2. Given $\begin{cases} x+y=5 \\ x^2y^2-4xy=12 \end{cases}$, to find x and y.

SOLUTION.

	$x+y=5,$	(1)
	$x^2y^2-4xy=12.$	(2)
Completing the square,	$x^2y^2-4xy+4=16,$	(3)
extracting square root,	$xy-2=\pm 4;$	(4)
whence,	$xy=6, \text{ or } -2,$	(5)
squaring (1),	$x^2+2xy+y^2=25,$	
subtracting 4 times (5),	$x^2-2xy+y^2=1, \text{ or } 33,$	
extracting square root,	$x-y=\pm 1, \text{ or } \pm\sqrt{33};$	
whence,	$x=3, \text{ or } 2, \text{ or } \frac{1}{2}(5\pm\sqrt{33}),$	
and	$y=2, \text{ or } 3, \text{ or } \frac{1}{2}(5\pm\sqrt{33}).$	

3. Given $\begin{cases} x+y=7 \\ x^3+y^3=91 \end{cases}$, to find x and y.

SOLUTION.

	$x+y=7,$	(1)
	$x^3+y^3=91.$	(2)
Dividing (2) by (1),	$x^2-xy+y^2=13,$	(3)
squaring (1),	$x^2+2xy+y^2=49,$	(4)
subtracting (3) from (4),	$3xy=36,$	(5)
dividing by 3,	$xy=12,$	(6)
subtracting (6) from (3),	$x^2-2xy+y^2=1,$	
extracting square root,	$x-y=\pm 1;$	
whence,	$x=4, \text{ or } 3,$	
and	$y=3, \text{ or } 4.$	

NOTES.—1. Let the pupils see that the values of x and y in these equations are not equal to each other; for when $x=4$, $y=3$; and when $x=3$, $y=4$. Their values are interchangeable.

2. The signs $\pm$ and $\mp$ are equivalent when used independently; but when taken in connection they are the reverse of each other. Thus, if $x=\pm a$ and $y=\mp b$, then when $x=+a$, $y=-b$; and when $x=-a$, $y=+b$.

EXAMPLES.

4. Given $\begin{cases} xy=20 \\ x-y=1 \end{cases}$. *Ans.* $\begin{cases} x=5, \text{ or } -4, \\ y=4, \text{ or } -5. \end{cases}$

5. Given $\begin{cases} \sqrt{(xy)}=2 \\ \sqrt{x}+\sqrt{y}=3 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{cases}$

6. Given $\begin{cases} x-y=3 \\ \dfrac{x^2}{y^2}+\dfrac{4x}{y}=32 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 2\frac{2}{3}, \\ y=1, \text{ or } -\frac{1}{3}. \end{cases}$

7. Given $\begin{cases} xy=15 \\ (x+y)^2-6(x+y)=16 \end{cases}$.
Ans. $\begin{cases} x=5, 3, \text{ or } -1\pm\sqrt{(-14)}, \\ y=3, 5, \text{ or } -1\mp\sqrt{(-14)}. \end{cases}$

8. Given $\begin{cases} x+\sqrt{(xy)}+y=9 \\ x^2+xy+y^2=27 \end{cases}$. *Ans.* $\begin{cases} x=3, \\ y=3. \end{cases}$

9. Given $\begin{cases} x-y=2 \\ x^3-y^3=152 \end{cases}$. *Ans.* $\begin{cases} x=6, \text{ or } -4, \\ y=4, \text{ or } -6. \end{cases}$

10. Given $\begin{cases} x^3-y^3=19 \\ x^2y-xy^2=6 \end{cases}$. *Ans.* $\begin{cases} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{cases}$

11. Given $\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=9 \\ x+y=6 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{cases}$

302. Equations which are not symmetrical may sometimes be so combined as to produce a symmetrical equation.

303. Equations that are not symmetrical with respect to the unknown quantities themselves may be symmetrical with respect to some multiple or power of these quantities.

1. Given $\begin{cases} x^2+xy=45 \\ y^2+xy=36 \end{cases}$, to find x and y.

SOLUTION.

