PROBLEMS

2. Mary's age is twice Sarah's, and the sum of their ages is a years; how old is each? *Ans.* Sarah, $\frac{a}{3}$; Mary, $\frac{2a}{3}$.

3. Find the age of each when $a=36$, by substituting the value of a in the result. *Ans.* 12 years; 24 years.

4. Divide the number m into two parts, such that the larger part will be 5 times the smaller. *Ans.* $\frac{m}{6}$; $\frac{5m}{6}$.

5. Find the value of each part when $m=144$, by substituting the value of m in the results. *Ans.* 24; 120.

6. The difference of two numbers is a, and 5 times the smaller equals the larger; what are the numbers? *Ans.* $\frac{a}{4}$; $\frac{5a}{4}$.

7. Find the value of each part when $a=24$, by substituting the value of a in the results. *Ans.* 6; 30.

8. What number is that to which if its one-third be added the sum will be b? *Ans.* $\frac{3b}{4}$.

9. Divide the number c into three such parts that the first shall be twice the second, and the second twice the third.

Ans. 1st, $\frac{4c}{7}$; 2d, $\frac{2c}{7}$; 3d, $\frac{c}{7}$.

10. If three times a number increased by n equals a, what is the number? *Ans.* $\frac{a-n}{3}$.

11. The sum of two numbers is a, and the smaller equals the larger diminished by c; what are the numbers?

Ans. $\frac{a+c}{2}$; $\frac{a-c}{2}$.

12. One-half the length of a pole is in the mud, one-third in the water, and h feet in the air; what is the length of the pole? *Ans.* $6h$.

NOTE.—Let the pupil give special values to the general quantities in each of the above problems, and find the results by Substitution.

ELEMENTARY ALGEBRA.

SECTION I.

DEFINITIONS AND EXPLANATIONS.

1. Mathematics is the science of quantity. It treats of the properties and relations of quantity.

2. Quantity is anything that can be measured. It is of two kinds, *Number* and *Extension.*

3. Arithmetic is the science of *Number; Geometry* is the science of *Extension.*

4. Algebra is a method of investigating quantity by means of general characters called *symbols.*

5. Algebraic Symbols are the characters used to represent quantities, their relations and the operations performed upon them.

6. The **Symbols of Algebra** are of three kinds, namely—

1. Symbols of Quantity; 2. Symbols of Operation;
3. Symbols of Relation.

NOTES.—1. With beginners we regard Algebra as restricted to *numbers,* or as a kind of *general Arithmetic.* They may afterward be led to see how general symbols introduce ideas not found in Arithmetic; and eventually, that Algebra is a general method of investigation that may be applied to all kinds of quantity.

2. Some writers divide Algebra into *Arithmetical* Algebra and *Symbolical* Algebra. Newton called it *Universal Arithmetic,* and many writers speak of it as *General Arithmetic.* D'Alembert divides Arithmetic into *Numérique,* Special Arithmetic, and *Algebra,* General Arithmetic.

SYMBOLS OF QUANTITY.

7. A **Symbol of Quantity** is a character used to represent a quantity.

8. The **Symbols of Quantity** generally used are the *figures* of arithmetic and the *letters* of the alphabet.

9. Known Quantities are represented by *figures* and the *first* letters of the alphabet, as 1, 2, 3, etc., and a, b, c, etc.

10. Unknown Quantities are usually represented by the *final* letters of the alphabet, as x, y, z, v, etc.

11. The **Symbol** 0, called *zero*, denotes the absence of quantity, or that which is less than any assignable quantity.

12. The **Symbol** ∞, called *infinity*, denotes that which is greater than any assignable quantity.

13. Accents are small marks used to denote different quantities which occupy similar positions in an operation; as a', a'', a''', etc. These are read *a prime*, *a second*, etc.

14. Subscript figures are sometimes used for the same purpose; as a_1, a_2, a_3, etc. These are read *a sub. one*, *a sub. two*, etc.

15. The **Sign of Continuation** is .... It denotes that the quantities are continued by the same law, and is read *and so on*. Thus, a, $2a$, $3a$ ...., means a, $2a$, $3a$, $4a$, $5a$, etc.

