SECTION VIII.

RATIO AND PROPORTION.

RATIO.

312. Ratio is the measure of the relation of two similar quantities. Thus, the ratio of 8 to 4 is 2.

313. The **Symbol** of ratio is the colon, :, read *to*, or *is to* Thus, $a : c$ indicates the ratio of a to c.

314. The **Terms** of a ratio are the two quantities compared. The first term is the *Antecedent;* the second term is the *Consequent.* The two terms together are called a *Couplet.*

315. A ratio is expressed by writing the two quantities with the symbol between them (Art. 27), or by writing the consequent under the antecedent in the form of a fraction.

Thus, the ratio of a to c is $a : c$, or $\frac{a}{c}$.

316. A **Simple Ratio** is the ratio of two quantities. A *Compound Ratio* is the product of two or more simple ratios. Thus, $(a : b)(c : d)$, or $\frac{a}{b} \times \frac{c}{d}$.

317. A **Compound Ratio** is usually expressed by writing the simple ratios one under another.

Thus, $\left\{\begin{matrix} a : b \\ c : d \end{matrix}\right\}$ expresses the ratio compounded of $a : b$ and $c : d$.

318. A **Duplicate Ratio** of two quantities is the ratio of their squares; as, $a^2 : c^2$. A **Triplicate Ratio** of two quantities is the ratio of their cubes; as, $a^3 : c^3$.

319. A **Ratio of Equality** exists when the two terms are equal. When the antecedent is the greater, it is called a ratio of *greater inequality* when less, a ratio of *less inequality.*

NOTES.—1. The symbol of ratio, :, is supposed to be a modification of the symbol of division.

2. Ratio is usually defined as the relation of two numbers. This is indefinite, however, for the ratio is the *measure* of the relation.

3. A few authors divide the second term by the first, calling it the *French Method.* This is wrong in method and name, as nearly all the French mathematicians, like the German, English, etc., divide the first term by the second.

PRINCIPLES.

1. *The ratio equals the quotient of the antecedent divided by the consequent.*

OPERATION.

Thus, if r represents the ratio of a to c, we have $r = a : c$, or r equals a divided by c (Art. 315). Therefore, etc.

$$r = a : c$$
$$r = \frac{a}{c}$$

2. *The antecedent is equal to the product of the consequent and ratio.*

OPERATION.

For, if $r = a$ divided by c, clearing of fractions, we have $a = r \cdot c$. Therefore, etc.

$$r = \frac{a}{c}$$
$$r \cdot c = a$$

3. *The consequent is equal to the quotient of the antecedent divided by the ratio.*

OPERATION.

For, if $r = a$ divided by c, clearing of fractions, we have $r \cdot c = a$; and dividing by r, we have $c = a$ divided by r. Therefore, etc.

$$r = \frac{a}{c}$$
$$r \cdot c = a$$
$$c = \frac{a}{r}$$

4. *Multiplying the antecedent or dividing the consequent multiplies the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and multiplying the numerator or dividing the denominator multiplies the fraction (Art. 129, Prin. 1). Therefore, etc.

5. *Dividing the antecedent, or multiplying the consequent, divides the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and dividing the numerator or multiplying the denominator divides the fraction (Art. 129, Prin. 2). Therefore, etc.

6. *Multiplying or dividing both terms of a ratio by any number does not change the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and multiplying or dividing both terms of a fraction does not change its value (Art. 129, Prin. 3). Therefore, etc.

NOTE.—These principles are restricted to simple ratio. Similar principles may be proved of compound ratio.

CASE I.