	$x^2+xy=45;$	(1)
	$y^2+xy=36.$	(2)
Adding,	$x^2+2xy+y^2=81;$	(3)
evolving,	$x+y=\pm 9;$	(4)
subtracting (2) from (1),	$x^2-y^2=9;$	(5)
dividing (5) by (4),	$x-y=\pm 1;$	(6)
adding (4) and (6),	$2x=\pm 10$	(7)
whence	$x=\pm 5;$	(8)
subtracting (6) from (4),	$2y=\pm 8;$	(9)
whence	$y=\pm 4.$	(10)

2. Given $\begin{cases} x+2y=13 \\ x^2+4y^2=109 \end{cases}$, to find x and y.

SOLUTION.

	$x+2y=13$	(1)
	$x^2+4y^2=109$	(2)
Squaring (1),	$x^2+4xy+4y^2=169$;	(3)
subtracting (2) from (3),	$4xy=60$;	(4)
subtracting (4) from (2),	$x^2-4xy+4y^2=49$;	(5)
evolving,	$x-2y=\pm 7$;	(6)
adding (1) and (6),	$2x=20$, or 6;	(7)
whence	$x=10$, or 3;	(8)
subtracting (6) from (1),	$4y=6$, or 20;	(9)
whence	$y=\frac{3}{2}$, or 5.	(10)

NOTE.—In Example 1, the sum of the two equations gives a symmetrical equation. Example 2 is symmetrical with respect to x and $2y$.

EXAMPLES.

3. Given $\begin{cases} x^2+y^2+2x=19 \\ xy+y=8 \end{cases}$. *Ans.* $\begin{cases} x=1, \text{ or } 3, \\ y=4, \text{ or } 2. \end{cases}$

4. Given $\begin{cases} x-3y=3 \\ x^2+9y^2=45 \end{cases}$. *Ans.* $\begin{cases} x=6, \text{ or } -3, \\ y=1, \text{ or } -2. \end{cases}$

5. Given $\begin{cases} xy=ab \\ \frac{x}{a}+\frac{y}{b}=2 \end{cases}$. *Ans.* $\begin{cases} x=a, \\ y=b. \end{cases}$

6. Given $\begin{cases} x^2+y^2=20 \\ xy-x-y=2 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{cases}$

7. Given $\begin{cases} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \end{cases}$. *Ans.* $\begin{cases} x=a, \text{ or } 0, \\ y=0, \text{ or } b. \end{cases}$

8. Given $\begin{cases} x^2+y^2=106 \\ x-y+\sqrt{(x-y)}=6 \end{cases}$. *Ans.* $\begin{cases} x=9, \text{ or } -5, \\ y=5, \text{ or } -9. \end{cases}$

9. Given $\begin{cases} x+y=a^3-b^3 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=a-b \end{cases}$. *Ans.* $\begin{cases} x=a^3, \text{ or } -b^3, \\ y=-b^3, \text{ or } a^3. \end{cases}$

10. Given $\begin{cases} xy^2-x^2y=-6 \\ x^3-y^3=19 \end{cases}$. *Ans.* $\begin{cases} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{cases}$

11. Given $\begin{cases} x^2y+xy^2=20 \\ x^3+y^3=65 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{cases}$

12. Given $\begin{cases} 2xy=2a^{\frac{1}{2}}b^{\frac{1}{2}} \\ x^4+y^4=a^2+b^2 \end{cases}$. *Ans.* $\begin{cases} x=\sqrt{a}, \text{ or } \sqrt{b}, \\ y=\sqrt{b}, \text{ or } \sqrt{a}. \end{cases}$

13. Given $\begin{cases} \sqrt[3]{x}-\sqrt[3]{y}=1 \\ x-y=7 \end{cases}$. *Ans.* $\begin{cases} x=8, \text{ or } -1, \\ y=1, \text{ or } -8. \end{cases}$

14. Given $\begin{cases} xy=36 \\ x-y=\sqrt{x}+\sqrt{y} \end{cases}$. *Ans.* $\begin{cases} x=9, \text{ or } 4, \\ y=4, \text{ or } 9. \end{cases}$

15. Given $\begin{cases} xy=6 \\ x^2-3x+3y=10-y^2 \end{cases}$. *Ans.* $\begin{cases} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{cases}$

