SUPPLEMENT.

SECTION X.

INEQUALITIES, INDETERMINATE AND HIGHER EQUATIONS.

INEQUALITIES.

384. An **Inequality** is an expression signifying that one quantity is greater or less than another; as $ax-b>c$.

385. The **First Member** of an inequality is the part on the left of the sign; the *second member* is the part on the right.

386. In treating inequalities the terms *greater* and *less* must be understood in their algebraic sense; thus,

1. *A negative quantity is regarded as less than* zero.

2. *Of two negative quantities, the greater is the one which has the less number of units.*

387. Two inequalities are said to exist *in the same sense* when the first member is greater in both or less in both; thus, $4>3$ and $6>5$.

388. Two inequalities are said to exist *in a contrary sense* when the first member is greater in one and less in the other; thus, $4>1$ and $3<5$.

389. The following examples will be readily solved by the student.

EXAMPLES.

1. Given $\frac{x}{2}+\frac{2x}{3}>\frac{3x}{4}+\frac{5}{3}$, to find a limit of x.

Solution.—Clearing of fractions, we have $6x+8x>9x+20$; transposing, etc., we have $5x>20$; hence, $x>4$.

2. Given $5x > \frac{3x}{2} + 14$; find the limit of x. *Ans.* $x > 4$.

3. Given $\frac{2x}{5} - \frac{2x}{3} < \frac{x}{4} - \frac{31}{12}$; find a limit of x. *Ans.* $x > 5$.

4. Given $\frac{2x}{5} - \frac{2x}{3} > \frac{2x}{5} - 2$; find a limit of x. *Ans.* $x < 3$.

5. Given $4 + \frac{x}{3} < 7 + \frac{x}{4}$; find a limit of x. *Ans.* $x < 36$.

6. Given $2x + 5y > 16$ and $2x + y = 12$; find the limits of x and y. *Ans.* $x < 5\frac{1}{2}$; $y > 1$.

7. Given $3x - 5 < 2x + 1$ and $4x + 1 > 13 + x$; find the value of x if integral. *Ans.* $x = 5$.

8. Given $x + 2y > 18$ and $2x + 3y = 34$; find limits of x and y. *Ans.* $x < 14$; $y > 2$.

9. Twice an integer, plus 5, is less than 3 times the integer, plus 3, and 4 times the integer, less 4, is greater than 6 times the integer, minus 12; required the integer. *Ans.* 3.

10. Twice a number, plus 7, is not greater than 19; and three times the number, minus 5, is not less than 13; what is the number? *Ans.* 6.

THEOREMS IN INEQUALITIES.

1. Prove that the sum of the squares of two unequal quantities, a and b, is greater than twice their product.

For, $(a-b)^2$ is positive whatever the values of a and b;

hence,	$(a-b)^2 > 0$;
or,	$a^2 - 2ab + b^2 > 0$.
Hence,	$a^2 + b^2 > 2ab$.

2. Prove that $a^2 + b^2 + c^2 > ab + ac + bc$.

By Theo. 1,	$a^2 + b^2 > 2ab$,
	$a^2 + c^2 > 2ac$,
	$b^2 + c^2 > 2bc$.
Hence, adding,	$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$.
Whence,	$a^2 + b^2 + c^2 > ab + ac + bc$.

3. Prove that $a + b > 2\sqrt{ab}$, unless $a = b$.

4. Prove that $a^3b + ab^3 > 2a^2b^2$, unless $a = b$.

5. Prove that $3a^2 + b^2 > 2a(a+b)$, unless $a = b$.

6. Prove that $a^3 + 1 > a^2 + a$, unless $a = 1$.

7. Prove that the sum of any fraction and its reciprocal is greater than 2.

8. Prove that $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}$, unless $a = b$.

9. Prove that $a - b > (\sqrt{a} - \sqrt{b})^2$, when $a > b$.

10. Prove that the ratio of $a^2 + b^2$ to $a^3 + b^3$ is less than the ratio of $a + b$ to $a^2 + b^2$.

11. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, which is greater, xy or $ac + bd$, and xy or $ad + bc$? *Ans.* xy.

