9. Given $\frac{ab^x - c}{n} = d$, to find x. *Ans.* $x = \frac{\log(nd+c) - \log a}{\log b}$.

10. Given $ab^{\frac{1}{x}} = c$, to find x. *Ans.* $x = \frac{\log b}{\log c - \log a}$.

11. Given $a^{mx+n} = b$, to find x. *Ans.* $x = \frac{\log b - n \log a}{m \log a}$.

12. Given $m^{ax} n^{bx} = p$, to find x. *Ans.* $x = \frac{\log p}{a \log m + b \log n}$.

13. Given $a^{2x} - 2a^x = 63$, to find x. *Ans.* $x = \frac{2 \log 3}{\log a}$.

14. Given $3^{2x} + 3^x = 6$, to find x. *Ans.* $x = 0.6308$.

15. Given $n^x + \frac{1}{n^x} = m$, to find x. *Ans.* $x = \frac{\log \frac{1}{2}(m \pm \sqrt{m^2 - 4})}{\log n}$.

16. Given $a^x + b^y = 2m$ and $a^x - b^y = 2n$, to find x and y.

Ans. $x = \frac{\log(m+n)}{\log a}$, $y = \frac{\log(m-n)}{\log b}$.

17. Given $x^y = y^x$, and $x^2 = y^3$, to find x and y.

Ans. $x = 3\frac{3}{8}$, $y = 2\frac{1}{4}$.

18. In a geometrical progression, given a, r and s, to find n.

Ans. See page 268.

19. In a geometrical progression, given l, r and s, to find n.

Ans. See page 268.

20. In compound interest, if P represents the principal, $R = 1 + r$, the rate, A the amount, and t the time, show that $A = P \times R^t = P(1+r)^t$.

21. From the above formula derive the following formulas:

1. $\log A = \log P + t \log(1+r)$; 3. $\log(1+r) = \frac{\log A - \log P}{t}$;

2. $\log P = \log A - t \log(1+r)$; 4. $t = \frac{\log A - \log P}{\log(1+r)}$.

NOTE.—Exponential equations of the form $x^x = a$ cannot be solved by elementary algebra. Numerical forms like $x^x = 10$ may be solved by Double Position.

SECTION XII.

PERMUTATIONS, COMBINATIONS, BINOMIAL THEOREM.

PERMUTATIONS.

442. Permutations are the different orders in which a number of things can be arranged.

Thus, the permutations of a and b are ab and ba; the permutations of a, b, and c, taken two at a time, are ab, ba, ac, ca, bc, cb.

443. Things may be arranged in sets of one, of two, of three, etc. Thus, the three letters a, b, and c may be arranged in sets as follows:

Of one,	a,	b,	c.
Of two,	ab, ac;	ba, bc;	ca, cb.
Of three,	abc, acb;	bac, bca;	cab, cba.

NOTES.—1. It is convenient to let P_2 represent the number of permutations when taken two together; P_3, the number when taken three together, etc.; P_r, the number when taken r together.

2. The term *permutations* is sometimes restricted to the case where the quantities are taken *all* together, while the term *arrangements* or *variations* is given to the grouping by twos, threes, etc., the number in the group being less than the whole number of things.

PROBLEMS.

444. *To find the number of permutations or arrangements that can be formed of* **n** *things taken two at a time, three at a time, etc.*

Let $a, b, c, d, \ldots . k$ represent n things.

First, if we reserve one of the n letters, as a, to place before each of the others, there will remain $n-1$ letters; and placing a before each of the $n-1$ letters, there will be $n-1$, in which a stands first; as ab, ac, $ad, \ldots . ak$. Similarly, if we put b before each of the other letters, there will be $n-1$ arrangements in which b stands first. Similarly, there will be $n-1$ arrangements in which c stands first. Hence, since each of the n letters may be first, in all there will be $n(n-1)$ arrangements of n things taken *two* at a time.

Second, let us now find the number of arrangements of the n letters taken *three* at a time. If we reserve one letter, as a, there remain $n-1$ letters; the number of permutations of $n-1$ letters taken two together, from what has been shown, is $(n-1)(n-2)$. Putting a before each of these, there will be $(n-1)(n-2)$ permutations of n letters taken *three* together, in which a stands first. Similarly, there are $(n-1)(n-2)$ permutations, in which b stands first, and so for each of the n letters; hence the whole number of permutations of n letters taken *three* together is $n(n-1)(n-2)$.