16. Given $\begin{cases} x^3+y^3=\frac{7}{16}(x+y)^3 \\ xy=3 \end{cases}$. *Ans.* $\begin{cases} x=3, \text{ or } 1, \\ y=1, \text{ or } 3. \end{cases}$

17. Given $\begin{cases} x^{-2}+y^{-2}=\frac{13}{36} \\ x^{-1}+y^{-1}=\frac{5}{6} \end{cases}$. *Ans.* $\begin{cases} x=2, \text{ or } 3, \\ y=3, \text{ or } 2. \end{cases}$

18. Given $\begin{cases} x^2+y^2=25 \\ x^4+y^4=337 \end{cases}$. *Ans.* $\begin{cases} x=\pm 3, \text{ or } \pm 4, \\ y=\pm 4, \text{ or } \pm 3. \end{cases}$

19. Given $\begin{cases} xy=6 \\ x^4+y^4=97 \end{cases}$. *Ans.* $\begin{cases} x=\pm 3, \text{ or } \pm 2, \\ y=\pm 2, \text{ or } \pm 3. \end{cases}$

20. Given $\begin{cases} x+y=5 \\ x^4+y^4=257 \end{cases}$. *Ans.* $\begin{cases} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{cases}$

PROBLEMS

PRODUCING QUADRATICS WITH TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is 7, and the sum of their squares is 25; required the numbers. *Ans.* 4 and 3.

2. The difference of two numbers is 2, and the difference of their squares is 20; required the numbers. *Ans.* 6 and 4.

3. Divide 97 into two such parts that the sum of the square roots of those parts may equal 13. *Ans.* 81 and 16.

4. The difference of two numbers is a, and the difference of their square roots is $\frac{1}{2}\sqrt{2a}$; required the numbers.
Ans. $\frac{9}{8}a$ and $\frac{a}{8}$.

5. Find two numbers whose product is 3 times their sum, and the sum of their squares is 160. *Ans.* 4 and 12.

6. Divide the number 10 into two such parts that the sum of the cubes of the parts may be 280. *Ans.* 6 and 4.

7. The difference of two numbers is 3, and the difference of their cubes is 117; required the numbers. *Ans.* 5 and 2.

8. Find two numbers whose product is 6 times their difference, and the sum of their squares is 13. *Ans.* 3 and 2.

9. The sum of two numbers is a, and the sum of their cubes is $4a^3$; required the numbers. *Ans.* $\frac{a}{2}(1 \pm \sqrt{5})$; $\frac{a}{2}(1 \mp \sqrt{5})$.

10. Two men, A and B, can together do a piece of work in 12 days; in how many days can each do it if it takes B 10 days longer than A? *Ans.* A, 20 days; B, 30 days.

11. A colonel forms his regiment of 1025 men into two squares, one of which has 5 men more in a side than the other; required the number of men in a side of each. *Ans.* 20; 25.

12. A farmer sold 7 calves and 12 sheep for $50; and the price received for each was such that 3 more calves were sold for $10 than sheep for $6; what was the price of each?
Ans. Calves, $2; sheep, $3.

13. Find two numbers such that their difference added to the difference of their squares shall equal 6, and their sum added to the sum of their squares shall equal 18. *Ans.* 3 and 2.

14. The expense of a sociable was $70, but before the bill was paid, 4 of the young men sneaked off, in consequence of which each of the others had to pay $2 more than his proper share; how many young men were there? *Ans.* 14.

15. A merchant sold some cloth for $24, and some silk at $1 less a yard for the same sum; required the number of yards of each, provided there were 2 yards of silk more than of cloth.
Ans. Cloth, 6; silk, 8.

16. A and B run a race; B, who runs slower than A by a mile in 2 hours, starts first by 2 minutes, and they get to the 4-mile stone together; required their rates of running.
Ans. A, 8 mi.; B, $7\frac{1}{2}$ mi. an hour.

17. A certain rectangle contains 300 square feet; a second rectangle is 8 feet shorter and 10 feet broader, and also contains 300 square feet; find the length and breadth of the first rectangle. *Ans.* Length, 20 ft.; breadth, 15 ft

18. A bought two pieces of cloth of different sorts; the finer cost 1 dollar more a yard than the coarser, and there were 10 yards more of the coarser than the finer; find how many yards there were in each piece, provided the coarser cost $80 and the finer $90. *Ans.* 30 yds.; 40 yds.