INDETERMINATE EQUATIONS.

390. An **Indeterminate Equation** is an equation in which the values of the unknown quantities are unlimited.

391. Thus, in the equation $2x + 3y = 35$, x and y may have different values; and if *any* value be assigned to one of the quantities, a *corresponding* value may be found of the other.

392. The *solution* of indeterminate equations, though the number of corresponding values is unlimited, is usually limited to finding *positive integral values.*

393. Of the several interesting cases that may arise we shall consider only two.

NOTE.—The treatment of indeterminate equations is usually called *Indeterminate Analysis.*

CASE I.

394. To find positive integral values of the unknown quantities in the equation.

1. Given $2x + 3y = 35$, to find positive integral values for x and y.

SOLUTION.

Given,	$2x + 3y = 35$.	(1)
Transposing,	$2x = 35 - 3y$.	(2)
Whence,	$x = \frac{35 - 3y}{2} = 17 - y + \frac{1-y}{2}$.	(3)

Since y is an integer, $17 - y$ is an *integer*; and since also x is an *integer*, $\frac{1-y}{2}$ is also an integer.

Let m represent this integer.

Then,	$\frac{1-y}{2}=m,$	(5)
and	$1-y=2m.$	(6)
Whence,	$y=1-2m.$	(7)
Sub. in (1),	$2x+3-6m=35.$	(8)
Whence,	$x=16+3m.$	(9)

In equation (7), for y to be *integral* and *positive*, m may be 0, or *negative*, but cannot be *positive*. In equation (9), for x to be *integral* and *positive*, m can be 0, or *positive*, or *negative* while less than 5. Hence m may be 0, -1, -2, -3, or -4.

Substituting these values of m in (7) and (9), we have

$$x=16,\ 13,\ 10,\ 7,\ 4,\ 1.$$
$$y=1,\ 3,\ 5,\ 7,\ 9,\ 11.$$

NOTE.—We shall use *Int.* to mean *an integer.*

2. Given $7x+9y=23$, to find positive integral values for x and y.

SOLUTION.

Here, $$x=\frac{23-9y}{7}=3-y+\frac{2-2y}{7}. \qquad (1)$$

Now, $\frac{2-2y}{7}$ must be an integer.

If we put $\frac{2-2y}{7}=m$, then $y=\frac{2-7m}{2}$, a *fractional* expression; but we wished to obtain an *integral* expression for the value of y. To avoid this difficulty, it is necessary to operate on $\frac{2-2y}{7}$, so as to make the coefficient of y a *unit.*

Since $\frac{2-2y}{7}$ is integral, any multiple of $\frac{2-2y}{7}$ is integral. Multiply, then, by some number that will make the coefficient of y contain the denominator with a *remainder of* 1.

Multiplying by 4, we have

$$\frac{2-2y}{7}\times 4=\frac{8-8y}{7}=1-y+\frac{1-y}{7}=Int.$$

Hence, $\frac{1-y}{7}=Int.=m$, and $y=1-7m.$

Sub. in (1), $x=2+9m.$

Here, for x and y to be positive integers, m can be only 0.

Substituting $m=0$, we have $x=2$ and $y=1$.

NOTE.—Other methods of reducing besides multiplying may be used, as may be seen in the following solution, the object being to obtain an integral form for the value of y.

3. Given $19x-14y=11$, to find integral values of x and y.

SOLUTION.