Similarly, we find that the number of permutations of n letters taken *four* at a time is $n(n-1)(n-2)(n-3)$. From these cases it may be inferred, by analogy, that the number of permutations of n things taken r at a time is

$$n(n-1)(n-2)(n-3) \ldots . (n-\overline{r-1}).$$

445. We shall now prove that the formula for the permutation of n things taken r at a time, derived above by analogy, is true.

Suppose it to be true that the number of permutations of n letters taken $r-1$ at a time is

$$n(n-1) \ldots . \{n-(\overline{r-1}-1)\},$$

or

$$n(n-1) \ldots . (n-\overline{r-1}+1),$$

then we can show that a similar formula will give the number of permutations of the letters taken r at a time. For, out of the $n-1$ letters $b, c, d, \ldots .$ we can form (Art. 444),

$$(n-1)(n-2) \ldots . (n-1-\overline{r-1-1}),$$

or,

$$(n-1)(n-2) \ldots . (n-\overline{r-1}),$$

permutations each of $r-1$ letters, in which a stands first. Similarly, we

have the same number when b stands first, and as many when c stands first, and so on. Hence, on the whole, there are

$$n(n-1)(n-2) \ldots . (n-\overline{r-1})$$

permutations of n letters taken r at a time.

This proves that if the formula holds when the letters are taken $r-1$ at a time, it will hold when they are taken r at a time. But it has been shown to hold when they are taken *three* at a time; hence it holds when they are taken *four* at a time; hence also it must hold when they are taken *five* at a time, and so on; therefore the formula holds universally.

NOTE.—The method of reasoning employed in Art. 445 is called *Mathematical Induction.* It is based on the principle that *a truth is universal if when it is true in* n *cases it is true in* n + 1 *cases.* It is regarded as a valid method of demonstration, while the method by analogy or pure induction is not.

446. *To find the number of permutations of* n *things taken all together.*

In the formula just proved put $r=n$, and we have

$$P_n=n(n-1)(n-2) \ldots . 1.$$

447. For the sake of brevity $n(n-1)(n-2) \ldots . 1$ is often written $\lfloor n$, which is read "*factorial n.*"

Thus, $\lfloor n$ denotes $1 \times 2 \times 3 \times \ldots . \times n$; that is, the product of the natural numbers from 1 to n inclusive.

448. *Any number of* r *things, or combination of* r *things, will produce* $\lfloor r$ *permutations.*

For by Article 446, the r things which make the given combination can be arranged in $\lfloor r$ different ways.

449. *To find the number of permutations of* n *things taken all together when some occur more than once.*

Let there be n letters, and suppose a occurs p times, b occurs q times, and c occurs r times, the rest, d, e, f, etc., occurring but once. Let N represent the required number of permutations.

Now, if in any of the permutations we suppose the p letters a to be changed into p new letters, different from the rest, then without changing the situation of any of the remaining letters we could, from their interchange with one another, produce $\underline{|p}$ permutations; hence if the p letters a were changed into p different letters, the whole number of permutations would be $N\times\underline{|p}$. Similarly, if the q letters b were also changed into q new letters, different from any of the rest, the whole number of permutations would be $N\times\underline{|p}\times\underline{|q}$. So also if the r letters c were also changed, the whole number of permutations would be $N\times\underline{|p}\times\underline{|q}\times\underline{|r}$. But this number would be equal to the number of permutations of n different things taken all together; that is, to $\underline{|n}$.

Thus, $$N\times\underline{|p}\times\underline{|q}\times\underline{|r}=\underline{|n}.$$

Hence $$N=\frac{\underline{|n}}{\underline{|p}\times\underline{|q}\times\underline{|r}}.$$

450. *To find the number of permutations of* n *things when each may occur once, twice, thrice, ... up to* r *times.*

Let there be n letters, a, b, c, ... Taking them one at a time, we shall have n arrangements. Taking them two at a time, a may stand before a, or before any one of the remaining letters; similarly, b may stand before b, or before any one of the remaining letters, and so on; thus there are $n\times n$, or n^2, different arrangements. Taking them three at a time, each one of the n letters may be combined with the n^2 arrangements, making $n\times n^2$, or n^3 in all. Similarly, when the letters are taken r at a time, the whole number of permutations will be n^r.

451. If they are taken n at a time, or all together, r becomes n, and the number of permutations becomes n^n.

EXAMPLES.