320. Problems which arise in simple ratio.

1. Find the ratio of $4a^2$ to $2a$. *Ans.* $2a$.

2. Find the ratio of 3 bushels to 2 pecks. *Ans.* 6.

3. Find the ratio of a^2-x^2 to $a+x$. *Ans.* $a-x$.

4. The ratio is $2a$ and the consequent $3ab$; required the antecedent. *Ans.* $6a^2b$.

5. The antecedent is $6a^3c^2$ and ratio $2a^2$; required the consequent. *Ans.* $3ac^2$.

6. The ratio is $\frac{a}{c}$ and the antecedent is $\frac{c^2}{a}$; required the consequent. *Ans.* $\frac{c^3}{a^2}$.

7. If the ratio of a to b is $\frac{8}{15}$, what is the ratio of $5a$ to $4b$? *Ans.* $\frac{2}{3}$.

8. If the ratio of $3a$ to $2b$ is $\frac{2}{3}$, what is the ratio of a to b? *Ans.* $\frac{4}{9}$.

9. If the ratio of $2m$ to $5n$ is $\frac{4}{5}$, what is the ratio of $5m$ to $2n$? *Ans.* 5.

10. If the ratio of a to c is $\frac{4}{5}$, what is the ratio of $a+c$ to $a-c$? *Ans.* -9.

11. The ratio of two numbers is $a+b$, and the consequent is $a-b$; required the antecedent. *Ans.* a^2-b^2.

CASE II.

321. Problems which arise in compound ratio.

1. Find the ratio compounded of $8:15$ and $21:24$.

SOLUTION. The ratio of 8 to 15 is $\frac{8}{15}$; the ratio of 21 to 24 is $\frac{21}{24}$; compounding them by taking their product, we have $\frac{8}{15}\times\frac{21}{24}=\frac{7}{15}$.

OPERATION. $r=\frac{8}{15}\times\frac{21}{24}=\frac{7}{15}$

2. Find the ratio compounded of $a:b$ and $b^2:3ax$. *Ans.* $\frac{b}{3x}$.

3. Required the value of—

$\left\{\begin{matrix}3:6\\8:15\end{matrix}\right\}$; of $\left\{\begin{matrix}a:b\\c:d\end{matrix}\right\}$; of $\left\{\begin{matrix}a^2:c^3\\c^2:d\end{matrix}\right\}$. *Ans.* $\frac{4}{15}$; $\frac{ac}{bd}$; $\frac{a^2}{cd}$.

4. Given the compound ratio of $x:8$ and $6:9$ equals $\frac{8}{15}$, to find the first antecedent. *Ans.* $x=6\frac{2}{5}$.

5. Given the compound ratio $\left\{\begin{matrix}9:12\\a:18\end{matrix}\right\}=\frac{1}{4}$, to find the second antecedent. *Ans.* 6.

6. Given the compound ratio $\left\{\begin{matrix}16:15\\21:c\end{matrix}\right\}=7$, to find the second consequent. *Ans.* $3\frac{1}{5}$.

7. The ratio of $2:3\frac{1}{3}$ equals the compound ratio $\left\{\begin{matrix}16:15\\18:c\end{matrix}\right\}$; required the second consequent. *Ans.* 32.

8. The duplicate of the ratio $x:1$ equals the ratio $27:x$; required the value of x. *Ans.* 3.

9. The ratio $a-x:b-x$ is the duplicate of the ratio $a:b$; required the value of x. *Ans.* $\frac{ab}{a+b}$.

10. The duplicate ratio of $x:a$ equals the compound ratio $\left\{\begin{matrix}x^2:a^2\\b^2:x\end{matrix}\right\}$; required the value of x. *Ans.* $x=b^2$.

PROPORTION.

322. A **Proportion** is an expression of equality between equal ratios. Thus, a formal comparison of the equal ratios 8 to 4 and 12 to 6, as $8 : 4 = 12 : 6$, is a proportion.

323. The **Symbol** of proportion is the double colon, $::$. Thus, $a : b :: c : d$ is read the ratio of a to b equals the ratio of c to d; or, a is to b as c is to d.

324. The **Terms** of a proportion are the four quantities compared. The first and fourth terms are the *extremes*, and the second and third are the *means*.

325. The **Couplets** are the two ratios compared. The *first couplet* consists of the first and second terms; the *second couplet* consists of the third and fourth terms.

326. A **Mean Proportional** of two quantities is a quantity which may be made the means of a proportion in which the two quantities are the extremes; as, $a : b :: b : c$.

327. A **Continued Proportion** is one in which each consequent is the same as the next antecedent; as, $a : b :: b : c :: c : d$.

328. Quantities are in proportion by *Alternation* when antecedent is compared with antecedent and consequent with consequent. Thus, if $a : b :: c : d$, by alternation, $a : c :: b : d$.