Here, $$x=\frac{14y+11}{19}=Int.;\ \text{also}\ \frac{19y}{19}=Int.$$

Subtracting, $$\frac{19y}{19}-\frac{14y+11}{19}=\frac{5y-11}{19}=Int.$$

Also, $$\frac{5y-11}{19}\times 4=\frac{20y-44}{19}=y-2+\frac{y-6}{19}=Int.$$

Hence, $\frac{y-6}{19}=m$, and $y=19m+6;$

and, $x=14m-5.$

Taking $m=0,\ 1,\ 2,\ 3,$ etc., we have

$$x=5,\ 19,\ 33,\ 47,\ \text{etc.},$$
$$y=6,\ 25,\ 44,\ 58,\ \text{etc.}$$

NOTES.—1. If the equation is in the form $ax+by=c$, the number of answers will be always limited, and in some cases a solution is impossible. The form $ax-by=\pm c$ will admit of an infinite number of answers.

2. If in $ax\pm by=c$, a and b have a common factor not common to c, there can be no integral solution.

4. How can 78 cents be paid with 5-cent and 3-cent pieces, and in how many ways?

SOLUTION. Let x = the number of 5-cent pieces, and y = the number of 3-cent pieces; then $5x+3y=78$, from which, by the method explained above, we find $x=15,\ 12,\ 9,\ 6,\ 3,\ 0$; and $y=1,\ 6,\ 11,\ 16,\ 21,\ 26$. Hence it can be paid in 5 ways when both kinds of pieces are used.

5. Given $2x+3y=25$, to find positive integral values for x and y. *Ans.* $x=2,\ 5,\ 8,\ 11$; $y=7,\ 5,\ 3,\ 1$.

6. Given $3x-8y=-16$, to find positive integral values for x and y. *Ans.* $x=8,\ 16$, etc.; $y=5,\ 8$, etc.

7. Given $8x+11y=49$, to find positive integral values for x and y. *Ans.* $x=2$; $y=3$.

8. Given $14x=5y+17$, to find the least positive integral values for x and y. *Ans.* $x=3$; $y=5$.

9. Given $19x-13y=17$, to find the least positive integral values for x and y. *Ans.* $x=5$; $y=6$.

10. Divide 100 into two such parts that one may be divided by 7, and the other by 11. *Ans.* 56 and 44.

11. In how many different ways may I pay a debt of £20 in half-guineas and half-crowns? *Ans.* 7 ways.

12. In how many ways can £100 be paid in guineas and crowns? *Ans.* 19.

13. What is the simplest way for a person who has only guineas to pay 10s. 6d. to another who has only half-crowns? *Ans.* In 3 guineas, receiving 21 half-crowns.

NOTE.—The *crown* equals 5 shillings, and the *guinea* equals 21 shillings.

CASE II.

395. To find the least integer which, divided by given numbers, shall leave given remainders.

1. Find the least integer which, being divided by 17, leaves a remainder of 7, and being divided by 26 leaves a remainder of 13.

SOLUTION.

Let $x =$ the required integer.

Then, $\frac{x-7}{17}$ and $\frac{x-13}{26} =$ integers.

Let $\frac{x-7}{17} = m$; then, $x = 17m+7$; substitute this in the second fraction,

$$\frac{17m+7-13}{26} = \frac{17m-6}{26} = Int.$$

Hence, $\frac{26m}{26} - \frac{17m-6}{26}$, or $\frac{9m+6}{26} = Int.$

And $\frac{9m+6}{26} \times 3 = \frac{27m+18}{26} = m + \frac{m+18}{26} = Int.$

Whence, $\frac{m+18}{26} = Int.$, which we represent by n.

Then, $\frac{m+18}{26} = n$; hence $m = 26n - 18$.

Now, if $n = 1$, we shall have $m = 8$.

Hence, $x = 17m + 7 = 17 \times 8 + 7 = 143$, the number.

ANOTHER SOLUTION. Let $N =$ the number.

Then $\frac{N-7}{17} = x$ (1), and $\frac{N-13}{26} = y$ (2).

Whence, $N = 17x + 7$ (3), and $N = 26y + 13$ (4);

and $17x + 7 = 26y + 13$,

or, $17x - 26y = 6$.

Then find x and y, as in Case I., and substitute in (3) and (4).

