EXAMPLES.

1. Solve $x^4-3x^3+4x^2-3x+1=0$.

SOLUTION.

Given $x^4-3x^3+4x^2-3x+1=0.$

Divide by x^2, $x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0.$

Whence $x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0.$

Or $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)=-2.$

Complete the sq., $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+\frac{9}{4}=\frac{1}{4}.$

Extract the root $\left(x+\frac{1}{x}\right)-\frac{3}{2}=\pm\frac{1}{2}.$

Whence $x=1,\ 1,\ \frac{1}{2}(1\pm\sqrt{-3}).$

NOTE.—It is sometimes simpler to substitute some other quantity, as z, for $x+\frac{1}{x}$, and find the value of x from that of z.

2. Solve $x^4+x^3+x+1=0$. *Ans.* $x=-1,\ -1,\ \frac{1}{2}(1\pm\sqrt{-3}.$

3. Solve $x^4-5x^3+6x^2-5x+1=0$.
Ans. $x=2\pm\sqrt{3},\ \frac{1}{2}(1\pm\sqrt{-3})$.

4. Solve $x^4-10x^3+26x^2-10x+1=0$.
Ans. $x=3\pm2\sqrt{2},\ 2\pm\sqrt{3}$.

5. Solve $x^4-3x^3+3x+1=0$. *Ans.* $x=1\pm\sqrt{2},\ \frac{1}{2}(1\pm\sqrt{5})$.

6. Solve $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$. *Ans.* $x=2,\ \frac{1}{2},\ \pm\sqrt{-1}$.

7. Solve $x^4-3x^3+3x-1=0$. *Ans.* $x=\pm1,\ \frac{1}{2}(3\pm\sqrt{5})$.

8. Solve $x^2+x^{-2}+x+x^{-1}-4=0$.
Ans. $x=1,\ 1,\ \frac{1}{2}(-3\pm\sqrt{5})$.

NOTES.—1. In equation 7 divide by x^2-1, and then reduce and find the values of x.

2. It may be readily shown that any two corresponding pair of roots are reciprocals of one another.

SECTION XI.

EXPONENTS AND LOGARITHMS.

THEORY OF EXPONENTS.

413. An **Exponent** denotes the power of a quantity or the number of times it is used as a factor.

Thus a^3 means $a\times a\times a$, or a used as a factor three times; and a^n means $a\times a\times a\times \ldots$ to n factors, or a used as a factor n times.

414. By the original conception of a *power* the exponent n could be conceived only as a *positive integer*, and the rules for multiplication, division, etc. were all based on this conception.

415. Subsequently it was seen that division gave rise to *negative exponents* and evolution to *fractional exponents*, and that these could be used the same as positive integral exponents.

NOTE.—We shall now give a complete logical discussion of the subject, assuming only the definition of an exponent and the rules of addition and subtraction.

POSITIVE EXPONENTS

PRIN. 1. When m and n are positive integers, $a^m\times a^n=a^{m+n}$.

For $a^m=a\times a\times a\times \ldots$ to m factors. (Def.)

And $a^n=a\times a\times a\times \ldots$ to n factors. (Def.)

Hence $a^m\times a^n=a\times a\times \ldots \times a\times a\times a \ldots$ to $m+n$ factors, which by the definition equals a^{m+n}.

PRIN. 2. When m and n are positive integers, and m is greater than n, $a^m\div a^n=a^{m-n}$.

For $a^{m-n} \times a^n = a^{m-n+n}$. (Prin. 1.)

Reducing, $a^{m-n} \times a^n = a^m$.

Dividing by a^n, $\frac{a^m}{a^n} = a^{m-n}$.

Prin. 3. When m and n are positive integers, $(a^m)^n$ equals a^{mn}.

For $(a^m)^n = a^m \times a^m \times a^m \times \ldots$ to n factors;

But $a^m = a \times a \times a \times \ldots$ to m factors.

Hence $(a^m)^n = a \times a \times a \times \ldots$ to $m \times n$ factors,

which is indicated thus, a^{mn}.

Prin. 4. When m and n are positive integers, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

For, since in raising a quantity to the nth power we multiply the exponent by n, in extracting the nth root we must divide the exponent by n; whence the nth root of a^m is $a^{m \div n}$, or $a^{\frac{m}{n}}$.

NEGATIVE EXPONENTS.

416. The **Negative Exponent** arises from division when the exponent of the divisor is greater than the exponent of the dividend.

Prin. 5. Prove that $a^{-n} = \frac{1}{a^n}$.

For $a^{m-n} = \frac{a^m}{a^n}$. Prin. 2.

Dividing by a^m, $a^{-n} = \frac{1}{a^n}$.

This may also be shown as follows:

Now, $a^m \times a^{-n} = a^{m-n} = \frac{a^m}{a^n}$. Prin. 2.

Hence, $a^{-n} = \frac{1}{a^n}$. Div. by a^m.

Prin. 6. Prove that $a^m \div a^n = a^{m-n}$ when m is less than n.

Now, $a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.

But, $\frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}$.

Prin. 7. Prove that $a^m \times a^n = a^{m+n}$ when one or both exponents are negative.

First, suppose either exponent, as n, is negative. Let $n = -s$.

Then $a^m \times a^n = a^m \times a^{-s} = a^m \times \frac{1}{a^s} = \frac{a^m}{a^s} = a^{m-s}$.

Substituting n for $-s$, $a^{m-s} = a^{m+n}$; hence $a^m \times a^n = a^{m+n}$.

Second, suppose both exponents are negative. Let $m = -r$ and $n = -s$.

Then, $a^m \times a^n = a^{-r} \times a^{-s} = \frac{1}{a^r} \times \frac{1}{a^s} = \frac{1}{a^{r+s}} = a^{-(r+s)} = a^{-r-s}$.

Substituting, $a^{-r-s} = a^{m+n}$; hence $a^m \times a^n = a^{m+n}$.

Prin. 8. Prove that $a^m \div a^n = a^{m-n}$ when either or both exponents are negative.

First, suppose either exponent, as n, is negative. Let $n = -s$.