329. Quantities are in proportion by *Inversion* when the antecedents are made consequents and the consequents antecedents. Thus, if $a : b :: c : d$, by inversion, $b : a :: d : c$.

330. Quantities are in proportion by *Composition* when the sum of antecedent and consequent is compared with either antecedent or consequent. Thus, if $a : b :: c : d$, by composition, $a\ \ a+b :: c : c+d$.

331. Quantities are in proportion by *Division* when the difference of antecedent and consequent is compared with antecedent or consequent. Thus, if $a : b :: c : d$, by division, $a : a-b :: c : c-d$.

NOTE.—Ratio arises from the comparison of two *quantities;* proportion from the comparison of two *ratios*. A proportion is therefore a comparison of the results of two previous comparisons.

SIMPLE PROPORTION.

332. A **Simple Proportion** is an expression of equality between simple ratios; as, $a : b :: c : d$.

333. A **Proportion** may be written in the form of an *equation*. Thus, $a : b :: c : d$ becomes $\frac{a}{b} = \frac{c}{d}$.

334. This **Equation** is called the *fundamental equation* of the proportion. It lies at the basis of the principles of proportion.

335. The **Principles** of simple proportion are expressed in the following theorems:

THEOREM I.

In every proportion the product of the extremes is equal to the product of the means.

Let	$a : b :: c : d;$
then (Art. 333),	$\frac{a}{b} = \frac{c}{d};$
clearing of fractions	$ad = bc.$

Therefore, etc.

THEOREM II.

Either extreme is equal to the product of the means divided by the other extreme.

Let	$a : b :: c : d;$
then (Theo. I.),	$ad = bc;$
hence,	$a = \frac{bc}{d}$; and $d = \frac{bc}{a}$.

Therefore, etc.

COR.—*Either mean equals the product of the extremes divided by the other mean.*

THEOREM III.

If the product of two quantities equals the product of two other quantities, two of them may be made the extremes, and the other two the means, of a proportion.

Let	$ad = bc$;
dividing by bd,	$\frac{a}{b} = \frac{c}{d}$
or (Art. 333),	$a : b :: c : a$.

Therefore, etc.

THEOREM IV.

A mean proportional between two quantities equals the square root of their product.

Let	$a : b :: b : c$;
then (Theo. I.),	$b^2 = ac$,
and	$b = \sqrt{ac}$.

Therefore, etc.

THEOREM V.

If four quantities are in proportion, they will be in proportion by ALTERNATION.

Let	$a : b :: c : d$;
then,	$ad = bc$;
dividing by dc,	$\frac{a}{c} = \frac{b}{d}$,
whence,	$a : c :: b : d$.

Therefore, etc.

THEOREM VI.

If four quantities are in proportion, they will be in proportion by INVERSION.

Let	$a : b :: c : d$;
then (Theo. I.),	$bc = ad$;
dividing by ac,	$\frac{b}{a} = \frac{d}{c}$;
whence,	$b : a :: d : c$.

Therefore, etc.

THEOREM VII.

If four quantities are in proportion, they will be in proportion by COMPOSITION.

Let	$a : b :: c : d$;
then will	$a+b : b :: c+d : d$.
For,	$\frac{a}{b} = \frac{c}{d}$;
adding 1 to each side,	$\frac{a}{b}+1 = \frac{c}{d}+1$;
reducing,	$\frac{a+b}{b} = \frac{c+d}{d}$;
whence,	$a+b : b :: c+d : d$.

Therefore, etc.

THEOREM VIII.

If four quantities are in proportion, they will be in proportion by DIVISION.

Let	$a : b :: c : d$;
then will	$a-b : b :: c-d : d$.
For,	$\frac{a}{b} = \frac{c}{d}$;
subtracting 1,	$\frac{a}{b}-1 = \frac{c}{d}-1$;
reducing,	$\frac{a-b}{b} = \frac{c-d}{d}$;
whence,	$a-b : b :: c-d : d$.

Therefore, etc.

THEOREM IX.

If four quantities are in proportion, like powers or roots of those quantities will be proportional.

Let	$a : b :: c : d$;
then	$\frac{a}{b} = \frac{c}{d}$;
raising to nth power,	$\frac{a^n}{b^n} = \frac{c^n}{d^n}$;
hence,	$a^n : b^n :: c^n : d^n$;
similarly,	$a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Therefore, etc.