16. *Quantities* represented by *letters* are called *Literal Quantities.* Quantities represented by *figures* are called *Numerical Quantities.*

NOTE.—These symbols are the *representatives* of quantities, but for convenience we speak of them as quantities, meaning the quantities which they represent. Thus we say, the quantity a, the quantity b, and also a and b; as, add a to b; subtract a from b, etc.

SYMBOLS OF OPERATION.

17. A **Symbol of Operation** is a character used to indicate the operations of quantities.

18. The **Sign of Addition** is +, called *plus*. Thus, $a+b$ indicates the addition of a and b, and is read *a plus b*.

19. The **Sign of Subtraction** is −, called *minus*. Thus, $a-b$ denotes the subtraction of b from a, and is read *a minus b*.

20. The **Sign of Multiplication** is ×, read *into*, *times* or *multiplied by*. Thus, $a\times b$ denotes that a is to be multiplied by b, and is read *a into b*, or *a times b*, or *a multiplied by b*.

Multiplication is also denoted by a simple point; thus, $a \,.\, b$ denotes the same as $a\times b$. With letters the sign is usually omitted; thus, $2ab$ denotes the same as $2\times a\times b$.

The *Coefficient* of a quantity is a number written before it to show how many times the quantity is taken. Thus, in $3ab$ the 3 is the coefficient, and shows that ab is taken 3 times; in ax, the a is the coefficient of x, showing that x is taken a times. When the coefficient is expressed by a figure, it is called a *numerical* coefficient; when it is expressed by a letter, it is called a *literal* coefficient.

21. The **Sign of Division** is ÷, read *divided by*. Thus, $a\div b$ denotes that a is to be divided by b, and is read *a divided by b*.

Division is also indicated by writing the dividend above and the divisor below a short horizontal line, as in a fraction, as $\frac{a}{b}$. The expressions $a\lfloor b$ and $a(b$ also denote the division of a by b.

22. The **Sign of Involution**, called the *Exponent*, is a number written at the right of and above a quantity to indicate its power.

The *Power* of a quantity is the product obtained by using the quantity as a factor any number of times. Thus, $a\times a$ is the second power of a; $a\times a\times a$ is the third power of a, etc.

The *Exponent* of a quantity is the number which indicates how often the quantity is used as a factor. Thus, in a^3 the ^{3} indicates that a is used as a factor *three* times; a^3 is equivalent to $a\times a\times a$. a^2 is read "*a square*," or "*a second power*;" a^3 is read "*a cube*," or "*a third power*," or "*a third*;" a^n is read "*a nth power*," or "*a nth.*"

When the exponent is expressed by a *figure*, it is called a *numerical* exponent; when it is expressed by a *letter*, it is called a *literal* exponent. When no exponent is written, the exponent 1 is understood.

23. The **Sign of Evolution** is $\sqrt{\ }$, called the *Radical Sign*. It indicates that some root of the quantity before which it is placed is to be extracted. Thus, $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$ indicate, respectively, the *square root*, the *cube root* and the *fourth root* of a.

The *Index* of the root is the number written in the angle of the radical sign to indicate the required root. When no index is written, 2 is understood; thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

A *Fractional Exponent* is also used to indicate some root of a quantity. Thus, $a^{\frac{1}{2}}$ indicates the square root of a, $a^{\frac{1}{3}}$ the cube root of a, etc.

24. The **Signs of Aggregation** are the *Vinculum*, ——; the *Bar*, |; the *Parentheses*, ( ); the *Brackets*, [ ], and the *Braces*, { }. These indicate that the quantities connected or enclosed are to be subjected to the same operation. Thus, $\overline{a+b}\times c$; $\left.\begin{array}{r}+a\\+b\end{array}\right|c$; $(a+b)c$; $[a+b]c$; $\{a+b\}c$, each indicates that $a+b$ is to be multiplied by c.

SYMBOLS OF RELATION.

25. A **Symbol of Relation** is a character used to indicate the relation of quantities.

26. The **Sign of Equality** is =, read *equals* or *equal to*. Thus, $x=a$ indicates the *equality* of x and a, and is read *x is equal to a*, or *x equals a*.

27. The **Sign of Ratio** is :, read *to* or *is to*. Thus, $a:b$ indicates the *ratio* of a to b, and is read *the ratio of a to b*.