Then, $a^m \div a^n = a^m \div a^{-s} = a^m \div \frac{1}{a^s} = a^m \times \frac{a^s}{1} = a^m a^s = a^{m+s}$.

Substituting, $a^{m+s} = a^{m-n}$.

Second, suppose both exponents are negative. Let $m = -r$ and $n = -s$.

Then, $a^m \div a^n = a^{-r} \div a^{-s} = \frac{1}{a^r} \div \frac{1}{a^s} = \frac{1}{a^r} \times a^s = \frac{a^s}{a^r} = a^{s-r}$.

Substituting, $a^{s-r} = a^{-n-(-m)} = a^{m-n}$.

Prin. 9. Prove that $(a^m)^n = a^{mn}$ when one or both exponents are negative.

First, suppose m is negative, and let $m = -r$.

Then, $(a^m)^n = (a^{-r})^n = \left(\frac{1}{a^r}\right)^n = \frac{1}{a^{rn}} = a^{-rn} = a^{mn}$.

Second, suppose n is negative, and let $n = -p$.

Then, $(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} = \frac{1}{a^{mp}} = a^{-mp} = a^{mn}$.

Third, suppose m and n are both negative, and let $m = -r$ and $n = -p$.

Then, $(a^m)^n = (a^{-r})^{-p} = \frac{1}{(a^{-r})^p} = \frac{1}{a^{-rp}} = a^{rp} = a^{-m \times -n} = a^{mn}$.

PRIN. 10. Prove that $\sqrt[n]{a^m}=a^{\frac{m}{n}}$, when either m or n, or both, are negative.

First, suppose m is negative, and let $m=-r$.

Then $\sqrt[n]{a^m}=\sqrt[n]{a^{-r}}=\sqrt[n]{\frac{1}{a^r}}=\frac{1}{a^{\frac{r}{n}}}=a^{-\frac{r}{n}}=a^{\frac{m}{n}}.$

Second, suppose n is negative, and $n=-r$. Let $x=\sqrt[n]{a^m}$.

Then $x^n=a^m$, and $x^{-r}=a^m$, or $\frac{1}{x^r}=a^m$; $x^r=\frac{1}{a^m}$; $x=\frac{1}{a^{\frac{m}{r}}}=a^{-\frac{m}{r}}=a^{\frac{m}{n}}$.

Third, suppose both m and n are negative; let $m=-p$, $n=-r$, and $x=\sqrt[n]{a^m}$.

Then $x^n=a^m$, $x^{-r}=a^{-p}$, $\frac{1}{x^r}=\frac{1}{a^p}$, $x^r=a^p$, $x=a^{\frac{p}{r}}=a^{\frac{m}{n}}$.

NOTE.—No practical significance is attached to a *negative index* of a root; but the form is a possible one, and the above demonstration proves the principle to be general.

417. Thus we see that whether m and n are positive or negative integers, we have the following:

I. $a^m \times a^n = a^{m+n}$. III. $(a^m)^n = a^{mn}$.

II. $a^m \div a^n = a^{m-n}$. IV. $\sqrt[n]{a^m}=a^{\frac{m}{n}}$.

FRACTIONAL EXPONENTS.

418. A **Fractional Exponent** arises from evolution by dividing the exponent of the power by the index of the root, when the former is not a multiple of the latter.

419. We shall show the meaning of the fractional exponent and prove a few principles to be used in showing the universality of the fundamental rules.

PRIN. 11. Prove that $(a^{\frac{m}{n}})^p=a^{\frac{mp}{n}}$ when $\frac{m}{n}$ and p are positive or negative.

First, suppose p is positive, $\frac{m}{n}$ being either positive or negative.

Raising $a^{\frac{m}{n}}$ to the pth power, we have $a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times \ldots$ to p factors, or $a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots \text{ to } p \text{ terms}}=a^{\frac{m}{n}\times p}=a^{\frac{mp}{n}}$.

Second, suppose p is negative, and let $p=-s$.

Then $(a^{\frac{m}{n}})^p=(a^{\frac{m}{n}})^{-s}=\frac{1}{(a^{\frac{m}{n}})^s}=\frac{1}{a^{\frac{ms}{n}}}=a^{-\frac{ms}{n}}=a^{-\frac{m}{n}\times -p}=a^{\frac{mp}{n}}.$

PRIN. 12. Prove that $a^{\frac{m}{n}}=\sqrt[n]{a^m}$, m and n being either positive or negative.

First, suppose m is positive or negative, n being positive.

By Prin. 11, $(a^{\frac{m}{n}})^n=a^m$; extract nth root, $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

Second, suppose n is negative, and let $n=-s$.

Then, $(a^{\frac{m}{n}})^n=(a^{-\frac{m}{s}})^{-s}=\frac{1}{(a^{-\frac{m}{s}})^s}=\frac{1}{a^{-m}}=a^m.$

Hence, $(a^{\frac{m}{n}})^n=a^m$, and $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

COR. Hence, $a^{\frac{1}{n}}=\sqrt[n]{a}$, or $\sqrt[n]{a}=a^{\frac{1}{n}}$; also $(a^{\frac{1}{n}})^n=a^{\frac{n}{n}}$, or a.

NOTE.—This principle was derived under the previous article, but is here proved by another process of reasoning.

PRIN. 13. Prove that $\sqrt[n]{a^m}=(a^m)^{\frac{1}{n}}$.

Let $a^m=x$; then $\sqrt[n]{a^m}=\sqrt[n]{x}=x^{\frac{1}{n}}$. (Prin. 12, Cor.)

But since $x=a^m$, $x^{\frac{1}{n}}=(a^m)^{\frac{1}{n}}$; hence $\sqrt[n]{a^m}=(a^m)^{\frac{1}{n}}$.

COR. Hence $a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}$.

PRIN. 14. Prove that $a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}$.

Let $x=a^{\frac{1}{n}}\times b^{\frac{1}{n}}$.

Then, $x^n=(a^{\frac{1}{n}}\times b^{\frac{1}{n}})^n=(a^{\frac{1}{n}})^n\times(b^{\frac{1}{n}})^n=ab.$

Hence, $x^n=(ab)$; therefore, $x=(ab)^{\frac{1}{n}}$. (Prin. 12, Cor.)