19. The area of a rectangular field is 2275 square rods; and if the length of each side is diminished by 5 rods, the area will be 1800 rods; required the dimensions of the field.
Ans. 65 rods; 35 rods.

20. There is a certain number, of two digits; the sum of the squares of the digits is equal to the number increased by the product of the digits, and if 36 be added to the number, the digits will be reversed; what is the number?
Ans. 48.

21. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per cubic yard as there were yards in the side of the other; and the greater stood on more ground than the less by 9 square yards; what was the price of each? *Ans.* £25 and £16.

22. A laborer dug two trenches for £17 16s., one of which was 6 yards longer than the other, and the digging of each trench cost as many shillings a yard as it was yards in length; what was the length of each? *Ans.* 10 yds.; 16 yds.

23. Required two numbers such that their sum, their product and the difference of their squares shall be equal to one another.
Ans. $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$; $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

24. Two partners, A and B, gained $18 by trade: A's money was in trade 12 months, and he received for his principal and gain $26; B's money, which was $30, was in trade 16 months. How much did A put in trade? *Ans.* $20.

25. The fore wheels of a carriage make 5 revolutions more than the hind wheels in going 60 yards; but if the circumference of each should be increased one yard, the fore wheels will make only 3 more revolutions than the hind wheels in the same distance; required the circumference of each.
Ans. 3 and 4 yards.

26. An English landholder received £7 4s. for a certain quantity of wheat, and an equal sum, at a price less by 1s. 6d.

per bushel, for a quantity of barley which exceeded the quantity of wheat by 16 bushels; how many bushels were there of each? *Ans.* 32 bu. wheat; 48 bu. barley.

27. A and B run a race around a two-mile course. In the first heat B reaches the winning-post 2 minutes before A; in the second heat A increases his speed 2 miles per hour, and B diminishes his as much, and A then arrives at the winning-post 2 minutes before B. Find at what rate each man ran in the first heat. *Ans.* 10 mi. per hour; 12 mi. per hour.

PRINCIPLES OF QUADRATIC EQUATIONS.

304. The PRINCIPLES of Quadratics are the relations which exist between a quadratic and its roots.

NOTE.—This subject may be omitted by young pupils, and even by older pupils until review, if the teacher prefers.

PRINCIPLE I.

Every quadratic equation has two roots, and only two.

FIRST. The general form of the complete quadratic is $x^2+2px=q$. Completing the square of the general quadratic, and finding the value of x, we have two values, $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$, which proves the principle.

OPERATION.

$$x^2+2px=q \qquad (1)$$
$$x^2+2px+p^2=q+p^2 \qquad (2)$$
$$x=-p+\sqrt{q+p^2} \qquad (3)$$
$$x=-p-\sqrt{q+p^2} \qquad (4)$$

SECOND. This proposition can also be demonstrated in another way, as follows:

Assume that $m^2=q+p^2$, or $m=\sqrt{q+p^2}$,
then we have $x^2+2px+p^2=m^2$;
or, in another form, $(x+p)^2=m^2$;
transposing, $(x+p)^2-m^2=0$;
factoring, $(x+p+m)(x+p-m)=0$;
making the second factor equal to zero, $x+p-m=0$;
from which we have $x=-p+m=-p+\sqrt{q+p^2}$;
making the first factor equal to zero, $x+p+m=0$;
from which we have $x=-p-m=-p-\sqrt{q+p^2}$;

and since equation (3) can be satisfied in these two ways, and in these two ways only, therefore x can have but two values, and the principle is true.

PRINCIPLE II.

The sum of the two roots of a quadratic of the form $x^2+2px=q$ is equal to the coefficient of the first power of x, *with its sign changed.*

For, solving the quadratic $x^2+2px=q$, we find the two roots are $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$; taking the sum of the two roots, we have $-2p$, which is the coefficient of the first power of x, with its sign changed. Therefore, etc.

OPERATION.

$$x^2+2px=q$$
$$x=-p+\sqrt{q+p^2}$$
$$\underline{x=-p-\sqrt{q+p^2}}$$
$$-2p$$

PRINCIPLE III.

The product of the two roots of a quadratic of the form $x^2+2px=q$ is equal to the known term with its sign changed.