28. The **Sign of Equality of Ratios** is ::, read *equals* or *as*. Thus, $a:b::c:d$ indicates the *equality of the ratios* of $a:b$ and $c:d$, and is read *the ratio of a to b equals the ratio of c to d*, or *a is to b as c is to d*.

29. The **Signs of Inequality** are >, read *is greater than*, and <, read *is less than*. Thus, $a>b$ and $a<b$ indicate the *inequality* of a and b; $a>b$ is read *a is greater than b*, and $a<b$ is read *a is less than b*.

30. The **Signs of Deduction** are ∴, read *therefore* or *hence*, and ∵, read *since* or *because*.

NOTE.—The signs of Deduction are used when the *relation* is inferred from some previous *relation*. It is evident, therefore, that they may be classed with Symbols of Relation.

ALGEBRAIC EXPRESSIONS.

31. An **Algebraic Expression** is the expression of a quantity by means of algebraic symbols. Thus, $a+3b-c$.

32. The **Terms** of an algebraic expression are the parts connected by the signs + and −. Thus, in $a+3b-c$ the terms are a, $3b$ and $-c$.

33. A **Positive Term** is one having the plus sign prefixed to it, as $+3a$. When no sign is expressed the sign + is understood.

34. A **Negative Term** is one having the minus sign prefixed to it; as $-3a$. This sign should not be omitted.

35. **Similar** or **Like Terms** are those which contain the same letters affected by the same exponents; as, $3ab^2$ and $-5ab^2$.

36. **Dissimilar** or **Unlike Terms** are those which contain different letters or exponents; as, $3ab^2$ and $-5a^2b^3c$.

37. A **Monomial** is an algebraic expression consisting of *one* term; as, a, $4a$, $5a^3$, etc.

38. A **Polynomial** is an algebraic expression consisting of *two* or *more* terms; as $a+b$, $a+b+c+d+e$, etc.

39. A **Binomial** is a polynomial consisting of *two* terms; as, $a+b$ and $3a+4b^3$.

40. A **Trinomial** is a polynomial consisting of *three* terms; as, $a+2ab+c$.

41. The **Degree** of a term is determined by the number of literal factors it contains. Thus, $2a$ is of the *first* degree, $3a^2$ or $3ab$ of the *second* degree.

42. **Homogeneous Terms** are those which are of the same degree. Thus, $3abc$ and $5ab^2$ are homogeneous.

43. A **Polynomial** *is homogeneous* when all of its terms are of the *same degree*; as, $a^4-4a^3b+a^2b^2$.

ALGEBRAIC LANGUAGE.

44. Algebraic Language is a method of expressing mathematical ideas by means of algebraic symbols.

45. Numeration is the art of translating algebraic expressions into common language.

46. Notation is the art of expressing mathematical ideas in algebraic language.

EXERCISES IN NUMERATION.

1. Read $a+b$.
2. Read a^2+2ab.
3. Read $(a+b)c$.
4. Read $\sqrt{(a+b)}$.
5. Read $2(a+b^3)$.
6. Read $x^2+2xy+y^2$.
7. Read $(a+x)(a-x)$.
8. Read $\frac{a-b}{a+b}\times\frac{a-x}{a+x}$.
9. Read $\sqrt{a}+(x-z)^2$.
10. Read $4\sqrt[3]{a+\sqrt{b^2-\sqrt[4]{c}}}$.

EXERCISES IN NOTATION.

Express in algebraic language—

1. The sum of a and b. *Ans.* $a+b$.
2. Three times b subtracted from a. *Ans.* $a-3b$.
3. The sum of a and b, minus c. *Ans.* $a+b-c$.
4. The product of a and b, minus c squared. *Ans.* $ab-c^2$.
5. The sum of a and b, multiplied by c. *Ans.* $(a+b)c$.
6. The square of m, minus m into n. *Ans.* m^2-mn.
7. The sum of a and b, into the difference of a and b. *Ans.* $(a+b)(a-b)$.
8. The square of a, plus the square root of a. *Ans.* $a^2+\sqrt{a}$.
9. The square of the sum of a and b. *Ans.* $(a+b)^2$.
10. Four times a square into b, minus three times c square into x cube. *Ans.* $4a^2b-3c^2x^3$.
11. The square of a plus b, divided by a minus b, plus four times a into b square. *Ans.* $\frac{(a+b)^2}{a-b}+4ab^2$.
12. The sum of a times x, and the square of b, divided by a minus x. *Ans.* $\frac{ax+b^2}{a-x}$.
13. The sum of the squares of b and c, divided by the difference of three times a and twice c. *Ans.* $\frac{b^2+c^2}{3a-2c}$.
14. The cube of $a-x$, diminished by the square root of a plus x. *Ans.* $(a-x)^3-\sqrt{a+x}$.
15. The cube of a, minus x, diminished by the sum of a and the square root of x. *Ans.* $a^3-x-(a+\sqrt{x})$.
16. A trinomial with its second term negative, and twice the product of the other two terms.
17. A homogeneous trinomial of the fifth degree, with the second term negative.