COR. 1. In the same way it may be shown that

$$a^{\frac{1}{n}} \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

COR. 2. Hence also $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$.

PRIN. 15. Prove that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

Let $x = a^{\frac{m}{n}} \times b^{\frac{m}{n}}$.

Then, $x^n = (a^{\frac{m}{n}} \times b^{\frac{m}{n}})^n = (a^{\frac{m}{n}})^n \times (b^{\frac{m}{n}})^n = a^m \times b^m = (ab)^m$.

Hence, $x^n = (ab)^m$; and $x = (ab)^{\frac{m}{n}}$; therefore, $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

PRIN. 16. Prove that $(a^{\frac{1}{m}})^n = (a^n)^{\frac{1}{m}}$.

By Prin. 11, $(a^{\frac{1}{m}})^n = a^{\frac{n}{m}}$; let $x = a^{\frac{n}{m}}$.

Then, $x^m = a^n$, and $x = (a^n)^{\frac{1}{m}}$. Prin. 12.

COR. In a similar way it may be shown that $(a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}}$.

PRIN. 17. Prove that $(a^m)^{\frac{1}{n}} \times (a^p)^{\frac{1}{n}} = (a^m \times a^p)^{\frac{1}{n}}$.

Let $x = (a^m)^{\frac{1}{n}} \times (a^p)^{\frac{1}{n}}$; then $x^n = a^m \times a^p$;

Hence, $x = (a^m \times a^p)^{\frac{1}{n}}$.

PRIN. 18. Prove that $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

Let $x = a^{\frac{m}{n}}$; then $x^n = a^m$, and $x^{np} = a^{mp}$.

Hence, $x = a^{\frac{mp}{np}}$; therefore, $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

420. We shall now proceed to show that the rules for multiplication, division, involution, and evolution apply to fractional exponents as well as integral.

PRIN. 19. Prove that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$.

$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}$. Prin. 18.

$= (a^{ps})^{\frac{1}{qs}} \times (a^{qr})^{\frac{1}{qs}}$. Prin. 13, Cor.

$= (a^{ps} \times a^{qr})^{\frac{1}{qs}}$. Prin. 17.

$= (a^{ps+qr})^{\frac{1}{qs}}$.

$= a^{\frac{ps}{qs}+\frac{qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}$.

COR. In the same way it may be shown that

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

PRIN. 20. Prove that $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let $x = (a^{\frac{p}{q}})^{\frac{r}{s}}$; then $x^s = (a^{\frac{p}{q}})^r = a^{\frac{pr}{q}}$.

Hence $x^{qs} = a^{pr}$; and $x = a^{\frac{pr}{qs}}$.

COR. In the same way it may be shown that

$$\sqrt[\frac{r}{s}]{a^{\frac{p}{q}}} = a^{\frac{p}{q} \div \frac{r}{s}} = a^{\frac{p}{q} \times \frac{s}{r}} = a^{\frac{ps}{qr}}.$$

SCHOLIUM. When the exponents in Prin. 19 and 20 are negative, we can let m and n represent the fractions, and since the principles are true for m and n, they are true for negative fractions. Or we can prove them by the methods used for negative integral exponents.

421. It is thus shown that the following relations are universal, m and n being positive or negative, integral or fractional.

I. $a^m \times a^n = a^{m+n}$. III. $(a^m)^n = a^{mn}$.

II. $a^m \div a^n = a^{m-n}$. IV. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

NOTE.—The student will be interested in noticing that this general discussion has introduced two forms of expression that are not usually employed in algebra—viz. *negative indices* and *fractional indices* of roots. Thus, since n is general, $\sqrt[n]{a}$ gives the forms $\sqrt[-2]{a}$; $\sqrt[\frac{2}{3}]{a}$; $\sqrt[-\frac{2}{3}]{a}$.

EXAMPLES.

1. Prove $a^{m-n}=\dfrac{1}{a^{n-m}}$.

2. Prove $(a)^{-\frac{1}{n}}=\dfrac{1}{a^{\frac{1}{n}}}$.

3. Prove $\sqrt[-n]{a^m}=\dfrac{1}{a^{\frac{m}{n}}}$.

4. Show the meaning of the negative exponent, a^{-n}.

5. Show the meaning of the fractional exponent, $a^{\frac{3}{4}}$ or $a^{\frac{n}{m}}$.

6. Show the meaning of a fractional index, $\sqrt[\frac{1}{2}]{a}$ or $\sqrt[\frac{n}{m}]{a}$.

7. Show the meaning of a negative index, $\sqrt[-2]{a}$ or $\sqrt[-n]{a}$.

8. Show the meaning of a negative fractional index, $\sqrt[-\frac{2}{3}]{a}$ or $\sqrt[-\frac{n}{m}]{a}$.

EXAMPLES IN REDUCTION.

1. $9^{-\frac{1}{2}}$.	*Ans.* $\frac{1}{3}$.	11. $(4a^{-\frac{2}{3}})^{-\frac{3}{2}}$.	*Ans.* $\dfrac{a}{8}$.
2. $4^{-\frac{3}{2}}$.	*Ans.* $\frac{1}{8}$.	12. $(64n^{-3})^{-\frac{2}{3}}$.	*Ans.* $\left(\dfrac{n}{4}\right)^2$.
3. $64^{-\frac{1}{3}}$.	*Ans.* $\frac{1}{4}$.		
4. $\dfrac{1}{81^{-\frac{3}{4}}}$.	*Ans.* 27.	13. $(32a^{-15})^{\frac{2}{5}}$.	*Ans.* $\dfrac{4}{a^6}$.
5. $(a^{-2})^{\frac{1}{3}}$	*Ans.* $\dfrac{1}{a^{\frac{2}{3}}}$.	14. $(5^{\frac{3}{4}}a^{-6})^{-\frac{2}{3}}$.	*Ans.* $\dfrac{a^4}{\sqrt{5}}$.
6. $\dfrac{1}{(a^2)^{-\frac{1}{3}}}$.	*Ans.* $a^{\frac{2}{3}}$.	15. $\sqrt[3]{a^{-3}}$.	*Ans.* $\dfrac{1}{a}$.
7. $(x^{-2})^{-3}$.	*Ans.* x^6.	16. $\sqrt[-2]{\dfrac{a^{-2}}{16}}$.	*Ans.* $4a$.
8. $\sqrt{a^{-4}}$.	*Ans.* $\dfrac{1}{a^2}$.		
		17. $\sqrt[\frac{2}{3}]{4a^2}$.	*Ans.* $8a^3$.
9. $(m^{-\frac{2}{3}})^{-\frac{3}{4}}$.	*Ans.* $m^{\frac{1}{2}}$.		
10. $(x^{-\frac{m}{n}})^{\frac{n}{m}}$.	*Ans.* $\dfrac{1}{x}$.	18. $\sqrt[-\frac{3}{4}]{8a^{-3}}$.	*Ans.* $\left(\dfrac{a}{2}\right)^4$.