2. Find the least number which, being divided by 3, 4, and 5, shall leave respectively the remainders 2, 3, and 4.

SOLUTION. Let $x =$ the integer, then $\frac{x-2}{3} = Int. = m$; whence $x = 3m+2$. Also, $\frac{x-3}{4} = Int.$; by substitution, $= \frac{3m-1}{4} = n$; whence, $m = n + \frac{n+1}{3}$. Placing $\frac{n+1}{3} = p$, we have $n = 3p-1$; $m = 4p-1$, and $x = 12p-1$.

But, $\frac{x-4}{5} = Int. = \frac{12p-5}{5} = 2p - 1 + \frac{2p}{5}$. Now, $\frac{2p}{5} = Int.$; hence, $\frac{2p}{5} \times 3 = \frac{6p}{5} = Int. = p + \frac{p}{5}$. Put, $\frac{p}{5} = q$; then, $p = 5q$, and $x = 60q - 1$. Now if $q = 1$, $x = 59$; if $q = 2$, $x = 119$, etc.

3. Find the least integer which, being divided by 6, shall leave the remainder 2, and divided by 13 shall leave the remainder 3. *Ans.* 68.

4. Find the least number which, being divided by 17 and 26, shall leave for remainders 7 and 13 respectively. *Ans.* 143.

5. What is the least integral number which, being divided by 3, 5, and 6, shall leave the respective remainders 1, 3, and 4? *Ans.* 28.

6. A man buys cows and colts for $1000, giving $19 for each cow and $29 for each colt; how many did he buy of each? *Ans.* 45 cows and 5 colts, or 16 cows and 24 colts.

7. A farmer bought 100 animals for $100: geese at 50 cents, pigs at $3, and calves at $10; how many were there of each kind? *Ans.* 94, 1, 5.

8. A farmer buys oxen, sheep, and ducks, 100 in all, for £100; required the number of each if the oxen cost £5, the sheep £1, and the ducks 1 shilling each. *Ans.* 19, 1, 80.

9. A lady bought 10 books of three different kinds for $30; the first kind cost $4¼ each, the second $2½ each, the third $2 each; required the number of each kind. *Ans.* 4, 2, 4.

10. A market-woman finds by counting her eggs by threes she has 2 over, and counting by fives has 4 over; how many had she if the number is between 40 and 60? *Ans.* 44 or 59.

11. A boy has between 100 and 200 marbles; when he counts them by 12s, 10 remain, but when he counts them by 15s, 4 remain; how many marbles had he? *Ans.* 154.

12. A person wishes to purchase 20 animals for £20: sheep at 31 shillings, pigs at 11s., and rabbits at 1s. each; how many of each kind can he buy?

Ans. { Sheep, 10, 11, 12.
Pigs, 8, 5, 2.
Rabbits, 2, 4, 6. }

NOTE.—The solution of indeterminate equations of a higher degree is called *Diophantine Analysis*.

HIGHER EQUATIONS.

396. A **Cubic Equation** is an equation in which the highest power of the unknown quantity is the third power; as $x^3+4x^2+5x=10$.

397. A **Biquadratic Equation** is an equation in which the highest power of the unknown quantity is the fourth power; as $x^4+3x^3+4x^2+5x=13$.

398. The general form of a higher equation is $x^n+ax^{n-1}+bx^{n-2}+\ldots\ tx+u=0$.

399. No general method of solving equations above the second degree, that is practicable, has yet been discovered.

NOTE.—Cardan's method for cubics and Ferrari's method for biquadratics fail in so many cases as not to be practically general. Abel has shown that a general solution above the fourth degree is impossible.

400. The following PRINCIPLES, which are demonstrated in higher algebra, may often be used in finding the roots of an equation.

PRIN. 1. *If a is a root of an equation (the unknown quantity being x), the equation is divisible by $x-a$.*

Thus, if 2 is a root, the equation is divisible by $x-2$; if -2 is a root, the equation is divisible by $x-(-2)$, or $x+2$.

