11. The sum of two numbers is 10, and the sum of their squares is to the difference of their squares as 13 to 5; required the numbers. *Ans.* 6 and 4.

12. The difference of two numbers is 6, and their product is to the sum of their squares as 2 to 5; what are the numbers? *Ans.* 12 and 6.

13. Two numbers are to each other as 3 to 2, and if 6 be added to the greater and subtracted from the less, the results will be as 3 to 1; what are the numbers? *Ans.* 24 and 16.

14. The product of two numbers is 12, and the difference of their cubes is to the sum of their cubes as 13 to 14; required the numbers. *Ans.* 6 and 2.

15. There are three numbers in continued proportion: the middle number is 60, and the sum of the others is 125; what are the numbers? *Ans.* 45; 60; 80.

16. A quantity of milk is increased by water in the ratio of 7 : 6, and then 8 gallons are sold; the remainder, when mixed with 8 gallons of water, is increased in the ratio of 7 to 5; how much milk was there at first? *Ans.* 24 gallons.

REVIEW QUESTIONS.

1. Define Ratio. The Terms. A Simple Ratio. A Compound Ratio. A Duplicate Ratio. A Triplicate Ratio. A Ratio of Equality. Of Inequality. State the Principles of Ratio.

2. Define a Proportion. The Terms of a Proportion; Extremes, Means; Couplets. A Mean Proportional. A Continued Proportion. Proportion by Alternation. By Inversion. By Composition. By Division. State the Fundamental Equation of Proportion. Enunciate the Theorems.

SECTION IX.

PROGRESSIONS.

338. A **Progression** is a series of quantities in which the terms vary according to some fixed law.

339. The **Terms** of a progression are the quantities of which it is composed.

340. The **Extremes** of a progression are the first and last terms; the *Means* are the terms between the extremes.

NOTE.—The general term for *Progression* is *series*. There are many different kinds of series; the only two appropriate for an elementary algebra are *arithmetical* and *geometrical progression*.

ARITHMETICAL PROGRESSION.

341. An **Arithmetical Progression** is a series of quantities which vary by a common difference.

342. The **Common Difference** is the quantity which, added to any term, will give the following term: thus, in 1, 3, 5, 7, the *common difference* is 2.

343. An **Ascending Progression** is one in which the quantities increase from left to right; as, 1, 3, 5, 7, 9, etc.

344. A **Descending Progression** is one in which the terms decrease from left to right; as, 12, 10, 8, 6, etc.

345. The **Terms** considered in Arithmetical Progression are *five*, any three of which being given the other two may be found.

THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The common difference, d;
5. The sum of the terms, S.

346. Prin.—*The common difference is positive in an ascending series, and negative in a descending series.*

CASE I.

347. Given the first term, the common difference and the number of terms, to find the last term.

1. Given a the first term, d the common difference, n the number of terms, to find the expression for l, the last term.

Solution. The 1st term is a; the 2d term is $a+d$; the 3d term is $a+d$ plus d, or $a+2d$; the 4th term is $a+3d$, and so on. By examining these terms, we see that *any term equals a, plus the product of d taken as many times as the number of terms less one;* hence the nth term equals $a+(n-1)d$; or, representing the nth term by l, we have $l=a+(n-1)d$.

OPERATION.

1st	term $=a$
2d	term $=a+d$
3d	term $=a+2d$
4th	term $=a+3d$
$\therefore$ nth	term $=a+(n-1)d$
or,	$l=a+(n-1)d$

Rule.—*To the first term add the product of the common difference multiplied by the number of terms less one.*

Note.—An ascending series of n terms may be written as follows:

$$a,\ a+d,\ a+2d,\ a+3d\ \ldots\ldots\ a+(n-1)d.$$

A descending series of n terms may be written as follows:

$$a,\ a-d,\ a-2d,\ a-3d\ \ldots\ldots\ a-(n-1)d.$$

EXAMPLES.

2. Find the 12th term of the series 2, 5, 8, 11, etc.

Solution. In this problem $a=2$, $d=3$ and $n=12$. The formula for the last term is $l=a+(n-1)d$; substituting the values of a, d and n, we have $l=2+(12-1)3$; and reducing, we have $l=35$.

OPERATION.

$l=a+(n-1)a$
$l=2+(12-1)3$
$l=2+33=35$

Note.—The problem may also be solved by the rule instead of substituting in the formula.