19. $\sqrt[-3]{\dfrac{a^6}{64}}$.	*Ans.* $\left(\dfrac{2}{a}\right)^2$.	23. $\sqrt[\frac{1}{n}]{a^{\frac{1}{n}}b^{\frac{m}{n}}}$.	*Ans.* ab^m.
20. $\sqrt[-3]{\dfrac{a^{-9}}{27}}$.	*Ans.* $3a^3$.	24. $\sqrt[-\frac{1}{n}]{a^{-2n}b^{4n}}$.	*Ans.* $a^{2n^2}b^{-4n^2}$.
21. $\sqrt[\frac{2}{3}]{\dfrac{a^4}{c^2}}$.	*Ans.* $\left(\dfrac{a^2}{c}\right)^3$.	25. $\sqrt[-\frac{m}{n}]{a^{-\frac{1}{n}}b^{\frac{1}{n}}}$.	*Ans.* $\left(\dfrac{a}{b}\right)^{\frac{1}{m}}$.
22. $\sqrt[-\frac{4}{5}]{a^{-8}b^4}$.	*Ans.* $\left(\dfrac{a^2}{b}\right)^5$.	26. $\left(\sqrt[-\frac{m}{n}]{a^{\frac{m}{n}}c^{\frac{n}{m}}}\right)^{-\frac{m}{n}}$.	*Ans.* $a^{\frac{m}{n}}c^{\frac{n}{m}}$.

EXAMPLES IN MULTIPLICATION.

1. $a^{\frac{1}{3}}\times a^{-\frac{1}{6}}$.	*Ans.* $a^{\frac{1}{6}}$.	8. $y^{\frac{3}{4}}\times\sqrt{y^{-1}}$.	*Ans.* $y^{\frac{1}{4}}$.
2. $a^{-2}\times\sqrt{a}$.	*Ans.* $\left(\dfrac{1}{a}\right)^{\frac{3}{2}}$.	9. $(4^m a)^{m-n+p}\times(4^m a)^{n-m-p}$.	*Ans.* 1.
3. $a^{\frac{n}{2}}\times a^{\frac{1}{2}(m-n)}$.	*Ans.* $\sqrt{a^m}$.	10. $a^{m-n}\times 3a^{3m-2}\times 4a^{n+5}c$.	
4. $a^2c^0\times a^{-2}c^{2n}$.	*Ans.* c^{2n}.		*Ans.* $12a^{4m+3}c$.
5. $x^{\frac{1}{2}}\times\sqrt{x^{\frac{1}{3}}}$.	*Ans.* $x^{\frac{2}{3}}$.	11. $(a^{-m}b^p\times a^nb^{-2}\times a^{m+n}b^4$.	
6. $m^{-\frac{1}{3}}\times\sqrt{m^{\frac{1}{2}}}$.	*Ans.* $\left(\dfrac{1}{m}\right)^{\frac{1}{12}}$.		*Ans.* $a^{2n}b^{p+2}$.
		12. $(a+b)^{m+n}c^n\times(a+b)^{m-n}c^{-n}$.	
7. $a^{\frac{1}{n}}\times a^{\frac{1}{m}}$.	*Ans.* $a^{\frac{m+n}{n}}$.		*Ans.* $(a+b)^{2m}$.

Multiply the following:

13. $a^{\frac{m}{2}}b^{-\frac{n}{3}}\times a^{m+n}\times(a^{\frac{1}{2}}b^{\frac{1}{3}})^{2m-2n}$. *Ans.* $a^{\frac{5m}{2}}b^{\frac{2m}{3}-n}$.

14. $(8^{-2}a^2x)^{m+3n}\times(8^3a^{-2}x^2)^{m+2n}$. *Ans.* $8^m a^{2n}x^{3m+7n}$.

15. $(a+b)^{-3}c^4\times(a+b)^{n+3}c^{-n}\times(a+b)^{1+n}c^{-2}$.
Ans. $(a-b)^{2n+1}c^{2-n}$.

16. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})$ by $a^{\frac{2}{3}}-b^{\frac{2}{3}}$. *Ans.* $a^{\frac{4}{3}}-b^{\frac{4}{3}}$.

17. $a^{m+n}+b^{m-n}$ by $a^{m+n}-b^{m-n}$. *Ans.* $a^{2(m+n)}-b^{2(m-2)}$.

18. x^n-y^{mn-n} by $x^{mn-n}y^n$. *Ans.* $x^{mn}y^n-x^{mn-n}y^{mn}$.

19. $a^{\frac{m}{n}}+b^{\frac{m}{n}}$ by $a^{\frac{m}{n}}-b^{\frac{m}{n}}$. *Ans.* $a^{\frac{2m}{n}}-b^{\frac{2m}{n}}$.

20. $x+x^{\frac{1}{2}}+2$ by $x+x^{\frac{1}{2}}-2$. *Ans.* $x^2+2x^{\frac{3}{2}}+x-4$.

21. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. *Ans.* $x-y$.

22. a^4+a^2+1 by $a^{-4}-a^{-2}+1$. *Ans.* a^4+1+a^{-4}.

23. $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$ by $a^{-\frac{1}{3}}-1$. *Ans.* $a^{-1}-1$.

24. $a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$. *Ans.* $a^2-3a^{\frac{2}{3}}+3a^{-\frac{2}{3}}-a^{-2}$.

