\sqrt{-7}$.

NOTE.—The 4 was subtracted to put the equation in the quadratic form. Sometimes this can be done by transposing a term or by adding or subtracting some quantity.

EXAMPLES.

3. Given $(x^2-3)^2+4(x^2-3)=5$, to find x. *Ans.* $x=\pm 2$.

4. Given $(x^2+3x)^2-6(x^2+3x)=216$, to find x.
Ans. $x=3$, or -6.

5. Given $(x^2-4x)^{\frac{1}{2}}+(x^2-4x)=3\frac{3}{4}$, to find x.
Ans. $x=4\frac{1}{2}$, or $-\frac{1}{2}$.

6. Given $\sqrt{(5+x)}+\sqrt[4]{(5+x)}=6$, to find x.
Ans. $x=11$, or 76.

7. Given $\left(\frac{4}{x}+x\right)^2+6\left(\frac{4}{x}+x\right)=40$, to find x.
Ans. $x=2$, or $-5\pm\sqrt{(21)}$.

8. Given $x-\sqrt{(x+5)}=1$, to find x. *Ans.* $x=4$, or -1.

9. Given $(x-4)^2-6\sqrt{(x-4)}=\frac{16}{x-4}$, to find x.
Ans. $x=8$, or $4+\sqrt[3]{4}$.

10. Given $(x^2+2x-3)^2+7(x^2+2x-3)=60$, to find x.
Ans. $x=2$, -4, or $-1\pm 2\sqrt{(-2)}$.

11. Given $(x^2-9)^2-11x^2+40=21$, to find x.
Ans. $x=\pm 5$, or ± 2.

12. Given $(x^2-4x+5)^2+4x^2-16x=-8$, to find x.
Ans. $x=3$, or 1, or $2\pm\sqrt{(-7)}$.

NOTE.—Add 59 to both members of Ex. 11; add 20 to both members of Ex. 12. Several of the problems in Art. 290 have other results than those given.

PROBLEMS

PRODUCING AFFECTED QUADRATICS.

1. Find two numbers such that their sum is 16 and their product is 60.

SOLUTION.

Let	$x=$ one number;
then	$16-x=$ the other number.
by the conditions,	$(16-x)x=60$;
which gives	$x^2-16x=-60$;
whence,	$x=10$, or 6,
and	$16-x=6$, or 10.

Hence the two numbers are 10 and 6.

2. A man sold a watch for \$24, and lost as much per cent. as the watch cost him; what did the watch cost him?

SOLUTION.

Let	$x=$ the cost of the watch;
then	$x=$ the loss per cent.,
and	$\frac{x}{100}\times x=\frac{x^2}{100}=$ the loss;
therefore,	$x-\frac{x^2}{100}=24$;
whence,	$x=60$, or 24.

Both of these values will satisfy the conditions of the problem. The pupil will show this by verification.

3. A lady divided \$144 equally among some poor persons: if there had been two more, each would have received \$1 less; required the number of persons.

SOLUTION.

Let $x=$ the number of persons; then $\frac{144}{x}=$ what each received; and $\frac{144}{x+2}=$ what each would have received if there had been two more; then $\frac{144}{x+2}=\frac{144}{x}-1$; whence $x=16$, or -18. The number of persons was therefore 16; the negative result will not satisfy the problem in an arithmetical sense.

NOTE.—If, in the problem, 2 *more* be changed to 2 *less*, and \$1 *less* to \$1 *more*, the correct result will be 18.

4. A gentleman divided $50 between his two sons in such a manner that the product of their shares was 600; what was the share of each? *Ans.* $30; $20.

5. The wall which encloses a rectangular garden is 128 yards long, and the area of the garden is 1008 square yards; what is its length and breadth? *Ans.* 36 yds.; 28 yds.

6. An officer wishes to arrange 1600 men in a solid body, so that each rank may exceed each file by 60 men; how many must be placed in rank and file? *Ans.* 20; 80.

7. A merchant bought a number of Bibles for $50, which he sold for $5.50 a piece, and thus gained as much as one Bible cost; how many Bibles did he buy? *Ans.* 10.

8. The perimeter of a room is 48 feet, and the area of the floor equals 35 times the difference of its length and breadth; what are the dimensions of the room? *Ans.* 14 ft.; 10 ft.

9. Two boys, A and B, bought 10 oranges for 24 cents, each paying 12 cents; if A paid 1 cent more apiece than B, how many oranges did each buy? *Ans.* A, 4; B, 6.

10. A lot of sheep cost $180, but on 2 of them being stolen, the rest averaged $1 more a head than at first; find the number of sheep.

11. A man walked 48 miles in a certain time: if he had gone 4 miles more per hour, he would have gone the distance in 6 hours' less time; how many miles did he travel per hour?
Ans. 4 miles.

12. In an orchard containing 180 trees there are 3 more trees in a row than there are rows; required the number of rows and the number of trees in a row.
Ans. 12 rows, and 15 trees in a row.