PRIN. 2. *The coefficient of the second term, ax^{n-1}, is the sum of the roots, with their signs changed.*

PRIN. 3. *The term independent of x, when in the first member, is the product of the roots, with their signs changed.*

Thus, a cubic equation, in which a, b, and c are the roots, is equivalent to $(x-a)(x-b)(x-c)=0$; and when developed is $x^3-(a+b+c)x^2+(ab+ac+bc)x-(abc)=0$.

NOTE.—Principles 2 and 3 will often enable us to conjecture the roots of an equation; and Prin. 1 will enable us to test any supposed root.

SOLUTION OF CUBIC EQUATIONS.

401. Cubic Equations can often be solved by special artifices, a few of which will be given.

CASE I.

402. Solution by inspection and the application of the above principles.

1. Given $x^3-6x^2+11x=6$, to find x.

SOLUTION. The factors of 6 are 1, 2, and 3; and their sum equals the coefficient of the 2d term; hence we may suppose one of these factors, as 3, to be a root of the equation. Transposing 6 to the first member, and dividing by $x-3$, we have $x^2-3x+2=0$; therefore, 3 is one root, and solving $x^2-3x=-2$, we find 1 and 2 to be the other roots.

2. Solve $x^3-9x^2+26x=24$. *Ans.* $x=2$, 3, and 4.
3. Solve $x^3-11x^2+38x=40$. *Ans.* $x=2, 4, 5$.
4. Solve $x^3-3x^2-10x=-24$. *Ans.* $x=2, -3, 4$.
5. Solve $x^3+4x^2+x=6$. *Ans.* $x=1, -2, -3$.
6. Solve $x^3-4x^2-7x=-10$. *Ans.* $x=1, -2, 5$.
7. Solve $x^3-2x^2+4x=8$. *Ans.* $x=2, 2\sqrt{-1}, -2\sqrt{-1}$.
8. Solve $x^3-7x^2+16x=10$.
Ans. $x=1, 3+\sqrt{-1}, 3-\sqrt{-1}$.

CASE II.

403. Solution by making both members a perfect cube.

1. Given $x^3+3x^2+3x=7$.

SOLUTION.

Given	$x^3+3x^2+3x=7$.	(1)
Adding 1,	$x^3+3x^2+3x+1=8$.	(2)
Whence	$x+1=2$, or $x=1$.	

Dividing Eq. 1 by $x-1$, we have $x^2+4x+7=0$.
Whence, $x=-2+\sqrt{-3}$ and $-2-\sqrt{-3}$.

2. Solve $x^3-3x^2+3x=9$. *Ans.* $x=3,\ +\sqrt{-3},\ -\sqrt{-3}$.

3. Solve $x^3+6x^2+12x=-16$.
Ans. $x=-4,\ -1+\sqrt{-3},\ -1-\sqrt{-3}$.

4. Solve $x^3+9x^2+27x=-35$.
Ans. $x=-5,\ -2+\sqrt{-3},\ -2-\sqrt{-3}$.

5. Solve $x^3-9x^2+27x=91$.
Ans. $x=7,\ 1+2\sqrt{-3},\ 1-2\sqrt{-3}$.

CASE III.

404. Solution when the equation is readily factored.

1. Given $x^3-6x=-4$.

SOLUTION.

Given	$x^3-6x=-4$.
Whence	$x^3-4x=2x-4$.
And	$x(x^2-4)=2(x-2)$.

Since this is divisible by $x-2$, $x-2=0$, or $x=2$.

Dividing by $x-2$, we have $x(x+2)=2$ or $x^2+2x=2$,

whence, $x=-1\pm\sqrt{3}$.