25. $-3a^{-5}+2a^{-4}b^{-1}$ by $-2a^{-3}-3a^{-4}b$.
Ans. $-4a^{-7}b^{-1}+9a^{-9}b$.

26. $a^{\frac{7}{2}}-a^3+a^{\frac{5}{2}}-a^2+a^{\frac{3}{2}}-a+a^{\frac{1}{2}}-1$ by $a^{\frac{1}{2}}+1$. *Ans.* a^4-1.

EXAMPLES IN DIVISION.

1. $a^{\frac{1}{2}} \div a^{-\frac{1}{3}}$. *Ans.* $a^{\frac{5}{6}}$.
2. $a^{-\frac{2}{3}} \div a^{\frac{3}{4}}$. *Ans.* $a^{-\frac{17}{12}}$.
3. $a^m \div a^{m-n}$. *Ans.* a^n.
4. $b^{\frac{1}{6}} \div b^{-\frac{1}{3}}$. *Ans.* $b^{\frac{1}{2}}$.
5. $a^{-n} \div a^{-2n+1}$. *Ans.* a^{n-1}.
6. $a^{m-n} \div a^{n+m}$. *Ans.* a^{-2n}.
7. $b^0 \div b^{-n}$. *Ans.* b^n.
8. $a^{\frac{1}{2}} \div \sqrt{a^{\frac{2}{3}}}$. *Ans.* $a^{\frac{1}{6}}$.
9. $\sqrt{a^{-2n}} \div (a^{-2})^n$. *Ans.* a^n.
10. $x^n \div \sqrt{x^{-4n}}$. *Ans.* x^{3n}.
11. $a^{\frac{n}{2}} \div a^{\frac{2n}{3}}$. *Ans.* $\left(\frac{1}{a^n}\right)^{\frac{1}{6}}$.
12. $(a+b)^{\frac{1}{2}} \div (a+b)^{\frac{1}{3}}$.
Ans. $(a+b)^{\frac{1}{6}}$.

13. $(a-x)^{-\frac{2}{3}} \div (a-x)^{\frac{1}{3}}$.
Ans. $(a-x)^{-1}$.
14. $6a^{\frac{2}{3}}b^{\frac{1}{3}} \div 3a^{-\frac{1}{3}}b^{\frac{2}{3}}$. *Ans.* $2ab^{-\frac{1}{3}}$.
15. $4a^{-\frac{3}{4}}b^{\frac{1}{3}} \div 2a^{\frac{1}{3}}b^{-\frac{3}{4}}$.
Ans. $2\left(\frac{b}{a}\right)^{\frac{13}{12}}$.
16. $(a-b)^{\frac{m}{2}+n} \div (a-b)^{2n-\frac{m}{2}}$.
Ans. $(a-b)^{m-n}$.
17. $a^n b^{-\frac{m}{2}} c^{-2} \div a^{-n} b^{\frac{m}{2}} c^{n-4}$.
Ans. $a^{2n}b^{-m}c^{2-n}$.
18. $(a^{3n}x^{9n})^{\frac{1}{3}} \div \sqrt[\frac{1}{n}]{a^3x}$.
Ans. $\left(\frac{x}{a}\right)^{2n}$.

Divide the following:

19. $a^{\frac{3}{4}}-b^{\frac{3}{4}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$. *Ans.* $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

20. $a^{\frac{2}{3}}-b^{\frac{2}{3}}$ by $a^{\frac{1}{6}}-b^{\frac{1}{6}}$. *Ans.* $a^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{1}{6}}+a^{\frac{1}{6}}b^{\frac{1}{3}}+b^{\frac{1}{2}}$.

21. $a-x$ by $a^{\frac{1}{3}}-x^{\frac{1}{3}}$. *Ans.* $a^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}$.

22. a^n-b^m by $a^{\frac{n}{4}}-b^{\frac{m}{4}}$. *Ans.* $a^{\frac{3n}{4}}+a^{\frac{2n}{4}}b^{\frac{m}{4}}+a^{\frac{n}{4}}b^{\frac{2m}{4}}+b^{\frac{3m}{4}}$.

23. $a^{2n}-b^{-3m}$ by $a^{\frac{n}{2}}-b^{-\frac{3m}{4}}$. *Ans.* $a^{\frac{3n}{2}}+a^nb^{-\frac{3m}{4}}+a^{\frac{n}{2}}b^{-\frac{3m}{2}}+b^{-\frac{9m}{4}}$.

24. $a^{m-2}b^{2-n}c^{m-n}$ by $a^{m-n}b^{n+2}c^{2-n}$. *Ans.* $a^{n-2}b^{-2n}c^{m-2}$.

25. $a^2b^{-2}+2+a^{-2}b^2$ by $ab^{-1}+a^{-1}b$. *Ans.* $ab^{-1}+a^{-1}b$

26. $8x^{-1}+27y^{-2}$ by $2x^{-\frac{1}{3}}+3y^{-\frac{2}{3}}$. *Ans.* $4x^{-\frac{2}{3}}-6x^{-\frac{1}{3}}y^{-\frac{2}{3}}+9y^{-\frac{4}{3}}$.

27. $a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}-a^{-\frac{n}{2}}$. *Ans.* a^n+1+a^{-n}.

28. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. *Ans.* $x+y$.

29. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ by $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. *Ans.* $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

30. $x^{-4}+x^{-2}y^{-2}+y^{-4}$ by $x^{-2}-x^{-1}y^{-1}+y^{-2}$.
Ans. $x^{-2}+x^{-1}y^{-1}+y^{-2}$.

31. $a^{\frac{2}{3}}+b^{\frac{2}{3}}-c^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$. *Ans.* $a^{\frac{1}{3}}+b^{\frac{1}{3}}-c^{\frac{1}{3}}$.