3. Find the 18th term of the series 1, 4, 7, 10, etc. *Ans.* 52.

4. Find the 17th term of the series 3, 7, 11, 15, etc. *Ans.* 67.

5. Find the 20th term of the series 1, $2\frac{1}{3}$, $3\frac{2}{3}$, 5, etc. *Ans.* $26\frac{1}{3}$.

6. Find the 14th term of the series 29, 27, 25, 23, etc. *Ans.* 3.

7. Find the 40th term of the series 1, $2\frac{2}{3}$, $4\frac{1}{3}$, 6, etc. *Ans.* 66

8. Find the 15th term of the series $\frac{7}{8}$, $\frac{13}{16}$, $\frac{3}{4}$, $\frac{11}{16}$, etc. *Ans.* 0.

9. Find the 30th term of the series a, $3a$, $5a$, $7a$, etc. *Ans.* $59a$.

10. Find the nth term of the series 2, 4, 6, 8, etc. *Ans.* $2n$.

11. Find the nth term of the series $2b$, $4b$, $6b$, etc. *Ans.* $2bn$.

12. Find the nth term of the series 1, 3, 5, 7, etc. *Ans.* $2n-1$.

13. Find the nth term of the series, 2, $2\frac{1}{3}$, $2\frac{2}{3}$, etc. *Ans.* $\frac{1}{3}(n+5)$.

14. If a body fall $16\frac{1}{12}$ feet the 1st second, 3 times as far the 2d second, 5 times as far the 3d second, and so on, how far will it fall the 20th second? *Ans.* $627\frac{1}{4}$ ft.

15. If a body fall n feet the 1st second, $3n$ feet the 2d, $5n$ feet the 3d, and so on, how far will it fall the tth second? *Ans.* $(2t-1)n$.

CASE II.

348. Given the first term, the last term, and the number of terms, to find the sum of the terms.

1. Given a, the first term; l, the last term; and n, the number of terms, to find the expression for S, the sum of the terms.

Solution.

We have the series, $S=a+(a+d)+(a+2d)+(a+3d)\ldots+l.$

inverting the series, $S=l+(l-d)+(l-2d)+(l-3d)\ldots+a.$

Adding the series, $2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots+(a+l)$

That is, $(a+l)$ taken as many times as there are terms, or n times.

Hence, $2S=(a+l)n$;

whence, $S=\left(\frac{a+l}{2}\right)n$, or $\frac{n}{2}(a+l)$.

Rule.—*Multiply the sum of the extremes by one-half of the number of terms.*

EXAMPLES.

2. Find the sum of the arithmetical series whose first term is 2, last term 35, and number of terms 12.

OPERATION.

$S=\frac{n}{2}(a+l)$

$S=\frac{12}{2}(2+35)$

$S=6\times37=222.$ *Ans.*

SOLUTION. In this problem, $a=2, n=12$, and $l=35$. Substituting the values of these terms in the formula, $S=\frac{n}{2}(a+l)$, we have $S=\frac{12}{2}(2+35)$, or 6×37, which is 222.

Find the sum—

3. When $a=3, l=40$ and $n=16$. *Ans.* 344.
4. Of 12 terms of the series $2+6+10+14$, etc. *Ans.* 288.
5. Of 16 terms of the series $3+7+11+15$, etc. *Ans.* 528.
6. Of 12 of the odd numbers $1+3+5+7$, etc. *Ans.* 144.
7. Of 12 of the even numbers $2+4+6+8$, etc. *Ans.* 156.
8. Of 18 terms of the series $\frac{1}{3}+1\frac{1}{3}+2\frac{1}{3}$, etc. *Ans.* 159.
9. Of 25 terms of the series $\frac{1}{2}+1+1\frac{1}{2}+2$, etc. *Ans.* $162\frac{1}{2}$.
10. Of 12 terms of the series $20+18+16$, etc. *Ans.* 108.
11. Of 17 terms of the series $.2+.25+.3+.35$, etc. *Ans.* 10.2.
12. Of n terms of the series $1+3+5+7$, etc. *Ans.* n^2.
13. Of n terms of the series $2+4+6+8$, etc. *Ans.* n^2+n.
14. Of n terms of $a+3a+5a+7a+$, etc. *Ans.* an^2.
15. Of 6 terms of $(a-5b)+(a-3b)+(a-b)+$, etc. *Ans.* $6a$.

16. If a body fall $16\frac{1}{12}$ feet the 1st second, 3 times as far the 2d second, 5 times as far the 3d second, and so on, how far will it fall in 20 seconds? *Ans.* $6433\frac{1}{3}$ ft.

17. If a body fall n feet the 1st second, $3n$ feet the 2d, $5n$ feet the 3d, and so on, how far will it fall in t seconds? *Ans.* t^2n ft.

CASE III.

349. Given any three of the five quantities to find either of the others.

350. The **Fundamental Formulas** of arithmetical progression are—

1. $l=a+(n-1)d;$
2. $S=\frac{n}{2}(a+l).$

These are called Fundamental because by means of them we can solve all the problems which may arise.