1. How many permutations can be formed of the letters in the word *chair*, taken three together?

SOLUTION. Here $n=5$, and $r=3$; hence substituting in the formula $P_2=n(n-1)$ . . . . $(n-\overline{r-1}))$, we have $5\times4\times3=60$.

2. In how many ways may the letters of the word *home* be written? *Ans.* 24.

3. In how many ways can 5 persons arrange themselves at table so as not to sit twice in the same order? *Ans.* 120.

4. In how many different ways, taken all together, can the 7 prismatic colors be arranged? *Ans.* 5040.

5. The number of permutations of a set of things taken *four* together is twice as great as the number taken *three* together; how many things in the set? *Ans.* 5.

6. In how many ways can 8 persons form a circle by joining hands? *Ans.* 5040.

7. How many permutations can be made of the letters of the word *Caraccas*, taken all together? *Ans.* 1120.

8. How many permutations can be made of the letters of the word *Mississippi*, taken all together? *Ans.* 34650.

9. The number of permutations of n things taken four together is six times the number taken three together; find the value of n. *Ans.* $n=9$.

10. The number of arrangements of 15 things, taken r together, is ten times the number taken $(r-1)$ together; find the value of r. *Ans.* $r=6$.

11. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice? *Ans.* $\dfrac{\underline{|10}}{\underline{|2}\ \underline{|3}\ \underline{|5}}$.

12. In how many different ways can six letters be arranged when taken singly, two at a time, three at a time, and so on, until they are taken all at a time? *Ans.* 1956.

NOTE.—Find the sum of the different permutations.

COMBINATIONS.

452. The **Combinations** of a set of things are the different collections that can be formed out of them without regarding the *order* in which they are placed.

Thus, the combinations of the letters a, b, c, taken *two* together, are *ab*, *ac*, *bc*; *ab* and *ba*, though different permutations, form the same combination.

453. Each *combination* of things when taken two together, as *ab*, gives *two permutations*; and when taken *three* together, as *abc*, gives $3\times2\times1$ permutations.

454. *In general, each combination or collection of* **r** *things gives* $|\underline{r}$ *permutations.*

Thus the combination *abc* gives the permutations *abc*, *acb*, etc.; that is, the permutations of three things taken all together, which by Art. 446 is $|\underline{3}$, or $3\times2\times1$. Similarly, it is seen that the combination of r things gives $|\underline{r}$ permutations.

455. *To find the number of combinations that can be formed out of* **n** *things taken* **r** *at a time.*

The number of *permutations* of n things taken r at a time is $n(n-1)\ldots\ldots n-\overline{r-1})$ (Art. 444), and each collection or combination of r things produces $|\underline{r}$ permutations (Art. 454); hence the number of combinations of n things taken r at a time equals the number of permutations divided by $|\underline{r}$; or, letting C_r represent the number of combinations, we have

$$C_r=\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-\overline{r-1})}{|\underline{r}}.$$

456. If we multiply both numerator and denominator of the previous expression by $|\underline{n-r}$, we have

$$C_r=\frac{|\underline{n}}{|\underline{r}\ |\underline{n-r}}.$$

457. *The number of combinations of* **n** *things taken* **r** *at a time is the same as the number of them taken* **n − r** *at a time.*

For the number of combinations of n things taken $n-r$ at a time is

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-\overline{n-r-1})}{|\underline{n-r}};$$

that is,

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (r-1)}{|\underline{n-r}}.$$

Multiply both numerator and denominator by $|\underline{r}$, and we obtain $C_{n-r}=\dfrac{|\underline{n}}{|\underline{r}\ |\underline{n-r}}$, which, by Art. 456, is the number of combinations of n things taken r at a time.

458. Hence in finding the number of combinations taken r together, when $r>\frac{1}{2}n$, the shortest way is to find the number taken $(n-r)$ together.

459. *To find the value of* **r** *from which the number of combinations of* **n** *things taken* **r** *at a time is greatest.*

The formula $C_r=\dfrac{n(n-1)(n-2)\ \ldots\ldots\ (n-\overline{r-1})}{1\times2\times3\times\ \ldots\ldots\ r}$ may be written

$$C_r=\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\ \ldots\ldots\ \frac{n-\overline{r-1}}{r}.$$

Now, it is seen that the numerators of this formula decrease from right to left by unity, and the denominators increase by unity; hence at some point in this series the factors become *less* than 1; therefore the value of C_r is greatest when the product includes *all* the factors *greater* than 1.