32. Find the value of $\{(a^{2m})^3 \times (a^{-n})^2\}^{\frac{1}{3m-n}}$. *Ans.* a^2.

33. Find the value of $[\{(a-b)^n\}^{n-\frac{1}{n}}]^{\frac{1}{n+1}}$. *Ans.* $(a-b)^{n-1}$.

34. Find the value of $\{(b^{m-n})^{-2} \times (b^{-m+2n})^{-3}\}^{\frac{a}{m-4n}}$. *Ans.* b^a.

35. Find the value of $[\{(a^{\frac{m}{n}})^{\frac{n}{m}}\}^{\frac{p}{q}})^{\frac{q}{p}} \times [\{(a^{\frac{n}{m}})^{\frac{m}{n}}\}^{\frac{q}{p}}]^{\frac{p}{q}}$.
Ans. a^2.

36. Find the value of $[\{(a^m)^{-\frac{1}{n}}\}^p]^{-\frac{1}{q}} \div [\{(a^{-m})^{\frac{1}{n}}\}^{-p}]^{\frac{1}{q}}$. *Ans.* 1.

LOGARITHMS.

422. The **Logarithm** of a number is the exponent denoting the power to which a fixed number must be raised to produce the first number.

Thus, if $B^x = N$, then x is called the logarithm of N.

423. The **Base** of the system is the *fixed number* which is raised to the different powers to produce the numbers.

Thus, in $B^x = N$, x is the logarithm of N to the base B; so in $4^3 = 64$, 3 is the logarithm of 64 to the base 4.

424. The term *logarithm*, for convenience, is usually written *log.* The expressions above may be written $\log N = x$ and $\log 64 = 3$.

425. In the **Common System** of logarithms the base is 10, and the nature of logarithms is readily seen with this base; thus,

$$10^2 = 100;\quad \text{hence } \log 100 = 2.$$
$$10^3 = 1000;\quad \text{hence } \log 1000 = 3.$$
$$10^4 = 10{,}000;\quad \text{hence } \log 10{,}000 = 4.$$
$$10^{2.369} = 234;\quad \text{hence } \log 234 = 2.369.$$

426. We shall first derive the general principles of logarithms, the base being *any number*, and then explain the common numerical system.

PRINCIPLES.

PRIN. 1. *The logarithm of 1 is 0, whatever the base.*

For, let B represent any base, then $B^0 = 1$; hence by the definition of a logarithm, 0 is the log. of 1, or $\log 1 = 0$.

PRIN. 2. *The logarithm of the base of a system of logarithms is unity.*

For, let B represent any base, then $B^1 = B$; hence 1 is the log. of B, or $\log B = 1$.

PRIN. 3. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.*

For, let	$m = \log M$, and $n = \log N$.
Then,	$B^m = M$, $\quad B^n = N$.
Multiplying,	$B^{m+n} = M \times N$.
Hence	$m + n = \log (M \times N)$.
Or,	$\log (M \times N) = \log M + \log N$.

PRIN. 4. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For, let	$m = \log M$, and $n = \log N$.
Then,	$B^m = M$, $\quad B^n = N$.
Dividing,	$B^{m-n} = M \div N$.
Hence,	$\log (M \div N) = m - n$.
Or	$\log (M \div N) = \log M - \log N$.

PRIN. 5. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, let	$m = \log M$.
Then,	$B^m = M$.
Raising to nth power,	$B^{n \times m} = M^n$.
Whence,	$\log M^n = n \times m$.
Or	$\log M^n = n \times \log M$.

PRIN. 6. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let	$m = \log M$.
Then,	$B^m = M$.
Taking nth root,	$B^{\frac{m}{n}} = M^{\frac{1}{n}}$.
Whence,	$\log M^{\frac{1}{n}} = \frac{m}{n}$.
Or	$\log M^{\frac{1}{n}} = \frac{\log M}{n}$.

427. These principles are illustrated by the following examples, which the pupil will work.

EXAMPLES.

1. $\text{Log } (a.b.c) = \log a + \log b + \log c.$
2. $\text{Log} \left(\frac{ab}{c}\right) = \log a + \log b - \log c.$
3. $\text{Log } a^n = n \log a.$
4. $\text{Log } (a^x b^y) = x \log a + y \log b.$
5. $\text{Log } \frac{a^x b^y}{c^z} = x \log a + y \log b - z \log c.$
6. $\text{Log } \sqrt{ab} = \frac{1}{2} \log a + \frac{1}{2} \log b.$
7. $\text{Log } (a^2 - x^2) = \log (a+x) + \log (a-x).$
8. $\text{Log } \sqrt{a^2 - x^2} = \frac{1}{2} \log (a+x) + \frac{1}{2} \log (a-x).$
9. $\text{Log } a^2 \sqrt[3]{a^{-2}} = \frac{4}{3} \log a.$
10. $\text{Log } \frac{\sqrt{a^2 - x^2}}{(a+x)^2} = \frac{1}{2} \{\log (a-x) - 3 \log (a+x)\}.$

COMMON LOGARITHMS.

428. The **Base** of the common system of logarithms is 10. This base is most convenient for numerical calculations, because our numerical system is decimal.

429. In this system every number is conceived to be some power of 10, and by the use of fractional exponents may be thus, approximately, expressed.

430. Raising 10 to different powers, we have

$10^0 = 1$; hence $0 = \log 1$.
$10^1 = 10$; hence $1 = \log 10$.
$10^2 = 100$; hence $2 = \log 100$.
$10^3 = 1000$; hence $3 = \log 1000$.
etc. etc.

Also, $10^{-1} = .1$; hence $-1 = \log .1$.
$10^{-2} = .01$; hence $-2 = \log .01$.
$10^{-3} = .001$; hence $-3 = \log .001$.

431. Hence the logarithms of all numbers

between 1 and 10 will be $0 + \text{a fraction}$;
between 10 and 100 will be $1 + \text{a fraction}$;
between 100 and 1000 will be $2 + \text{a fraction}$;
between 1 and .1 will be $-1 + \text{a fraction}$;
between .1 and .01 will be $-2 + \text{a fraction}$;
between .01 and .001 will be $-3 + \text{a fraction}$.

432. Thus it has been found that the log. of 76 is 1.8808, and the log. of 458 is 2.6608. This means that

$$10^{1.8808} = 76, \text{ and } 10^{2.6608} = 458.$$

433. When the logarithm consists of an integer and a decimal, the integer is called the *characteristic*, and the decimal part the *mantissa*. Thus, in 2.660865, 2 is the *characteristic*, and .660865 is the *mantissa*.