For, multiplying the two roots together, we have the result $p^2-(q+p^2)$, which, reduced, equals $-q$, which is the known term with its sign changed. Therefore, etc.

OPERATION.

$$x=-p+\sqrt{q+p^2}$$
$$\underline{x=-p-\sqrt{q+p^2}}$$
$$p^2-p\sqrt{q+p^2}$$
$$\underline{+p\sqrt{q+p^2}-(q+p^2)}$$
$$p^2-(q+p^2)=-q$$

305. Principles II. and III. enable us to construct quadratic equations from their roots.

1. Find the quadratics whose roots are 4 and 3.

SOLUTION. Since the sum of the two roots with its sign changed equals the *coefficient* of x, and their product, with its sign changed, equals the known term, the equation whose roots are 4 and 3 must be $x^2-(4+3)x=-(4\times3)$, which, reduced, becomes $x^2-7x=-12$.

OPERATION.

$$x^2-(4+3)x=-4\times3$$
$$x^2-7x=-12$$

EXAMPLES.

Find the quadratic whose—

2. Roots are 5 and 4. *Ans.* $x^2-9x=-20$
3. Roots are 7 and -3. *Ans.* $x^2-4x=21$.
4. Roots are 4 and -9. *Ans.* $x^2+5x=36$.

5. Roots are -2 and -8. *Ans.* $x^2+10x=-16$.

6. Roots are $2+\sqrt{3}$ and $2-\sqrt{3}$. *Ans.* $x^2-4x=-1$.

7. Roots are a and b. *Ans.* $x^2-(a+b)x=-ab$.

8. Roots are $a+b$ and $a-b$. *Ans.* $x^2-2ax=b^2-a^2$.

9. Roots are $a+b\sqrt{c}$ and $a-b\sqrt{c}$. *Ans.* $x^2-2ax=b^2c-a^2$.

PRINCIPLE IV.

A quadratic equation of the form $x^2+2px=q$ may be resolved into two binomial factors, of which the first term in each is x, *and the second term the roots with their signs changed.*

For, suppose the two roots are r and r'; then (Prin. II. and III.) we have $x^2-(r+r')x=-rr'$; transposing rr' to the first member, we have $x^2-(r+r')x+rr'=0$; factoring, we have $(x-r)(x-r')=0$; which proves the principle.

OPERATION.

Let $x=r$ and r'

$$x^2-(r+r')x=-rr'$$
$$x^2-(r+r')x+rr'=0$$
$$(x-r)(x-r')=0$$

306. Quadratic equations may also be constructed from their roots by Principle IV.

1. Find an equation whose roots are 3 and 5.

SOLUTION. Since the roots are 3 and 5, we have (Prin. IV.) $(x-3)(x-5)=0$. Expanding, we have $x^2-8x+15=0$; transposing, we have $x^2-8x=-15$.

OPERATION.

$$(x-3)(x-5)=0$$
$$x^2-8x+15=0$$
$$x^2-8x=-15$$

EXAMPLES.

Find the equation whose—

2. Roots are 6 and -4. *Ans.* $x^2-2x=24$.

3. Roots are $+3$ and -8. *Ans.* $x^2+5x=24$.

4. Roots are $2a$ and $3a$. *Ans.* $x^2-5ax=-6a^2$.

5. Roots are $+a$ and $-a$. *Ans.* $x^2=a^2$.

6. Roots are $\frac{m}{2}$ and $-\frac{n}{2}$. *Ans.* $x^2+\frac{1}{2}(n-m)x=\frac{1}{4}mn$.

7. Roots are $a+2\sqrt{n}$ and $a-2\sqrt{n}$. *Ans.* $x^2-2ax=4n-a^2$

FORMS OF QUADRATICS.

307. There are four distinct forms of the complete quadratic, depending upon the sign of $2p$ and q. Thus,

1st form,	$x^2+2px=q$.	3d form,	$x^2+2px=-q$.
2d form,	$x^2-2px=q$.	4th form,	$x^2-2px=-q$.

PRINCIPLE V.

In a quadratic of the first form one root is positive and the other negative, the negative root being the greater.

For, since q is the product of the two roots with its sign changed, and is *positive*, one root must be positive and the other negative; and since $2p$ is the sum of the roots with its sign changed, and is *positive*, the *negative* root must be the greater.