351. There are three classes of problems, since the four quantities may all be in the 1st formula, or all in the 2d formula, or part in the first and part in the 2d.

352. CLASS I. When the four quantities are all contained in the first fundamental formula.

1. Find the formula for d, having given a, n and l.

OPERATION.

$$l=a+(n-1)d \quad (1)$$
$$l-a=(n-1)d \quad (2)$$
$$d=\frac{l-a}{n-1} \quad (3)$$

SOLUTION. The formula $l=a+(n-1)d$ contains all the four quantities; we will therefore find the value of d from this formula in terms of the other quantities. Transposing, we have equation (2); dividing by the coefficient of d and transposing, we have equation (3), which is the value of d required.

EXAMPLES.

2. Find the value of a, having given n, d and l.
3. Find the value of n, having given a, d and l.
4. The first term is 90, the last term 34, and the common difference 4; required the number of terms. *Ans.* 15.
5. What term of the series 2, 5, 8, etc., is 35? What term of the series 29, 27, 25, etc., is 3? *Ans.* 12th; 14th.
6. The nth term of a series whose common difference is 2, is $2n$; what is the first term? *Ans.* 2.
7. The nth term of a series whose first term is 1, is $2n-1$, required the first four terms of the series. *Ans.* 1, 3, 5, 7.

353. Class II. When the four quantities are all contained in the second fundamental formula.

1. Given a, l and S, to find n.

Solution. The formula $S=\frac{n}{2}(a+l)$ contains all the four quantities; we can therefore find the value of n from this formula in terms of a, l and S. Clearing of fractions, we have equation (2); dividing by $a+l$ and transposing, we have equation (3), which is the value of n required.

OPERATION

$$S=\frac{n}{2}(a+l) \quad (1)$$
$$2S=n(a+l) \quad (2)$$
$$n=\frac{2S}{a+l} \quad (3)$$

EXAMPLES.

2. Find a, having given n, l and S.
3. Find l, having given a, n and S.
4. The first term is 2, the last term 35, and the sum of the terms 222; required the number of terms. *Ans.* 12.
5. The last term is 27, the number of terms 12, and the sum of the terms 180; required the first term. *Ans.* 3.
6. Required the last term of the series whose first term is 1, number of terms n, and sum of terms n^2. *Ans.* $2n-1$.
7. Required the first four terms of the series whose first term is 1, number of terms n, and sum n^2. *Ans.* 1, 3, 5, 7.
8. Required the last term of the series whose first term is a, number of terms n, and sum of the terms an^2. *Ans.* $a(2n-1)$.

354. Class III. When part of the quantities are in the first and part in the second fundamental formula.

1. Given d, n, l, to find S.

Solution. The two fundamental formulas contain one quantity, namely, a, not involved in this problem; hence we may combine these formulas by eliminating a, and obtain an equation containing the four quantities, d, n, l and S, from which we can find the value of S.

From the first formula we find equation (3); substituting this value in equation (2), we have equation (4); reducing, we have equation (5), which expresses the value of S in terms of d, n and l

OPERATION.

$$l=a+(n-1)d \quad (1)$$
$$S=\frac{n}{2}(a+l) \quad (2)$$
$$a=l-(n-1)d \quad (3)$$
$$S=\frac{n}{2}\{l-(n-1)d+l\} \quad (4)$$
$$S=\frac{n}{2}\{2l-(n-1)d\} \quad (5)$$

Note.—The superfluous quantity may be eliminated by comparison or substitution, as is most convenient. For examples, see the table.

TABLE OF FORMULAS.

355. Since there are five quantities, any three of which being given a fourth may be found, there are twenty cases in all. These cases are presented in the following table:

No.	Given.	To Find.	Formulas.	
1	a, d, n	l, S	$l=a+(n-1)d$;	$S=\frac{1}{2}n[2a+(n-1)d]$.
2	l, d, n	a, S	$a=l-(n-1)d$;	$S=\frac{1}{2}n[2l-(n-1)d]$.
3	a, n, l	d, S	$d=\frac{l-a}{n-1}$;	$S=\frac{1}{2}n(a+l)$.
4	d, n, S	a, l	$a=\frac{2S-n(n-1)d}{2n}$;	$l=\frac{2S+n(n-1)d}{2n}$.
5	a, n, S	d, l	$d=\frac{2(S-an)}{n(n-1)}$;	$l=\frac{2S}{n}-a$.
6	l, n, S	d, a	$d=\frac{2(nl-S)}{n(n-1)}$;	$a=\frac{2S}{n}-l$.
7	a, d, l	n, S	$n=\frac{l-a}{d}+1$;	$S=\frac{(l+a)(l-a+d)}{2d}$.
8	a, l, S	n, d	$n=\frac{2S}{a+l}$;	$d=\frac{(l+a)(l-a)}{2S-(l+a)}$.
9	a, d, S	l	$l=-\frac{1}{2}d\pm\sqrt{2dS+(a-\frac{1}{2}d)^2}$.	
		n	$n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d}$.	
10	l, d, S	a	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS}$.	
		n	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$.	