PRINCIPLES OF COMMON LOGARITHMS.

PRIN. 1. *The characteristic of a logarithm of a number is one less than the number of integral places in the number.*

For, from Art. 430, $\log 1 = 0$ and $\log 10 = 1$; hence the logarithm of numbers from 1 to 10 (which consist of *one* integral place) will have 0 for the characteristic. Since $\log 10 = 1$ and $\log 100 = 2$, the logarithm of numbers from 10 to 100 (which consist of *two* integral places) will have *one* for the characteristic, and so on; hence *the characteristic is always one less than the number of integral places.*

PRIN. 2. *The characteristic of the logarithm of a decimal is negative, and is equal to the number of the place occupied by the first significant figure of the decimal.*

For, from Art. 430, $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$; hence the logarithms of numbers from .1 to 1 will have -1 for a characteristic; the logarithms of numbers between .01 and .1 will have -2 for a characteristic, and so on; hence *the characteristic of a decimal is always negative, and equal to the number of the place of the first significant figure of the decimal.*

PRIN. 3. *The logarithm of the product of any number multiplied by* 10 *is equal to the logarithm of the number increased by* 1.

For, suppose $\log M = m$; then, by Prin. 3, Art. 426,

$\log (M \times 10) = \log M + \log 10$; but $\log 10 = 1$;

Hence $\log (M \times 10) = m + 1$.

Thus, $\log (76 \times 10) = 1.880814 + 1$; or $\log 760 = 2.880814$.

PRIN. 4. *The logarithm of the quotient of any number divided by* 10 *is equal to the logarithm of the number diminished by* 1.

For, suppose $\log M = m$; then, by Prin. 4, Art. 426,

$\log (M \div 10) = \log M - \log 10$;

Hence, $\log (M \div 10) = m - 1$.

Thus, $\log (458 \div 10) = 2.660865 - 1$; or $\log 45.8 = 1.660865$.

PRIN. 5. *In changing the decimal point of a number we change the characteristic, but do not change the mantissa of its logarithm.*

This follows from Principles 3 and 4. To illustrate:

$\log 234 = 2.369216$.	$\log .234 = \bar{1}.369216$.
$\log 23.4 = 1.369216$.	$\log .0234 = \bar{2}.369216$.
$\log 2.34 = 0.369216$.	

Thus we see that the *characteristic becomes negative, but not the mantissa.* The minus sign is written *over* the characteristic to show that it only is negative.

EXERCISES ON LOGARITHMS.

434. Common Logarithms are used to facilitate the operations of multiplying, dividing, etc. Tables of logarithms are constructed and used for this purpose.

435. We shall give the logarithms of a few prime numbers to four decimal places, and show how they are used.

$\log 2 = 0.3010$	$\log 7 = 0.8451$	$\log 17 = 1.2304$
$\log 3 = 0.4771$	$\log 11 = 1.0414$	$\log 19 = 1.2787$
$\log 5 = 0.6990$	$\log 13 = 1.1139$	$\log 23 = 1.3617$

MULTIPLICATION WITH LOGARITHMS.

436. Numbers are multiplied by means of logarithms by taking the sum of their logarithms. (See Art. 426.)

1. Find the logarithm of 2×5.

SOLUTION. From Prin. 3, Art. 426, the log. of 2×5 equals the log. of 2 plus the log. of 5; $\log 2 = 0.3010$, $\log 5 = 0.6990$; their sum is 1.0000.

OPERATION.

$\log 2 = 0.3010$
$\log 5 = 0.6990$
$\log 10 = 1.0000$

Find by the use of the logarithms given in Art. 435 the following:

2. $\log (3 \times 7)$.	5. $\log 5 \times 10$.	8. $\log 2 \times 7 \times 13$.
3. $\log (5 \times 7)$.	6. $\log 7 \times 10$.	9. $\log (5 \times 17 \times 23)$.
4. $\log (7 \times 11)$.	7. $\log 13 \times 10$.	10. $\log (7 \times 19 \times 23)$.

NOTE.—In actual practice with a table we find the number corresponding to the logarithm of the product, and thus obtain the product of the numbers.

437. The logarithms above given will enable us to find the logarithms of many numbers of which the prime numbers are factors. Find the following:

1. log 4.	5. log 20.	9. log 56.	13. log 1.15.
2. log 6.	6. log 26.	10. log 85.	14. log .230.
3. log 10.	7. log 30.	11. log 8.5.	15. log .380.
4. log 15.	8. log 42.	12. log 115.	16. log .0035.

DIVISION WITH LOGARITHMS.

438. Numbers are divided by means of logarithms by subtracting the logarithm of the divisor from the logarithm of the dividend.

1. Find the log. of $5 \div 2$.

SOLUTION. From Prin. 4, Art. 426, the log. of the quotient of 5 divided by 2 equals log 5 minus log 2; $\log 5 = 0.6990$, etc.

OPERATION.

$\log 5 = 0.6990$
$\log 2 = 0.3010$
$\log (5 \div 2) = 0.3980$

Find by the logs. given in Art. 435 the logs. of the following:

2. $\frac{3}{7}$.	6. $\frac{3}{5}$.	10. $\frac{.02}{11}$.	14. $\frac{10}{17}$.
3. $\frac{5}{7}$.	7. $\frac{1.7}{3}$.	11. $\frac{.03}{5}$.	15. $\frac{1.7}{.10}$.
4. $\frac{7}{2}$.	8. $\frac{15}{7}$.	12. $\frac{23}{17}$.	16. $\frac{3\times .5}{2\times .7}$.
5. $\frac{11}{3}$.	9. $\frac{21}{5}$.	13. $\frac{.05}{.003}$.	17. $\frac{2\times .3\times .05}{.7\times .11+22}$.

NOTE.—For an explanation of the nature and use of the *arithmetical complement* see Trigonometry.

POWERS AND ROOTS WITH LOGARITHMS.

439. The powers or roots of numbers are readily obtained by logarithms, according to Prin. 5, Art. 422.

1. Find the log. of 7^3.

SOLUTION. By Prin. 5, Art. 426, log 7^3 equals log 7 multiplied by 3; log 7 = 0.8451; multiplying by 3, we have 2.5353; hence log 7^3, or log 343 = 2.5353.