THEOREM X.

Equimultiples of two quantities are proportional to the quantities themselves.

Let a and b be any two quantities.

Then $\frac{a}{b}=\frac{a}{b}$;

multiplying by m, $\frac{ma}{mb}=\frac{a}{b}$;

whence, $ma : mb :: a : b$.

Therefore, etc.

THEOREM XI.

If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.

Let $a : b :: c : d$;

then $\frac{a}{b}=\frac{c}{d}$,

and $\frac{ma}{mb}=\frac{nc}{nd}$;

whence, $ma : mb :: nc : nd$.

Therefore, etc.

THEOREM XII.

If two proportions have a couplet in each the same, the other couplets will form a proportion.

Let $a : b :: c : d$,

and $a : b :: e : f$;

then, $\frac{a}{b}=\frac{c}{d}$ and $\frac{a}{b}=\frac{e}{f}$;

hence, $\frac{c}{d}=\frac{e}{f}$;

or, $c : d :: e : f$.

Therefore, etc.

THEOREM XIII.

The products of the corresponding terms of two proportions are proportional.

Let $a : b :: c : d$,

and $m : n :: p : q$;

then, $\frac{a}{b}=\frac{c}{d}$,

and $\frac{m}{n}=\frac{p}{q}$;

multiplying, $\frac{am}{bn}=\frac{cp}{dq}$;

whence, $am : bn :: cp : dq$.

Therefore, etc.

THEOREM XIV.

If any number of quantities are in proportion, any antecedent will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $a : b :: c : d :: e : f$, etc.;

then will $a : b :: a+c+e : b+d+f$.

For Theo. I., $ad=bc$,

and $af=be$;

also, $ab=ba$.

Adding, $ab+ad+af=ba+bc+be$,

factoring, $a(b+d+f)=b(a+c+e)$;

whence, $a : b :: a+c+e : b+d+f$.

Therefore, etc.

ADDITIONAL THEOREMS.

336. These theorems will afford pupils an opportunity to exercise original thought in applying the principles of proportion. Part of them may be omitted until review.

1. If $a : b :: c : d$, prove that $am : bn :: cm : dn$.
2. If $a : b :: c : d$, prove that $a^2 : b^2 :: ac : bd$.

3. If $a:b::c:d$, prove that $a^2:c^2::ab:cd$.

4. If $a:b::c:d$, prove that $a:a-b::c:c-d$.

5 If $a:b::c:d$, prove that $a+b:c+d::a-b:c-d$.

6 If $a:b::c:d$, prove that $a:na+mb::c:nc+md$.

7. If $a:b::b:c$, prove that $a^2:b^2::a:c$.

8. If $a:b::b:c$, prove that $a:c::b^2:c^2$.

9. If $a:b::b:c$, prove that $a^2-b^2:a::b^2-c^2:c$.

10. If $a:b::b:c$, prove that $a^2+b^2:a^2-b^2::a+c:a-c$.

11. If $a:b::b:c$, prove that $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$.

PROBLEMS IN RATIO AND PROPORTION.

337. These problems in ratio and proportion can be readily solved by an application of the previous principles.

1. The product of two numbers is 15, and the difference of their squares is to the square of their difference as 4 to 1; required the numbers.

SOLUTION.

Let	$x=$ the greater number,	
and	$y=$ the less number;	
then,	$xy=15,$	(1)
and	$x^2-y^2:(x-y)^2::4:1.$	(2)
Dividing 1st couplet by $x-y$,	$x+y:x-y::4:1,$	
by composition and division,	$2x:2y::5:3,$	
dividing 1st couplet by 2,	$x:y::5:3;$	
hence, Theo. I.,	$3x=5y,$	
substituting in eq. (1),	$\frac{5y^2}{3}=15;$	
whence,	$y=3,$	
and	$x=5.$	

2. The product of two numbers is 24, and the sum of their squares is to the square of their sum as 13 to 25; what are the numbers?

SOLUTION.

Let x and y represent the numbers.