NUMERICAL VALUES.

47. The **Numerical Value** of an algebraic expression is the result obtained by substituting for its letters definite numerical values, and then performing the operations indicated.

1. Find the numerical value of $(a^2-ab)c$ when $a=5$, $b=4$, and $c=3$.

SOLUTION. Substituting for a, b and c their assigned values, we have $(5^2-5\times4)\times3$; performing a part of the operations indicated, we have $(25-20)\times3$, which equals 5×3, or 15.

OPERATION.

$(a^2-ab)c=(5^2-5\times4)\times3$
$=(25-20)\times3=5\times3=15.$

EXAMPLES.

Find the numerical value of the following expressions when $a=6$, $b=5$, $c=4$, $m=3$, $n=2$:

2. a^2-ab. *Ans.* 6.
3. $(a^2-bc)n$. *Ans.* 32.
4. $ab+3a^2-5cn$. *Ans.* 98.

5. $(a+b)(a-b)$. *Ans.* 11.
6. $(a+b)m-(a-b)n$. *Ans.* 31.
7. $(a^2-b^2)(c-n)$. *Ans.* 22.
8. $\left(\frac{a+c}{a-n}\right)^2(b+m)$. *Ans.* 20.
9. $(m^2+n)\times(m^2-n)$. *Ans.* 77.
10. $\overline{m+c}\times\overline{a-c+n}$. *Ans.* 28.
11. $\overline{a+b+c}\div\overline{a-m}$. *Ans.* 5.
12. $\sqrt{(a+b)^2}-2n$. *Ans.* 7.
13. $\sqrt{a+2b}+3mn$. *Ans.* 22.
14. $(2a+2b+3)^{\frac{1}{2}}$. *Ans.* 5.
15. $\frac{ab+bc+mn}{a+2b-nc}$. *Ans.* 7.

POSITIVE AND NEGATIVE QUANTITIES.

48. A quantity with the *plus* sign prefixed is called an *Additive* or *Positive* quantity; a quantity with the *minus* sign prefixed is called a *Subtractive* or *Negative* quantity.

A *Positive Quantity* indicates *addition*, or that, when used, something is to be *increased* by it. A *Negative Quantity* indicates *subtraction*, or that, when used, something is to be *diminished* by it.

Positive and *Negative* quantities, being thus *opposite* in meaning, may be conveniently used to represent quantities reckoned in *opposite directions*.

Thus, if we use + to represent a person's *gains* in business, we may use – to represent his *losses; north* latitude may be represented by +, *south* latitude by –; *future* time by +, *past* time by –, etc.

The symbols + and – may therefore indicate the *nature* of the quantities to which they are prefixed, as well as the *operations* to be performed upon them.

49. The *Absolute Value* of a quantity is its value taken independently of the sign prefixed to it. Two quantities are evidently *equal* when they have the *same absolute value* and the *same sign*.

If I take any number, as 8, and *increase* it by 5, and then *diminish* it by 5, the value of 8 will remain unchanged; hence I may infer that *uniting* a positive and a negative quantity of the *same absolute* value gives *nothing* for the result.

If I unite 8 with +4, the result is 12; and if I unite 8 with –6, the result is 2; hence, since +4 united with 8 gives a *greater* result than –6 united with 8, I may infer that +4 is greater than –6, and in general that a *positive* quantity in Algebra may be regarded as greater than a *negative* quantity.

50. The above explanations may be formally stated in the following principles.

PRINCIPLES.