Now, when n is an *odd* number, the numerator and denominator of each factor will be alternately both odd and both even; so that the factor greater than 1, but nearest to 1, will be that factor whose numerator exceeds the denominator by 2. Hence, in this case the value of r must be such that

$$n-\overline{r-1}=r+2,\ \text{or}\ r=\frac{n-1}{2}.$$

When n is *even*, the numerator of the first factor will be even and the denominator odd; the numerator of the second factor will be odd and the denominator even, and so on alternately; hence the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 1. Hence, in this case the value of r must be such that

$$n-\overline{r-1}=r+1,\ \text{or}\ r=\frac{n}{2}.$$

NOTE.—For other principles of Permutations and Combinations see works on Higher Algebra.

EXAMPLES.

1. How many combinations can be formed from the letters of the word *Prague*, taken three together?

SOLUTION. Here $n=6$ and $r=3$; hence substituting in the formula, we have $C_3=\dfrac{n(n-1)\ \ldots\ldots\ n(n-\overline{r-1})}{|\underline{r}}=\dfrac{6\times5\times4}{1\times2\times3}=20.$

2. How many combinations may be made of the letters a, b, c, d, e, f, taken three together? four together? *Ans* 20; 15.

3. How many spans of horses can be selected from 20 horses? How many double spans? *Ans.* 190; 4845.

4. How many combinations of 3 or of 5 letters can be made out of 8 letters? *Ans.* 56.

5. How many combinations can be made of the letters of the word *longitude*, taken four at a time? *Ans.* 126.

6. How many combinations may be formed of 16 things taken 5 at a time? *Ans.* 4368.

7. How many different parties of 6 men can be formed out of a company of 20 men? *Ans.* 38760.

8. A guard of 5 soldiers is to be formed by lot out of 20 soldiers; in how many ways can this be done? how often will any one soldier be on guard? *Ans.* 15504; 3876.

9. In how many different ways can a class of 6 boys be placed in a line, one being denied the privilege of the head of the class? *Ans.* 600.

10. The number of the combinations of n things taken four together is to the number taken two together as 15 : 2; find the value of n. *Ans.* $n = 12$.

11. The number of permutations of n things taken 5 at a time is equal to 120 times the number of combinations taken 3 at a time; find n. *Ans.* $n = 8$.

12. A and B have each the same number of horses, and find that A can make twice as many different teams by taking 3 horses together as B can by taking 2 horses together; how many horses has each? *Ans.* 8.

THE BINOMIAL THEOREM.

460. The **Binomial Theorem** is derived on page 162 by induction; we shall now demonstrate it by a more rigid method of reasoning called *mathematical induction.*

461. By actual multiplication we obtain

(1) $(x+a)(x+b) = x^2 + (a+b)x + ab.$

(2) $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$

(3) $(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3.$
$\quad + (ab+ac+ad+bc+bd+cd)x^2.$
$\quad + (abc+bcd+cda+dab)x + abcd.$

462. Examining these results, we observe certain *laws* in the development:

1. *The number of terms is one more than the number of binomial factors involved.*

2. *The exponent of x in the first term is the same as the number of binomial factors, and decreases by unity in each succeeding term.*

3. *The coefficient of x in the first term is unity.*

The coefficient of x in the second term is the sum of the second letters, a, b, c, of the binomial factors.

The coefficient of x in the third term is the sum of the products of the second letters taken two at a time.

The coefficient of the fourth term is the sum of the products of the second letters taken three at a time, and so on.

4. *The last term is the product of all the second letters of the binomial factors.*

463. We shall now show that these laws always hold, whatever be the number of the binomial factors.

Suppose the laws to hold for $n-1$ factors, so that

$$(x+a)(x+b) \ldots (x+k) = x^{n-1} + px^{n-2} + qx^{n-3} + rx^{n-4} + \ldots u,$$

where p = the sum of the letters a, b, c, ... k.
q = the sum of the products of these letters taken two at a time.
r = the sum of the products of these letters taken three at a time.
. . . .
u = the product of all these letters.

Then multiply by another factor, $x+l$, and arrange the product according to the powers of x; thus,

$$(x+a)(x+b)(x+c) \ldots (x+k)(x+l) = x^n + (p+l)x^{n-1}$$
$$+ (q+pl)x^{n-2} + (r+ql)x^{n-3} + \ldots + ul.$$

The laws (1) and (2) evidently hold in this expression.

Now, $p+l \ = a+b+c+ \ldots . k+l$

$=$ the sum of the letters $a, b, c, \ldots . k, l$.