PRINCIPLE VI.

In a quadratic of the second form one root is positive and the other negative, the positive root being the greater.

For, since q is the product of the two roots with its sign changed, and is *positive*, one root must be positive and the other negative; and since $2p$ is their sum with its sign changed, and is *negative*, the *positive* root must be the *greater*.

PRINCIPLE VII.

In a quadratic of the third form both roots are negative.

For, since q is the product of the two roots with its sign changed, and is *negative*, the two roots must be *both negative* or *both positive;* and since $2p$ is the sum of the roots with its sign changed, and is *positive, both* roots must be *negative*.

PRINCIPLE VIII.

In a quadratic of the fourth form both roots are positive.

For, since q is the product of the two roots with its sign changed, and is *negative*, the roots must be *both positive* or *both negative;* and since $2p$ is the sum of the roots with its sign changed, and is *negative, both* roots must be *positive*.

EXAMPLES.

Required the form where the—

1. Roots are -8 and 5. *Ans.* 1st form.
2. Roots are 9 and -6. *Ans.* 2d form.
3. Roots are 6 and 7. *Ans.* 4th form.
4. Roots are -5 and -4. *Ans.* 3d form.
5. Roots are $-3a$ and $2a$. *Ans.* 1st form.

VALUES OF p AND q.

308. The quantities p and q are general, and may therefore have any values whatever. We will now discuss the equation by assigning different values to each.

FIRST.—Suppose $q=0$.

Solving the equation $x^2+2px=q$, and substituting 0 for q in the root, we have $x=-p\pm p$, whence $x=0$, or $-2p$.

1ST OPERATION.

$$x^2+2px=q$$
$$x=-p\pm\sqrt{q+p^2}$$
$$\text{Let}\quad q=0$$
$$x=-p\pm p$$
$$x=0,\text{ or }-2p$$

2D OPERATION.

$$x^2+2px=q$$
$$\text{Let}\quad q=0$$
$$x^2+2px=0$$
$$x(x+2p)=0$$
$$x=0$$
$$x=-2p$$

Making the same substitution in the equation, we have $x^2+2px=0$; factoring, we have $x(x+2p)=0$; dividing by $x+2p$, we have $x=0$; dividing by x, we have $x+2p=0$, or $x=-2p$.

The third form gives the same result; the second and fourth forms give $x=0$ and $x=+2p$.

SECOND.—Suppose $2p=0$.

If $2p=0$, by substituting 0 for $2p$ in the root of the quadratic we have $x=\pm\sqrt{q}$.

1ST OPERATION.

$$x^2+2px=q$$
$$x=-p\pm\sqrt{q+p^2}$$
$$\text{Let}\quad 2p=0$$
$$x=\pm\sqrt{q}$$

2D OPERATION.

$$x^2+2px=q$$
$$\text{Let}\quad 2p=0$$
$$x^2=q$$
$$x=\pm\sqrt{q}$$

Making the same substitution in the equation, it reduces to a pure

quadratic, $x^2=q$. Solving this, we have the same result as before, $x=\pm\sqrt{q}$.

In the first and second forms this result will be *real*; in the third and fourth forms, in which q is negative, the result will be imaginary.

THIRD.—Suppose $p^2=q$ when q is negative.

Take the quadratic of the third form, $x^2+2px=-q$. Substituting p^2 for q in the root, we have $x=-p+0$, or $-p$; and $x=-p-0$, or $-p$; hence x has two values, both of which are $-p$.

1ST OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{p^2-q}$$
$$\text{Let}\quad p^2=q$$
$$x=-p+0=-p$$
$$x=-p-0=-p$$

2D OPERATION.

$$x^2+2px=-q$$
$$\text{Let}\quad p^2=q$$
$$x^2+2px+p^2=0$$
$$(x+p)^2=0$$
$$(x+p)(x+p)=0$$
$$x=-p$$
$$x=-p$$

Making the same substitution in the equation and reducing, we have $(x+p)(x+p)=0$; dividing by the first factor, we have $x+p=0$, or $x=-p$; dividing by the second factor, we have $x+p=0$, or $x=-p$.

In the first and second forms the results will be different.