CASE IV.

356. Given two terms, to insert any number of arithmetical means between them.

1. Insert 3 arithmetical means between 4 and 16.

SOLUTION. If we insert m terms between 4 and 16, the entire series will consist of $m+2$ terms; hence in the formula for d, n will equal $m+2$. Substituting $m+2$ for n in the formula for d, and reducing, we have $d=\frac{l-a}{m+1}$. Substituting in this formula $l=16$, $a=4$ and $m=3$, we have $d=3$; hence the series will be 4, 7, 10, 13, 16.

OPERATION.

$$d=\frac{l-a}{n-1}$$

$$d=\frac{l-a}{m+2-1}=\frac{l-a}{m+1}$$

$$d=\frac{16-4}{3+1}=3$$

Series = 4, 7, 10, 13, 16

EXAMPLES.

2. Insert 4 arithmetical means between 5 and 20. *Ans.* 8, 11, 14, 17.

3. Required the arithmetical mean between 4 and 18. *Ans.* 11.

4. Insert 2 arithmetical means between $\frac{1}{2}$ and $\frac{1}{3}$. *Ans.* $\frac{7}{18}$; $\frac{4}{9}$.

5. Required the arithmetical mean between a and b. *Ans.* $\frac{a+b}{2}$.

6. Insert 2 arithmetical means between a and b. *Ans.* $\frac{2a+b}{3}$; $\frac{a+2b}{3}$

7. Insert 3 arithmetical means between a and b. *Ans.* $\frac{3a+b}{4}$; $\frac{a+b}{2}$; $\frac{a+3b}{4}$.

8 If the arithmetical mean between two numbers, one of which is 5, is 19, what are the numbers? *Ans.* 5 and 33

PROBLEMS

IN ARITHMETICAL PROGRESSION.

357. In Arithmetical Progression problems arise in which the terms are not directly given, but are implied in the given conditions.

358. In solving these problems we may represent the unknown terms and form equations by means of the principles of arithmetical progression.

359. An arithmetical series in which x represents the first term and y the common difference is generally represented thus:

$$x,\ x+y,\ x+2y,\ x+3y, \text{ etc.}$$

360. When there are *three terms*, it will often be found convenient to represent them thus:

$$x-y,\ x,\ x+y.$$

361. When there are *four terms*, the following method will often facilitate the solution:

$$x-3y,\ x-y,\ x+y,\ x+3y.$$

362. When there are *five terms*, the following method will often facilitate the solution:

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y.$$

NOTE.—The advantage of these methods is, that the sum of the series, or the sum or difference of the extremes, or of any two terms equally distant from the extremes, will contain but one quantity.

1. Find the series where the nth term is $2n-1$.

SOLUTION. Since n represents any term, the formula $2n-1$ is true for any value of n. When $n=1$, $2n-1=1$, hence the first term is 1; when $n=2$, $2n-1=3$, hence the second term is 3; when $n=3$, $2n-1=5$, etc. Hence the series is 1, 3, 5, 7, 9, etc.

2. The sum of three numbers in arithmetical progression is 12, and the sum of their squares is 66; find the numbers.

SOLUTION.

Let $x-y=$ first term,
$x=$ second term,
$x+y=$ third term.

By the 1st condition,	$3x=12;$	(1)
by the 2d condition,	$3x^2+2y^2=66;$	(2)
from equation (1) we have,	$x=4;$	(3)
substituting in equation (2),	$48+2y^2=66;$	(4)
from which we have,	$y=3;$	(5)
hence we have,	$x-y=1$, first term,	
and	$x+y=7$, third term.	

Therefore the numbers are 1, 4 and 7.

EXAMPLES.

3. The nth term of an arithmetical progression is $\frac{1}{3}(n+5)$; required the series. *Ans.* 2; $2\frac{1}{3}$; $2\frac{2}{3}$, etc.

4. The nth term of an arithmetical progression is $\frac{1}{6}(3n-1)$; find the first term, the common difference and the sum of n terms. *Ans.* $\frac{1}{3}$; $\frac{1}{2}$; $\frac{n}{12}(3n+1)$.

5. The nth term of an arithmetical progression is $(2n-1)a$; find the series and its sum. *Ans.* a, $3a$, $5a$, etc.; sum, an^2.

6. Three numbers are in arithmetical progression; their sum is 12 and their product 48; required the numbers.

Ans. 2, 4, 6.

7. The sum of three numbers in arithmetical progression is 18; the product of the first and second is 24; what are the numbers? *Ans.* 4, 6, 8.