Then,	$xy=24,$	(1)
and	$x^2+y^2:x^2+2xy+y^2::13:25.$	
By division, Theo. VIII.,	$2xy:(x+y)^2::12:25,$	
substituting for $2xy$,	$48:(x+y)^2::12:25,$	
dividing antecedents by 12,	$4:(x+y)^2::1:25;$	
extracting the square root,	$2:x+y::1:5;$	
from Theo. I.,	$x+y=10.$	

From which the values of x and y can readily be found.

3. What is the ratio of $6a$ inches to b yards? *Ans.* $\frac{a}{6b}$.

4. Two numbers are in the ratio of 2 to 3, and if 3 be added to each, the ratio is that of 5 to 7; find the numbers. *Ans.* 12 and 18.

5. Two numbers are in the ratio of 4 to 5, and if 6 be taken from each, the ratio is that of 3 to 4; find the numbers. *Ans.* 24 and 30.

6. Two numbers are in the ratio of 3 to 5, and if 2 be taken from the less and 5 be added to the greater, the ratio is that of $2:5$; find the numbers. *Ans.* 12 and 20.

7. Find the number which added to each term of the ratio $5:3$ makes it $\frac{3}{4}$ of what it would have been if the same number had been taken from each term. *Ans.* 1.

8. Find two numbers in the ratio of 2 to 3, such that their difference bears the same relation to the difference of their squares as 1 to 25. *Ans.* 10 and 15.

9. Find two numbers in the ratio of 3 to 4, such that their sum has to the sum of their squares the ratio of 7 to 50. *Ans.* 6 and 8.

10. Find two numbers in the ratio of 5 to 6, such that their sum has to the difference of their squares the ratio of 1 to 7. *Ans.* 35 and 42.

11. The sum of two numbers is 10, and the sum of their squares is to the difference of their squares as 13 to 5; required the numbers. *Ans.* 6 and 4.

12. The difference of two numbers is 6, and their product is to the sum of their squares as 2 to 5; what are the numbers? *Ans.* 12 and 6.

13. Two numbers are to each other as 3 to 2, and if 6 be added to the greater and subtracted from the less, the results will be as 3 to 1; what are the numbers? *Ans.* 24 and 16.

14. The product of two numbers is 12, and the difference of their cubes is to the sum of their cubes as 13 to 14; required the numbers. *Ans.* 6 and 2.

15. There are three numbers in continued proportion: the middle number is 60, and the sum of the others is 125; what are the numbers? *Ans.* 45; 60; 80.

16. A quantity of milk is increased by water in the ratio of 7 : 6, and then 8 gallons are sold; the remainder, when mixed with 8 gallons of water, is increased in the ratio of 7 to 5; how much milk was there at first? *Ans.* 24 gallons.

REVIEW QUESTIONS.

1. Define Ratio. The Terms. A Simple Ratio. A Compound Ratio. A Duplicate Ratio. A Triplicate Ratio. A Ratio of Equality. Of Inequality. State the Principles of Ratio.

2. Define a Proportion. The Terms of a Proportion; Extremes, Means; Couplets. A Mean Proportional. A Continued Proportion. Proportion by Alternation. By Inversion. By Composition. By Division. State the Fundamental Equation of Proportion. Enunciate the Theorems.

SECTION IX.

PROGRESSIONS.

338. A **Progression** is a series of quantities in which the terms vary according to some fixed law.

339. The **Terms** of a progression are the quantities of which it is composed.

340. The **Extremes** of a progression are the first and last terms; the *Means* are the terms between the extremes.

NOTE.—The general term for *Progression* is *series*. There are many different kinds of series; the only two appropriate for an elementary algebra are *arithmetical* and *geometrical progression*.

ARITHMETICAL PROGRESSION.

341. An **Arithmetical Progression** is a series of quantities which vary by a common difference.

342. The **Common Difference** is the quantity which, added to any term, will give the following term: thus, in 1, 3, 5, 7, the *common difference* is 2.

343. An **Ascending Progression** is one in which the quantities increase from left to right; as, 1, 3, 5, 7, 9, etc.

344. A **Descending Progression** is one in which the terms decrease from left to right; as, 12, 10, 8, 6, etc.

345. The **Terms** considered in Arithmetical Progression are *five*, any three of which being given the other two may be found.

THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The common difference, d;
5. The sum of the terms, S.