13. In a purse containing 52 coins of silver and copper, each silver coin is worth as many cents as there are copper coins, and each copper coin is worth as many cents as there are silver coins, and the whole is worth $2; how many are there of each? *Ans.* 2 silver; 50 copper.

14. A person distributed $6 equally among a number of

paupers; and as there were 5 less than he supposed, they each received 10 cents apiece more than they otherwise would; how many paupers were there? *Ans.* 15.

15. The expenses of a party amount to $10; and if each pays 30 cents more than there are persons, the bill will be settled; how many persons are there? *Ans.* 20.

16. There is a number consisting of two digits whose sum is 10 and the sum of whose squares is 52; it is required to find the number. *Ans.* 46, or 64.

17. Mr. Leslie sold his horse for $171, and gained as much per cent. as the horse cost him; what was the first cost of the horse? *Ans.* $90.

18. A person laid out a certain sum of money for goods, which he sold again for $24, and lost as much per cent. as the goods cost him; what was the first cost? *Ans.* $40, or $60.

19. A yacht sails 90 miles down a river whose current moves 3 miles an hour, and is gone 16 hours; required the rate of sailing. *Ans.* 12 miles an hour.

20. A farmer bought a number of sheep for $80; if he had bought 4 more for the same money, he would have paid $1 less for each; how many did he buy? *Ans.* 16 sheep.

21. A man bought a quantity of meat for $2.16. If meat were to rise in price 1 cent per pound, he would get 3 pounds less for the same sum. How much meat did he buy? *Ans.* 27 lbs.

22. The plate of a mirror, 18 inches by 12, is to be set in a frame of uniform width, whose surface is to be equal to the surface of the glass; required the width of the frame. *Ans.* 3 inches.

23. Todhunter gives the following beautiful little problem: Find the price of eggs per dozen when two less for 12 cents raises the price 1 cent per dozen. *Ans.* 8 cts.

24. A and B start at the same time to travel 90 miles; A travels 1 mile an hour faster than B, and arrives 1 hour earlier; at what rate per hour did each travel?
Ans. A, 10 mi.; B, 9 mi.

25. A person bought cloth for $72, which he sold again at $$6\frac{1}{2}$ a yard, and gained by the bargain as much as one yard cost him; required the number of yards. *Ans.* 12.

QUADRATIC EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

292. The **Degree** of an equation containing two or more unknown quantities is determined by the greatest sum of the exponents of the unknown quantities contained in any term. Thus,

$2ax+3xy=4a$ is an equation of the 2d degree;

$3xy+4x^2y=12$ is an equation of the 3d degree.

293. A **Homogeneous Equation** is one in which the sum of the exponents of the unknown quantities in each term which contains them is the same. Thus,

$$3x^2+2xy+y^2=31$$

and

$$x^3+3x^2y+3xy^2+y^3=27$$

are each homogeneous equations.

294. A **Symmetrical Equation** is one in which the unknown quantities are similarly involved, or one in which they can change places without destroying the equation. Thus,

$$x^2+y^2=13, \text{ and } \frac{x}{y}+\frac{y}{x}=2\tfrac{1}{2}$$

and

$$x^2+y^2-xy+3x+3y=22$$

are each symmetrical.

295. Quadratic Equations containing two unknown quantities can generally be solved by the rules for quadratics if they come under one of the following cases:

I. When one of the equations is simple and the other quadratic.

II. When each equation is homogeneous and quadratic.

III. When each equation is symmetrical.

NOTE.—Two quadratics containing two unknown quantities usually produce a biquadratic in elimination; hence all quadratics containing two unknown quantities cannot be solved by the rules for quadratics.

CASE I.

296. When one of the equations is simple and the other quadratic.

297. Equations of this case can generally be solved by substituting in the quadratic equation an expression for the value of one unknown quantity found from the simple equation.

1. Given $\left\{\begin{array}{l} 3x+y=9 \\ x^2+y^2=13 \end{array}\right\}$, to find x and y.

SOLUTION.

	$3x+y=9,$	(1)
	$x^2+y^2=13.$	(2)
From (1),	$y=9-3x;$	(3)
squaring (3),	$y^2=81-54x+9x^2;$	(4)
substituting in (2),	$x^2+81-54x+9x^2=13;$	(5)
reducing (5),	$x^2-\frac{27}{5}x=-\frac{34}{5};$	
completing the square,	$x^2-\frac{27}{5}x+(\frac{27}{10})^2=\frac{49}{100};$	
extracting sq. root,	$x-\frac{27}{10}=\pm\frac{7}{10};$	
whence,	$x=2$, or $3\frac{2}{5}$,	
and	$y=3$, or $-1\frac{1}{5}$.	

EXAMPLES.