2. Solve $x^3-3x=-2$. *Ans.* $x=1,\ 1,\ -2$.
3. Solve $x^3-3x=2$. *Ans.* $x=-1,\ -1,\ 2$.
4. Solve $x^3-7x=-6$. *Ans.* $x=1,\ 2,\ -3$.
5. Solve $x^3-a^2x-x=-a$. *Ans.* $x=a,\ \frac{1}{2}(-a\pm\sqrt{a^2+4})$.
6. Solve $x^3-x^2-2x=-2$. *Ans.* $x=1,\ +\sqrt{2},\ -\sqrt{2}$.
7. Solve $x^3-2x=\sqrt{3}$. *Ans.* $x=\sqrt{3},\ \frac{1}{2}(-\sqrt{3}\pm\sqrt{-1})$.
8. Solve $3x^3-7x^2-7x=-3$. *Ans.* $x=\frac{1}{3},\ 3,\ -1$.

CASE IV.

405. Solution by reducing to a biquadratic, and then changing to make both sides squares.

1. Given $x^3-7x=-6$, to find x.

SOLUTION.

Given	$x^3-7x=-6$.	(1)
Whence	$x^4-7x^2=-6x$.	(2)
Add $4x^2$	$x^4-3x^2=4x^2-6x$.	
Complete square,	$x^4-3x^2+\left(\frac{3}{2}\right)^2=4x^2-6x+\left(\frac{3}{2}\right)^2$	
Hence	$x^2-\frac{3}{2}=2x-\frac{3}{2}$, or $-2x+\frac{3}{2}$.	
Whence	$x=2$, or $x^2+2x=3$.	
	$\therefore x=1$, or -3.	

2. Solve $x^3+3x=14$. *Ans.* $x=2,\ -1\pm\sqrt{-6}$.
3. Solve $x^3-12x=16$. *Ans.* $x=4,\ -2,\ -2$.
4. Solve $x^3+5x=6$. *Ans.* $x=1,\ \frac{1}{2}(-1\pm\sqrt{-23})$.
5. Solve $x^3-4x=48$. *Ans.* $x=4,\ 2(-1\pm\sqrt{-2})$.
6. Solve $x^3-13x=-12$. *Ans.* $x=1,\ 3,\ -4$.
7. Solve $x^3+6x=20$. *Ans.* $x=2,\ -1\pm3\sqrt{-1}$.

SOLUTION OF BIQUADRATIC EQUATIONS.

406. Biquadratic Equations may often be solved by special artifices, a few of which we present.

CASE I.

407. Solution by inspection and applying the principles of equations.

1. Given $x^4-10x^3+35x^2-50x=-24$, to find x.

SOLUTION. We notice that $24=1\times2\times3\times4$, and $10=1+2+3+4$; hence, we presume that some of these factors are roots of the equation. Dividing by $x-1$, we see that the equation is divisible by $x-1$; hence 1 is a root; and in a similar way we find that 2, 3, and 4 are roots.

2. Solve $x^4-5x^3+5x^2+5x=6$. *Ans.* $x=1,\ -1,\ 2,\ 3$.
3. Solve $x^4+x^3-7x^2-x=-6$. *Ans.* $x=1,\ -1,\ 2,\ -3$.
4. Solve $x^4-6x^3-x^2+54x=72$. *Ans.* $x=2,\ 3,\ -3,\ +4$.
5. Solve $x^4-4x^3-9x^2+16x=-20$.
Ans. $x=-1,\ 2,\ -2,\ 5$.
6. Solve $x^4-4x^3-8x^2+4x=-7$. *Ans.* $1,\ -1,\ 2\pm\sqrt{11}$.

CASE II.

408. Solution by factoring when the factors can be readily obtained.

1. Given $x^4+4x^2-8x=32$, to find x.

SOLUTION.