8. Find three numbers in arithmetical progression such that their sum shall be 15 and the sum of their cubes 645.

Ans. 2, 5, 8.

9. There are four numbers in arithmetical progression; the sum of the two extremes is 8, and the product of the two means is 15; what are they? *Ans.* 1, 3, 5, 7.

10. There are four numbers in arithmetical progression; the product of the first and fourth is 22, and of the second and third, 40; what are the numbers? *Ans.* 2, 5, 8, 11.

11. The sum of four numbers in arithmetical progression is 30, and the sum of the cubes of the two means is 945; what are the numbers? *Ans.* 3, 6, 9, 12.

12. The sum of four numbers in arithmetical progression is 22, and their continued product is 280; what are the numbers? *Ans.* 1, 4, 7, 10.

13. Find four numbers in arithmetical progression such that the sum of the squares of the first and fourth shall be 200, and of the second and third, 136. *Ans.* 2, 6, 10, 14.

14. There are four numbers in arithmetical progression; the product of the first and fourth is 45, and of the second and third 77; what are the numbers? *Ans.* 3, 7, 11, 15.

15. Find four numbers in arithmetical progression such that the sum of the first and fourth shall be 17, and the difference of the squares of the two means shall be 51. *Ans.* 4, 7, 10, 13.

16. There are four numbers in arithmetical progression such that the sum of the squares of the means is 164, and the sum of the squares of the extremes is 180; what are they?

Ans. 6, 8, 10, 12.

17. There are five numbers in arithmetical progression; their sum is 40, and the sum of their squares 410; what are the numbers? *Ans.* 2, 5, 8, 11, 14.

18. There are seven numbers in arithmetical progression such that the sum of the first and fifth shall be 16, and the product of the fourth and seventh 160; required the numbers.

Ans. 4, 6, 8, 10, 12, 14, 16.

19. If the sum of n terms of an arithmetical progression is always equal to n^2, find the first term and the common difference. *Ans.* First term, 1; com. dif. $=2$.

20. If the sum of n terms of an arithmetical progression is always equal to $\frac{1}{2}n(n+1)$, find the series. *Ans.* 1, 2, 3, 4, etc.

NOTE.—In the 19th Example take $S=\frac{1}{2}n\{2a+(n-1)d\}=n^2$; then find the first term by supposing $n=1$ and $d=0$, etc.

GEOMETRICAL PROGRESSION.

363. A **Geometrical Progression** is a series of quantities which vary by a constant multiplier.

364. The **Ratio** or rate of the progression is the constant multiplier by which the terms vary; thus, in 1, 2, 4, 8 the ratio is 2.

365. An **Ascending Progression** is one that increases from left to right; as 2, 4, 8, 16, etc.

366. A **Descending Progression** is one that decreases from left to right; as 32, 16, 8, $\frac{1}{4}$, etc.

367. The **Terms** considered are five, any three of which being given, the other two may be found.

THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The ratio, r;
5. The sum of the terms, S.

368. Principle.—*The ratio is greater than a unit in an ascending series, and less than a unit in a descending series.*

CASE I.

369. Given the first term, the ratio and the number of terms, to find the last term.

1. Given a, the first term; r, the ratio; and n, the number of terms, to find an expression for l, the last term.

Solution. The *first* term is a; the *second* term equals $a \times r$, or ar; the *third* term equals $ar \times r$, or ar^2; the *fourth* term equals $ar^2 \times r$, or ar^3, etc. Examining these terms, we see that each term equals the *first term into r raised to a power one less than the number of the term;* hence the nth term will equal ar^{n-1}; and since l represents the nth or last term, we have $l = ar^{n-1}$.

OPERATION.

1st term $= a$
2d term $= ar$
3d term $= ar^2$
4th term $= ar^3$
nth term $= ar^{n-1}$
$l = ar^{n-1}$

Rule.—*Multiply the first term by the ratio raised to a power whose index is one less than the number of terms.*

Note.—An ascending series of n terms may be written as follows:

$$a,\ ar,\ ar^2,\ ar^3 \ldots .\ ar^{n-2},\ ar^{n-1}.$$

EXAMPLES.

2. Find the 8th term of the series 2, 4, 8, etc. *Ans.* 256.
3. The first term is 3 and ratio 4; what is the 7th term? *Ans.* 12288.
4. The first term is 729, the ratio $\frac{1}{3}$; required the 12th term. *Ans.* $\frac{1}{243}$.
5. Find the nth term of the series 1, 2, 4, 8, etc. *Ans.* 2^{n-1}.
6. Find the nth term of the series $2a$, $4a^2$, $8a^3$, etc. *Ans.* $(2a)^n$.
7. Find the nth term of the series 2, $4a$, $8a^2$, etc. *Ans.* $2^n a^{n-1}$.
8. If a merchant doubles his capital every 4 years, and begins with $4000, how much has he at the end of 20 years? *Ans.* $128000.