1. *A Positive quantity indicates that, when used, some quantity is to be* INCREASED *by it, and a Negative quantity that some quantity is to be* DIMINISHED *by it.*
2. *Positive and Negative quantities are sometimes used to indicate quantities reckoned in opposite directions.*
3. *A Positive and a Negative quantity of the same absolute value, united, amount to nothing.*
4. *In Algebra a Positive quantity is regarded as greater than a Negative quantity, whatever may be their absolute values.*

ALGEBRAIC REASONING.

51. All **Reasoning** is *comparison*. The reasoning in Algebra consists principally of the comparison of *equals*.

This **comparison** gives rise to the *equation*. The *equation* is therefore the fundamental idea in Algebra; it is the basis of all its investigations.

Comparison is controlled by certain laws called *axioms*, and gives rise to certain operations called *processes*.

52. Algebraic Reasoning is employed in the *solution* of *problems* and the *demonstration* of *theorems*.

53. A **Problem** is a question to be solved. A *solution* of a problem is the process of obtaining a required result.

54. A **Theorem** is a truth to be demonstrated. A *demon-*

stration of a theorem is a course of reasoning employed in establishing its truth.

55. An **Axiom** is a self-evident truth. *Axioms* are the laws which control the reasoning processes.

AXIOMS.

1. If equals be added to equals, the sums will be equal.
2. If equals be subtracted from equals, the remainders will be equal.
3. If equals be multiplied by equals, the products will be equal
4. If equals be divided by equals, the quotients will be equal.
5. If a quantity be both increased and diminished by another, the value of the former will not be changed.
6. If a quantity be both multiplied and divided by another, the value of the former will not be changed.
7. Quantities which are equal to the same quantity are equal to each other.
8. Like powers of equal quantities are equal. Like roots of equal quantities are equal.

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Mathematics. Quantity. Arithmetic. Geometry. Algebra. Symbols of Algebra. State the classes of symbols.

Define a Symbol of Quantity. Name Symbols of Quantity. Of Known Quantities. Of Unknown Quantities. Use of 0; of ∞. Of Accents. Of Subscript figures. The sign of continuation.

Define a Symbol of Operation. Explain the sign of Addition, Subtraction, etc. Define Coefficient. Power. Exponent. Index.

Define a Symbol of Relation. Explain the sign of Equality, etc.

Define an Algebraic Expression, the Terms, etc.

Define Algebraic language. Numeration. Notation. Numerical Value. State principles of positive and negative quantities. Define Reasoning. A Problem. A Theorem. An Axiom. Enunciate the Axioms.

SECTION II.

FUNDAMENTAL OPERATIONS.

ADDITION.

56. Addition is the process of finding the sum of two or more algebraic quantities.

57. The **Sum** of several algebraic quantities is a single quantity equal in value to the several quantities united.

NOTE.—The symbol + was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

CASE I.

58. To add when the terms are similar.

CLASS I. When the terms have the same sign.

1. Find the sum of $2a$, $3a$ and $5a$.

OPERATION.

$$\begin{array}{r} 2a \\ 3a \\ 5a \\ \hline 10a \end{array}$$

SOLUTION. $5a$, plus $3a$, are $8a$; and $8a$, plus $2a$, are $10a$. Hence the sum of $2a$, $3a$ and $5a$ is $10a$.

Rule.—*Add the coefficients, and prefix the sum with its proper sign to the common literal part.*

EXAMPLES.

(2.)	(3.)	(4.)	(5.)	(6.)
$3a$	$5x$	$-5ax$	$4a^2c$	$-9a^2b^3c$
$4a$	$3x$	$-7ax$	$5a^2c$	$-18a^2b^3c$
$5a$	$7x$	$-6ax$	$12a^2c$	$-a^2b^3c$
$7a$	$8x$	$-8ax$	$15a^2c$	$-6a^2b^3c$
$19a$	$23x$	$-26ax$	$36a^2c$	$-34a^2b^3c$

7. Find the sum of $4a$, $6a$ and $7a$. *Ans.* $17a$.
8. Find the sum of $-2a$, $-3a$ and $-5a$. *Ans.* $-10a$.
9. Find the sum of $3ab$, $5ab$, $6ab$ and $8ab$. *Ans.* $22ab$.
10. Find the sum of $-3ac$, $-4ac$, $-7ac$ and $-9ac$. *Ans.* $-23ac$.