Given	$x^4+4x^3-8x=32.$
Transposing,	$x^4+4x^3=8x+32.$
Factoring.	$x^3(x+4)=8(x+4),$
or,	$(x^3-8)(x+4)=0.$
Whence,	$x^3-8=0$ and $x+4=0,$
and	$x=-4$ or $2.$
Dividing,	x^3-8 by $x-2$, we obtain a quadratic,
from which	$x=-1\pm\sqrt{-3}.$

2. Solve $x^4-2x^3-x=-2$. *Ans.* $x=1,\ 2,\ \frac{1}{2}(-1\pm\sqrt{-3})$.
3. Solve $x^4-3x^3-8x=-24$. *Ans.* $x=2,\ 3,\ -1\pm\sqrt{-3}$.
4. Solve $x^4-ax^3-n^3x=-an^3$.

Ans. $x=a,\ n,\ \frac{n}{2}(-1\pm\sqrt{-3})$.

5. Solve $x^4+3x^3-3x=9$.

Ans. $x=-3;\ \sqrt[3]{3},\ \frac{\sqrt[3]{3}}{2}(-1\pm\sqrt{-3})$.

CASE III.

409. Solution by reducing to a quadratic form.

1. Given $x^4+2x^3-3x^2-4x=5$, to find x.

SOLUTION.

Given	$x^4+2x^3-3x^2-4x=5.$
Whence,	$(x^2+x)^2-4(x^2+x)+4=9,$
or	$x^2+x-2=\pm 3.$

From which x can be found.

2. Solve $x^4-4x^3+8x^2-8x=12$. *Ans.* $1\pm\sqrt{3},\ 1\pm\sqrt{-5}$.
3. Solve $x^4+2x^3-7x^2-8x=-12$. *Ans.* $1,\ 2,\ -2,\ -3$.
4. Solve $x^4+2x^3-3x^2-4x=-4$. *Ans.* $1,\ 1,\ -2,\ -2$.
5. Solve $x^4-6x^3+11x^2-6x=8$.

Ans. $\frac{1}{2}(3\pm\sqrt{17}),\ \frac{1}{2}(3\pm\sqrt{-7})$.

CASE IV.

410. Solution by reducing both members to a binomial square.

1. Solve $x^4-6x^3+12x^2-10x=-3$.

SOLUTION.

Given	$x^4-6x^3+12x^2-10x=-3.$
Whence,	$(x^2-3x)^2+3x^2-10x=-3,$
	$+4x^2-12x=x^2+2x-3,$
	$+4(x^2-3x)+4=x^2+2x+1;$
or	$(x^2-3x)^2+4(x^2-3x)+4=x^2+2x+1.$
Whence,	$(x^2-3x)+2=x+1.$
Whence,	$x=1;\ 1;\ 1;\ 3.$

2. Solve $x^4-4x^3-19x^2+46x=-120$.

Ans. $x=-2,\ -3,\ 4,\ 5$.

3. Solve $x^4+4x^3-x^2-16x=12$.

Ans. $x=-1,\ -2,\ 2,\ -3$.

4. Solve $x^4-9x^3+30x^2-46x=-24$.

Ans. $x=1,\ 4,\ 2\pm\sqrt{-2}$.

5. Solve $x^4+4x^3-6x^2+4x=7$. *Ans.* $x=\pm\sqrt{-1},-2\pm\sqrt{11}$.
6. Solve $x^4-12x^3+48x^2-68x=-15$. *Ans.* $x=3,\ 5,\ 2\pm\sqrt{3}$.

NOTE.—Cubics and biquadratics, when any of their roots are integral, can usually be solved by artifices similar to those we have explained.

The more general methods of finding approximate roots of numerical equations are those of *Double Position*, *Newton's Method of Approximation*, and *Horner's Method*, for an explanation of which the student is referred to works on Higher Algebra.

RECIPROCAL EQUATIONS.

411. A **Reciprocal Equation** is one in which the reciprocal of x may be substituted for x without altering the equation.

412. Thus, in $x^4-3x^3+4x^2-3x+1=0$, if we substitute $\frac{1}{x}$ for x, we shall obtain the same equation.

NOTES.—1. Such equations are also called *recurring equations*, because the coefficients recur in the same order.