CASE II.

370. Given the first term, the last term and the number of terms, to find the sum of the terms.

1. Given a, the first term; l, the last term; and n, the number of terms, to find an expression for S, the sum of the terms.

Solution.

We have $S = a + ar + ar^2 + ar^3 + \ldots .\ ar^{n-1}$; (1)

multiplying (1) by r, $rS = ar + ar^2 + ar^3 + \ldots .\ ar^{n-1} + ar^n$. (2)

Subtracting (1) from (2), $rS - S = ar^n - a$; (3)

factoring, $S(r-1) = ar^n - a$; (4)

whence $S = \dfrac{ar^n - a}{r - 1}$. (5)

This may be put in another form by substituting a value for ar^n.

From Case I. we have $l = ar^{n-1}$;

multiplying by r, $rl = ar^n$;

substituting in (5), $S = \frac{rl - a}{r - 1}$.

Rule.—*Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.*

EXAMPLES.

Find the sum of the series—

2. When $a = 2$, $l = 256$, and $r = 2$. *Ans.* 510.
3. When $a = 3$, $l = 12288$, and $r = 4$. *Ans.* 16383.
4. Of 9 terms of the series 2, 4, 8, 16, etc. *Ans.* 1022.
5. Of 12 terms of the series 1, 2, 4, 8, etc. *Ans.* 4095.
6. Of 10 terms of the series 1, 3, 9, 27, etc. *Ans.* 29524.
7. Of n terms of the series $1 + 2 + 4 + 8$, etc. *Ans.* $2^n - 1$.
8. Of n terms of the series 1, 3, 9, 27, etc. *Ans.* $\frac{1}{2}(3^n - 1)$.
9. Of n terms of the series $a + 2a + 4a + 8a$, etc. *Ans.* $a(2^n - 1)$.
10. Of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc. *Ans.* $\frac{2^n - 1}{2^{n-1}}$.
11. Of n terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$, etc. *Ans.* $\frac{1}{2}\left(\frac{3^n - 1}{3^{n-1}}\right)$.
12. Of n terms of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$, etc. *Ans.* $\frac{1}{3}\left(\frac{2^n - 1}{2^{n-1}}\right)$, or $\frac{1}{3}\left(\frac{2^n + 1}{2^{n-1}}\right)$.
13. A laborer agreed to work one year at the rate of \$1 for January, \$2 for February, \$4 for March, and so on; how much did he receive in the year? *Ans.* \$4095.
14. A servant-girl saved \$160 one year. Now, if it were possible for her to save half as much again every year as the previous year for 8 years, how much would she save? *Ans.* \$11981.87½.

INFINITE SERIES.

371. An **Infinite Series** is a series in which the number of terms is infinite; as, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

372. The **Sum** of a decreasing geometrical series to infinity is the limit toward which the series approaches as the number of terms increases.

1. Find the sum of a decreasing geometrical series to infinity.

OPERATION.

$$S = \frac{rl - a}{r - 1} \quad (1)$$

$$S = \frac{a - rl}{1 - r} \quad (2)$$

When $rl = 0$,

$$S = \frac{a}{1 - r} \quad (3)$$

SOLUTION. In a decreasing series, r is less than 1; hence, for a decreasing series we change formula (1) to formula (2), that the denominator may be positive.

Now, as the number of terms increases, the value of l decreases; hence, when the number of terms is *infinite*, l must become infinitely small; that is, 0; hence, $rl = 0$, and the formula for S becomes a divided by $1 - r$.

Rule.—*Divide the first term by 1 minus the ratio.*

EXAMPLES.

Find the sum of the infinite—

2. Series $1 + \frac{1}{2} + \frac{1}{4} +$, etc. *Ans.* 2.
3. Series $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$, etc. *Ans.* $1\frac{1}{2}$.
4. Series $\frac{1}{3} + \frac{2}{9} + \frac{4}{27}$, etc. *Ans.* 1.
5. Series $1 - \frac{1}{2} + \frac{1}{4} -$, etc. *Ans.* $\frac{2}{3}$.
6. Series $1 - \frac{2}{5} + \frac{4}{25} -$, etc. *Ans.* $\frac{5}{7}$.
7. Of the circulate .333, etc. ($= \frac{3}{10} + \frac{3}{100} +$, etc.). *Ans.* $\frac{1}{3}$.
8. Of the circulate .22727, etc. *Ans.* $\frac{5}{22}$.
9. Series $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, etc. *Ans.* $\frac{1}{a - 1}$.
10. Series $1 + x^{-2} + x^{-4} +$, etc. *Ans.* $\frac{x^2}{x^2 - 1}$.
11. Series $a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3}$, etc. *Ans.* $\frac{a^2}{a + b}$.