FOURTH.—Suppose q to be greater than p^2 when q is negative.

If in the third or fourth form we assume q numerically greater than p^2, the quantity under the radical becomes a negative quantity, and the value of x is therefore imaginary.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{p^2-q}$$

Hence, the root of an equation in the third and fourth forms is imaginary when q is numerically greater than p^2.

EXAMPLES.

1. Find a number such that its square increased by four times the number equals zero. *Ans.* -4.
2. Required the number such that its square, plus 6 times that number, shall equal minus 9. *Ans.* -3.
3. Divide 8 into two such parts that their product shall be equal to 20. *Ans.* $4\pm2\sqrt{-1}$.

Why does the root in the last equation become imaginary? Which supposition does Ex. 2 illustrate? What does Ex. 1 illustrate?

IMAGINARY ROOT.

309. An **Imaginary Root** of a quadratic is a root which contains an imaginary quantity.

310. The **Imaginary Root** occurs in the third and fourth forms of a quadratic upon a certain supposition.

311. We shall now discuss the imaginary root under three distinct heads:

FIRST.—When does a quadratic give an imaginary root?

PRIN. 1. *A quadratic gives an imaginary root when the known term is negative and numerically greater than the square of half the coefficient of the first power of* x.

For, if q is negative and numerically greater than p^2, the quantity under the radical is *negative*, and we shall have the square root of a negative quantity, which is imaginary. Therefore, etc.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{(p^2-q)}$$

SECOND.—What is assumed by a quadratic which gives an imaginary root?

PRIN. 2. *A quadratic which gives an imaginary root assumes that the product of two quantities is greater than the square of half their sum.*

For, since $2p$ is the *sum* of the two roots with its sign changed, p^2 is the *square of half the sum* of two quantities, and q is the *product* of the two roots, with its sign changed; hence, when q is negative and greater than p^2, the quadratic assumes that the product of two quantities is greater than the square of half their sum.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{(p^2-q)}$$

THIRD.—Prove that this assumption is false.

PRIN. 3.—*The product of two quantities can never be greater than the square of half their sum; hence, the above assumption is false.*

Let $2p$ represent any number, and let it be divided into two parts, $p+z$ and $p-z$; the product of the two parts is p^2-z^2; the sum of the parts is $2p$, and the square of half their sum is p^2. Now, p^2 is greater than p^2-z^2; hence the product of two numbers can never be greater than the square of half their sum.

OPERATION.

$$2p=(p+z)+(p-z);$$

Product. $(p+z)(p-z)=p^2-z^2$;

Sum, $(p+z)+(p-z)=2p$;

$\left(\frac{\text{Sum}}{2}\right)^2$, $\left(\frac{2p}{2}\right)^2=p^2$.

Now, $p^2>p^2-z^2$;

hence, $\left(\frac{\text{Sum}}{2}\right)^2>$ Product.

From the above discussion we see that a quadratic of the form $x^2\pm 2px=-q$, in which q is greater than p^2, assumes that the product of two quantities is greater than the square of half their sum, which is absurd. When a problem furnishes such an equation, the problem is impossible.

EXAMPLES.

1. Divide the number 12 into two parts such that their product shall be 40. *Ans.* $6\pm 2\sqrt{-1}$.

2. A farmer thought to enclose 40 square rods in rectangular form by a fence whose entire length shall be 20 rods; required its length and breadth. *Ans.* $5\pm\sqrt{-15}$; $5\mp\sqrt{-15}$.

Why do these problems give an imaginary result? What is incorrect in the first? What is incorrect in the second?

REVIEW QUESTIONS.

Define a Quadratic Equation. State the two classes of Quadratics. Define a Pure Quadratic. An Affected Quadratic. Give examples of each. How do we solve a pure quadratic? How solve an affected quadratic? State each method of completing the square. Explain each method.

Define a Quadratic of two Unknown Quantities. A Homogeneous Equation. A Symmetrical Equation. What cases of quadratics of two unknown quantities can be solved? Give examples. Show the method of solution.

Define principles of Quadratics. State the principles of Pure Quadratics. State the first four principles of Incomplete Quadratics. State the four forms of a quadratic. State the principles of the forms. Define an Imaginary Root. State the principles of an Imaginary Root.