Find the values of x and y in the following equations:

2. Given $\left\{\begin{array}{l} xy=15 \\ x+y=8 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \text{ or } 3, \\ y=3, \text{ or } 5. \end{array}\right.$

3. Given $\left\{\begin{array}{l} x-y=3 \\ x^2-y^2=21 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=2. \end{array}\right.$

4. Given $\left\{\begin{array}{l} x+y=6 \\ x^2+y^2=20 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{array}\right.$

5. Given $\left\{\begin{array}{l} xy=8 \\ 4x-3y=10 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } -1\frac{1}{2}, \\ y=2, \text{ or } -5\frac{1}{3}. \end{array}\right.$

6. Given $\left\{\begin{array}{l} 2x+y=11 \\ 3x^2-y^2=2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } +41. \\ y=5, \text{ or } -71. \end{array}\right.$

7. Given $\left\{\begin{array}{l} xy=18 \\ 3y-2x=12 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } -9, \\ y=6, \text{ or } -2. \end{array}\right.$

8\. Given $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=5\\ \frac{1}{x^2}+\frac{1}{y^2}=13\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{1}{2}, \text{ or } \frac{1}{3}\\ y=\frac{1}{3}, \text{ or } \frac{1}{2}\end{array}\right.$

9 Given $\left\{\begin{array}{l}xy=35\\ x^2-y^2=24\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 7 \text{ or } \pm 5\sqrt{(-1)},\\ y=\pm 5, \text{ or } \pm 7\sqrt{(-1)}.\end{array}\right.$

10\. Given $\left\{\begin{array}{l}x^3-y^3=28(x-y)\\ x+y=6\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=4, \text{ or } 2,\\ y=2, \text{ or } 4\end{array}\right.$

CASE II.

298. When each equation is homogeneous and quadratic.

299. Equations in this case are usually most conveniently solved by substituting for one unknown quantity the product of the other by a third unknown.

1\. Given $\left\{\begin{array}{l}x^2+xy=10\\ xy+2y^2=24\end{array}\right\}$, to find x and y.

SOLUTION.

	$x^2+xy=10,$	(1)
	$xy+2y^2=24.$	(2)
Let	$y=vx,$	(3)
substituting in (1),	$x^2+vx^2=10,$	(4)
substituting in (2),	$vx^2+2v^2x^2=24,$	(5)
from (4),	$x^2=\frac{10}{1+v},$	(6)
from (5),	$x^2=\frac{24}{v+2v^2},$	(7)
equating (6) and (7),	$\frac{10}{1+v}=\frac{24}{v+2v^2},$	(8)
clearing of fractions, etc.,	$10v^2-7v=12,$	(9)
solving (9),	$v=\frac{3}{2}, \text{ or } -\frac{4}{5}.$	(10)
substituting in (6),	$x^2=\frac{10}{1+\frac{3}{2}}=4,$	
and	$x^2=\frac{10}{1-\frac{4}{5}}=50;$	
whence,	$x=\pm 2, \text{ or } \pm 5\sqrt{2},$	
and	$y=\pm 3, \text{ or } \pm 4\sqrt{2}.$	

2\. Given $\left\{\begin{array}{l}x^2-2xy=5\\ x^2-y^2=21\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 5,\\ y=\pm 2.\end{array}\right.$

3\. Given $\left\{\begin{array}{l}x^2-y^2=12\\ x^2-xy+y^2=12\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 4,\\ y=\pm 2.\end{array}\right.$

4\. Given $\left\{\begin{array}{l}x^2y(x+y)=20\\ x^2y(2x-3y)=20\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2\sqrt{2}, \text{ or } \pm 2\sqrt{-2},\\ y=\pm \frac{1}{2}\sqrt{2}, \text{ or } \pm \frac{1}{2}\sqrt{-2}.\end{array}\right.$

5\. Given $\left\{\begin{array}{l}x^2-xy=8\\ x^2-y^2=12\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 4, \pm\sqrt{\infty},\\ y=\pm 2, \pm\sqrt{\infty}.\end{array}\right.$

6\. Given $\left\{\begin{array}{l}x^2+xy=10\\ x^2+y^2=13\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2, \text{ or } \pm \frac{5}{2}\sqrt{2},\\ y=\pm 3, \text{ or } \mp \frac{1}{2}\sqrt{2}.\end{array}\right.$

7\. Given $\left\{\begin{array}{l}x^2-y^2=3\\ x^2-2xy+2y^2=2\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2, \text{ or } \pm \frac{4}{5}\sqrt{5},\\ y=\pm 1, \text{ or } \pm \frac{1}{5}\sqrt{5}.\end{array}\right.$

CASE III.

300. When each equation is symmetrical.

301. There is no general method for the solution of equations in this case. The various expedients employed depend upon the powers of binomials and principles of factoring.

1\. Given $\left\{\begin{array}{l}xy=24\\ x+y=10\end{array}\right\}$, to find x and y.

SOLUTION.

	$xy=24,$	(1)
	$x+y=10.$	(2)
Squaring (2),	$x^2+2xy+y^2=100,$	(3)
multiplying (1) by (4),	$4xy=96,$	(4)
subtracting (4) from (3),	$x^2-2xy+y^2=4,$	(5)
extracting square root,	$x-y=\pm 2,$	(6)
uniting (6) and (2),	$x=6, \text{ or } 4,$	
and	$y=4, \text{ or } 6.$	