OPERATION.

$$\begin{array}{r} \log 7 = 0.8451 \\ 3 \\ \hline \log 7^3 \text{ or } 343 = 2.5353 \end{array}$$

Find the logarithms of the following:

2. 3^2.	6. 13^5.	10. $3^{\frac{1}{2}}$.	14. $2^3\times 3^2$.
3. 5^3.	7. 15^4.	11. $7^{\frac{2}{3}}$.	15. $3^2\div .05^3$.
4. 7^4.	8. 17^3.	12. $.07^{\frac{3}{4}}$.	16. $.07^3\times .014^2$.
5. 11^5.	9. 19^2.	13. $.01^{\frac{1}{4}}$.	17. $.09^{\frac{1}{2}}\div .021^{\frac{1}{3}}$.

NOTE.—Teachers who wish to give their pupils a knowledge of the use of the tables and numerical computation with logarithms will find the subject presented in my Geometry and Trigonometry.

EXPONENTIAL EQUATIONS.

440. An **Exponential Equation** is an equation in which the unknown quantity is an exponent; as,

$$a^x = b, \qquad x^x = a, \qquad b^{ax} = c, \text{ etc.}$$

441. Such equations are most readily solved by means of logarithms.

1. Given $a^x = b$, to find x.

SOLUTION.

Given, $a^x = b$.

Taking log. of members, $x \log a = \log b$.

Whence, $x = \frac{\log b}{\log a}$.

2. Given $5^x = 10$, to find x.

SOLUTION.

Given, $5^x = 10$.

Taking log., $x \log 5 = \log 10$.

Whence, $x = \frac{\log 10}{\log 5} = \frac{1.000}{0.6990} = 1.4306$.

3. Given $5^{\frac{2}{x}} = \frac{7}{3}$, to find x.

SOLUTION.

Given, $5^{\frac{2}{x}} = \frac{7}{3}$.

Raising to x power, $5^2 = \frac{7^x}{3^x}$.

Taking log., $2 \log 5 = x \log 7 - x \log 3$.

Whence, $x = \frac{2 \log 5}{\log 7 - \log 3}$.

Or, $x = \frac{2 \times .6990}{.8451 - .4771} = 3.7989$.

EXAMPLES.

4. Given $5^x = 8$, to find x. *Ans.* $x = 1.2918$.
5. Given $4^{2x} = 8^{\frac{2}{3}}$, to find x. *Ans.* $x = 0.5$.
6. Given $a^x = bc$, to find x. *Ans.* $x = \frac{\log b + \log c}{\log a}$.
7. Given $a^x = b^2c^3$, to find x. *Ans.* $x = \frac{2 \log b + 3 \log c}{\log a}$.
8. Given $5^{\frac{2}{x}} = 30$, to find x. *Ans.* $x = 0.9464$.

9. Given $\frac{ab^x - c}{n} = d$, to find x. *Ans.* $x = \frac{\log(nd+c) - \log a}{\log b}$.

10. Given $ab^{\frac{1}{x}} = c$, to find x. *Ans.* $x = \frac{\log b}{\log c - \log a}$.

11. Given $a^{mx+n} = b$, to find x. *Ans.* $x = \frac{\log b - n \log a}{m \log a}$.

12. Given $m^{ax} n^{bx} = p$, to find x. *Ans.* $x = \frac{\log p}{a \log m + b \log n}$.

13. Given $a^{2x} - 2a^x = 63$, to find x. *Ans.* $x = \frac{2 \log 3}{\log a}$.

14. Given $3^{2x} + 3^x = 6$, to find x. *Ans.* $x = 0.6308$.

15. Given $n^x + \frac{1}{n^x} = m$, to find x. *Ans.* $x = \frac{\log \frac{1}{2}(m \pm \sqrt{m^2 - 4})}{\log n}$.

16. Given $a^x + b^y = 2m$ and $a^x - b^y = 2n$, to find x and y.

Ans. $x = \frac{\log(m+n)}{\log a}$, $y = \frac{\log(m-n)}{\log b}$.

17. Given $x^y = y^x$, and $x^2 = y^3$, to find x and y.

Ans. $x = 3\frac{3}{8}$, $y = 2\frac{1}{4}$.

18. In a geometrical progression, given a, r and s, to find n.

Ans. See page 268.

19. In a geometrical progression, given l, r and s, to find n.

Ans. See page 268.

20. In compound interest, if P represents the principal, $R = 1 + r$, the rate, A the amount, and t the time, show that $A = P \times R^t = P(1+r)^t$.

21. From the above formula derive the following formulas:

1. $\log A = \log P + t \log(1+r)$; 3. $\log(1+r) = \frac{\log A - \log P}{t}$;

2. $\log P = \log A - t \log(1+r)$; 4. $t = \frac{\log A - \log P}{\log(1+r)}$.

NOTE.—Exponential equations of the form $x^x = a$ cannot be solved by elementary algebra. Numerical forms like $x^x = 10$ may be solved by Double Position.

SECTION XII.

PERMUTATIONS, COMBINATIONS, BINOMIAL THEOREM.

PERMUTATIONS.

442. Permutations are the different orders in which a number of things can be arranged.

Thus, the permutations of a and b are ab and ba; the permutations of a, b, and c, taken two at a time, are ab, ba, ac, ca, bc, cb.

443. Things may be arranged in sets of one, of two, of three, etc. Thus, the three letters a, b, and c may be arranged in sets as follows:

Of one,	a,	b,	c.
Of two,	ab, ac;	ba, bc;	ca, cb.
Of three,	abc, acb;	bac, bca;	cab, cba.

NOTES.—1. It is convenient to let P_2 represent the number of permutations when taken two together; P_3, the number when taken three together, etc.; P_r, the number when taken r together.

2. The term *permutations* is sometimes restricted to the case where the quantities are taken *all* together, while the term *arrangements* or *variations* is given to the grouping by twos, threes, etc., the number in the group being less than the whole number of things.

PROBLEMS.

444. *To find the number of permutations or arrangements that can be formed of* **n** *things taken two at a time, three at a time, etc.*