12. Suppose a body move 12 feet the first second, 6 feet the next second, 3 feet the next second, and so on until it stops; what is the entire distance it can reach? *Ans.* 24 ft.

13. If an ivory ball falls 12 feet to the floor and bounds back 6 feet, then, falling, bounds back 3 feet, and so on, how far will it move before it comes to rest? *Ans.* 36 ft.

14. A dog and rabbit, 20 rods apart, run so that when the dog runs the distance between them the rabbit will run $\frac{1}{10}$ of that distance; how far will the dog run to catch the rabbit?

Ans. $22\frac{2}{9}$ rods.

CASE III.

373. Given any three of the five quantities, to find either of the others.

374. The Fundamental Formulas of geometrical progression are—

$$1.\ l = ar^{n-1}; \qquad 2.\ S = \frac{rl - a}{r - 1}.$$

By means of these we are enabled to solve all the cases which arise. As in arithmetical progression, there are three classes of problems.

375. Class I. When the four quantities are all contained in the first fundamental formula.

EXAMPLES

1. Find a, given l, r and n. *Ans.* $a = \frac{l}{r^{n-1}}$.

2. Find r, given a, l and n. *Ans.* $r = \sqrt[n-1]{\frac{l}{a}}$.

3. A person agreed to labor for wages doubling every month what did he receive the first month if he received $512 the tenth month? *Ans.* $1.

4. If a man saves $6.40 the first year, and increases his savings each year in geometrical proportion, what is the rate of increase if he saves $109.35 the eighth year? *Ans.* $1\frac{1}{2}$.

376. Class II. When the four quantities are all contained in the second fundamental formula.

EXAMPLES.

1. Find a, given r, l and S. *Ans.* $a = rl - (r-1)S$.

2. Find l, given a, r and S. *Ans.* $l = \frac{a + (r-1)S}{r}$.

3. Find r, given a, l and S. *Ans.* $r = \frac{S-a}{S-l}$.

4. If a person agrees to labor for wages doubling every month, and receives $4095 in a year, how much did he receive the first month? *Ans.* $1.

5. If I discharge a debt in 10 months by monthly payments in geometrical progression, allowing the first payment to be $1 and the last $512, what will be the ratio? *Ans.* 2.

377. Class III. When some of the quantities are in the first and some in the second fundamental formula.

Note.—The four formulas for n require a knowledge of logarithms Four others, when n exceeds 2, require a knowledge of higher equations.

EXAMPLES.

1. Find S, given l, r and n. *Ans.* $S = \frac{lr^n - l}{r^n - r^{n-1}}$.

2. Find l, given r, n and S. *Ans.* $l = \frac{(r-1)Sr^{n-1}}{r^n - 1}$.

3. A man bought 10 yards of cloth for $295.24, giving three times as much for each yard as for the preceding yard; what did he pay for the first yard? *Ans.* 1 cent.

TABLE OF FORMULAS.

378. Since there are five quantities, any three of which being given a fourth may be found, there are twenty cases in all. These cases are presented in the following table:

No.	Given.	Required.	Formulas.	
1	a, r, n	l, S	$l=ar^{n-1};$	$S=\frac{ar^n-a}{r-1}.$
2	l, r, n	a, S	$a=\frac{l}{r^{n-1}};$	$S=\frac{lr^n-l}{r^n-r^{n-1}}.$
3	r, n, S	a, l	$a=\frac{(r-1)S}{r^n-1};$	$l=\frac{(r-1)Sr^{n-1}}{r^n-1}.$
4	a, n, l	r, S	$r=\sqrt[n-1]{\frac{l}{a}};$	$S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}.$
5	a, n, S	r, l	$ar^n-rS=a-S;$	$l(S-l)^{n-1}=a(S-a)^{n-1}.$
6	l, n, S	r, a	$(S-l)r^n-Sr^{n-1}=-l;$	$a(S-a)^{n-1}=l(S-l)^{n-1}.$
7	a, r, l	n, S	$n=\frac{\log l-\log a}{\log r}+1;$	$S=\frac{lr-a}{r-1}.$
8	a, l, S	n, r	$n=\frac{\log l-\log a}{\log (S-a)-\log (S-l)}+1;$	$r=\frac{S-a}{S-l}.$
9	a, r, S	n, l	$n=\frac{\log [a+(r-1)S]-\log a}{\log r};$	$l=\frac{a+(r-1)S}{r}.$
10	l, r, S	n, a	$n=\frac{\log l-\log [lr-(r-1)S]}{\log r}+1;$	$a=lr-(r-1)S.$

NOTE.—Pupils who have the time will be interested in deriving the formulas of the above table. The formulas for the values of n can be derived after completing the subject of logarithms.