2. It can be shown that a reciprocal equation of an *odd* degree is divisible by $x-1$ or $x+1$, according as the last term is *positive* or *negative*.

3. Also, a reciprocal equation of an *even* degree is divisible by x^2-1 when its last term is *positive*.

EXAMPLES.

1. Solve $x^4-3x^3+4x^2-3x+1=0$.

SOLUTION.

Given $\qquad x^4-3x^3+4x^2-3x+1=0.$

Divide by x^2, $\qquad x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0.$

Whence $\qquad x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0.$

Or $\qquad \left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)=-2.$

Complete the sq., $\qquad \left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+\frac{9}{4}=\frac{1}{4}.$

Extract the root $\qquad \left(x+\frac{1}{x}\right)-\frac{3}{2}=\pm\frac{1}{2}.$

Whence $\qquad x=1,\ 1,\ \frac{1}{2}(1\pm\sqrt{-3}).$

NOTE.—It is sometimes simpler to substitute some other quantity, as z, for $x+\frac{1}{x}$, and find the value of x from that of z.

2. Solve $x^4+x^3+x+1=0$. $\qquad$ *Ans.* $x=-1,\ -1,\ \frac{1}{2}(1\pm\sqrt{-3}$.
3. Solve $x^4-5x^3+6x^2-5x+1=0$.
   Ans. $x=2\pm\sqrt{3},\ \frac{1}{2}(1\pm\sqrt{-3})$.
4. Solve $x^4-10x^3+26x^2-10x+1=0$.
   Ans. $x=3\pm2\sqrt{2},\ 2\pm\sqrt{3}$.
5. Solve $x^4-3x^3+3x+1=0$. $\qquad$ *Ans.* $x=1\pm\sqrt{2},\ \frac{1}{2}(1\pm\sqrt{5})$.
6. Solve $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$. $\qquad$ *Ans.* $x=2,\ \frac{1}{2},\ \pm\sqrt{-1}$.
7. Solve $x^4-3x^3+3x-1=0$. $\qquad$ *Ans.* $x=\pm1,\ \frac{1}{2}(3\pm\sqrt{5})$.
8. Solve $x^2+x^{-2}+x+x^{-1}-4=0$.
   Ans. $x=1,\ 1,\ \frac{1}{2}(-3\pm\sqrt{5})$.

NOTES.—1. In equation 7 divide by x^2-1, and then reduce and find the values of x.

2. It may be readily shown that any two corresponding pair of roots are reciprocals of one another.

SECTION XI.

EXPONENTS AND LOGARITHMS.

THEORY OF EXPONENTS.

413. AN **Exponent** denotes the power of a quantity or the number of times it is used as a factor.

Thus a^3 means $a\times a\times a$, or a used as a factor three times; and a^n means $a\times a\times a\times\ .\ .\ .\ .$ to n factors, or a used as a factor n times.

414. By the original conception of a *power* the exponent n could be conceived only as a *positive integer*, and the rules for multiplication, division, etc. were all based on this conception.

415. Subsequently it was seen that division gave rise to *negative exponents* and evolution to *fractional exponents*, and that these could be used the same as positive integral exponents.

NOTE.—We shall now give a complete logical discussion of the subject, assuming only the definition of an exponent and the rules of addition and subtraction.

POSITIVE EXPONENTS

PRIN. 1. When m and n are positive integers, $a^m\times a^n=a^{m+n}$.

For $\qquad a^m=a\times a\times a\times\ .\ .\ .\ .$ to m factors. $\qquad$ (Def.)

And $\qquad a^n=a\times a\times a\times\ .\ .\ .\ .$ to n factors. $\qquad$ (Def.)

Hence $a^m\times a^n=a\times a\times\ .\ .\ .\ .\ \times a\times a\times a\ .\ .\ .\ .$ to $m+n$ factors, which by the definition equals a^{m+n}.

PRIN. 2. When m and n are positive integers, and m is greater than n, $a^m\div a^n=a^{m-n}$.