PROBLEMS

IN GEOMETRICAL PROGRESSION.

379. In **Geometrical Progression** problems arise in which the terms are not directly given, but are implied in the conditions.

380. In solving these problems we may represent the unknown terms and form equations by means of the principles of Geometrical Progression.

381. A geometrical series, where x represents the first term and y the ratio, is generally represented thus:

$$x,\ xy,\ xy^2,\ xy^3, \text{ etc.}$$

382. When *three* terms are considered in the problem, they may be represented thus:

$$x,\ \sqrt{xy},\ y;$$
$$\text{or } x^2,\ xy,\ y^2.$$

383. When *four* terms are considered in the problem, they may be represented thus:

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}.$$

NOTE.—The method most convenient in any case will depend upon the nature of the problem.

EXAMPLES.

1. Find the series whose nth term is 2^{n-1}.

SOLUTION. Since n represents any term, the formula 2^{n-1} is true for any value of n. When $n=1$, $2^{n-1}=2^{1-1}=2^0$, or 1; hence the first term of the series is 1. When $n=2$, $2^{n-1}=2^{2-1}=2$; when $n=3$, $2^{n-1}=2^{3-1}=2^2$ or 4, etc.; hence the series is 1, 2, 4, 8, etc.

2. The sum of three numbers in geometrical progression is 14, and the sum of their squares is 84; what are the numbers?

SOLUTION.

Let x, $\sqrt{xy}$, and y represent the series. Then

By 1st condition,	$x+\sqrt{xy}+y=14;$	(1)
by 2d condition,	$x^2+xy+y^2=84.$	(2)
Dividing (2) by (1),	$x-\sqrt{xy}+y=6;$	(3)
adding (1) and (3),	$2x+2y=20;$	(4)
dividing by 2,	$x+y=10;$	(5)
subtracting (5) from (1),	$\sqrt{xy}=4.$	

3. Find the series whose nth term is 6^{n-1}. *Ans.* 1, 6, 36, etc.

4. Find the series whose nth term is 3^n. *Ans.* 3, 9, 27, etc.

5. Find the series whose nth term is $(2a)^n$.
Ans. $2a$, $4a^2$, $8a^3$, etc.

6. Find the series in which the sum of n terms is $\frac{1}{2}(3^n-1)$.
Ans. 1, 3, 9, 27, etc.

7. Find the series in which the sum of n terms is $a(2^n-1)$.
Ans. a, $2a$, $4a$, etc.

8. Find the series in which the sum of n terms is $\dfrac{2^n-1}{2^{n-1}}$.
Ans. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

9. Find three numbers in geometrical progression such that their sum shall be 28 and the sum of their squares 336.
Ans. 4, 8, 16.

10. The product of three numbers in geometrical progression is 216, and the sum of their squares is 364; required the numbers. *Ans.* 2, 6, 18.

11. The sum of the first and third of three numbers in geometrical progression is 10, and the sum of the cubes of the first and third is 520; required the numbers. *Ans.* 2, 4, 8.

12. There are three numbers in geometrical progression; the sum of the first and second is 32, and the sum of the second and third is 96; what are the numbers? *Ans.* 8, 24, 72.

13. The product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 153; required the numbers. *Ans.* 3, 6, 12.

14. The sum of three numbers in geometrical progression is 39, and the sum of the extremes multiplied by the mean is 270; what are the numbers? *Ans.* 3, 9, 27.

15. There are three numbers in geometrical progression whose sum is 52, and the sum of their squares is 1456; what are the numbers? *Ans.* 4, 12, 36.

16. It is required to find three numbers in geometrical progression such that the sum of the first and last is 30, and the square of the mean is 144. *Ans.* 6, 12, 24.

17. Of four numbers in geometrical progression the sum of the first and third is 20, and the sum of the second and fourth is 60; what are the numbers? *Ans.* 2, 6, 18, 54.

18. Required to find four numbers in geometrical progression such that the sum of the first two is 10, and of the last two is 160. *Ans.* 2, 8, 32, 128.

In the 15th Example divide the 2d equation by the first. In the 18th, let x, xy, xy^2, xy^3 represent the numbers.

REVIEW QUESTIONS.

Define Progression. Arithmetical Progression. The Terms. Extremes. Means. Ascending Progression. Descending Progression. How many terms? State the four cases. The rule for each case. The formula for each case. How many cases are possible?

Define Geometrical Progression. What is the value of the ratio in an ascending progression? In a descending progression? State the three cases. Give the rule and formula for Case I. and Case II. Define an Infinite Series. State the rule for the sum of the terms of an infinite series. How many cases are possible?