Also, $q+pl = q+l(a+b+c+ \ldots . +k)$

$=$ the sum of the products of all the letters $a, b, c, \ldots . k, l$, taken two at a time.

Also, $r+ql = r+l(ab+ac+bc+ \ldots . .)$

$=$ the sum of the products of all the letters $a, b, c, \ldots . k, l$, taken three at a time.

Also, $ul =$ the product of all the letters.

Hence, if the laws hold for $n-1$ factors, they hold for n factors. But it has been shown that they hold for *four* factors; therefore they hold for *five* factors, and therefore for *six* factors, and so on. Therefore they hold universally.

464. We shall now proceed to find the general formula for the expansion of $(x+a)^n$.

Let P, Q, R etc. represent the coefficients in the above formula, and we have

$$(x+a)(x+b) \ldots . (x+k)(x+l) = x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \ldots . + V.$$

Here $P =$ the sum of the letters $a, b, c, \ldots . k, l$, which are n in number.

$Q =$ the sum of the products of these letters taken two at a time, so that there are $\frac{n(n-1)}{1 . 2}$ of these products, ab, ac, etc.

$R =$ the sum of the products of these letters taken three at a time, so that there are $\frac{n(n-1)(n-2)}{1 . 2 . 3}$ of these products, abc, abd, etc., and so on.

$V = abc \ldots . kl$.

Now, suppose $b, c \ldots . k, l$ are each equal to a; then,

First, $(x+a)(x+b) \ldots . (x+k)(x+l)$ equals $(x+a)^n$;

Second, P will equal a taken n times, or na;

Third, ab, ac, etc. will each become a^2, and Q will equal a^2 taken $\frac{n(n-1)}{1 . 2}$ times, or $\frac{n(n-1)}{1 . 2}a^2$.

Fourth, abc, abd, etc. will each become a^3, and R will equal a^3 taken $\frac{n(n-1)(n-2)}{1 . 2 . 3}$ times, or $\frac{n(n-1)(n-2)}{1 . 2 . 3}a^3$; and so on.

Finally, $abc \ldots . kl$ will equal a taken as a factor n times, or a^n, and V will equal a^n. Therefore,

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 . 2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1 . 2 . 3}a^3x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}a^4x^{n-4} + \ldots . + a^n.$$

465. If in the formula a and x be interchanged, the development will proceed by ascending powers of x; thus,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 . 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}a^{n-4}x^4 + \ldots . + x^n.$$

466. From the examination of this formula several laws will be observed, as follows:

1. *The sum of the exponents of a and x in each term equals n.*

2. *If x is negative, every odd power of x will be negative and the even powers positive; thus,*

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1 . 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-3}x^3 + \text{etc.}$$

467. The coefficient of the rth term of the development of $(a+x)^n$ is

$$\frac{n(n-1)(n-2) \ldots . (n-r+2)}{1 . 2 . 3 \ldots . r-1}.$$

For, the coefficient of the second term is n, which is the combination of n things taken singly; the coefficient of the third term is $\frac{n(n-1)}{1 . 2}$, which is the combination of n things taken two at a time; and generally, the coefficient of the rth term is the number of combinations of n things taken $r-1$ at a time, which, by Art. 455, is equal to

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{1\cdot 2\cdot 3\ \ldots\ r-1}.$$

468. This is called the *general coefficient;* and by making $r=2, 3, 4$, etc., all the others can be derived from it.

Thus, suppose we wish to find the 5th term of $(a-x)^7$. Here $r=5$ and $n=7$; and the coefficient of the term required is $\frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}=35$, and the 5th term is $35a^3x^4$.

469. In the expansion of $(a+x)^n$ the *coefficients of terms equally distant from the beginning and end are the same.*

For the coefficient of the rth term from the beginning is

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{\lfloor r-1},$$

which, by multiplying both numerator and denominator by $\lfloor n-r+1$ becomes

$$\frac{\lfloor n}{\lfloor r-1\ \lfloor n-r+1}.$$

The rth term from the end is the $(n-r+2)$th term from the beginning, and its coefficient is

$$\frac{n(n-1)(\ \ldots\ )\{n-(n-r+2)+2\}}{\lfloor n-r+1},\ \text{or}\ \frac{n(n-1)\ \ldots\ r}{\lfloor n-r+1}.$$

Multiplying both terms by $\lfloor r-1$, this becomes also

$$\frac{\lfloor n}{\lfloor r-1\ \lfloor n-r+1}.$$

470. The expansion of a binomial can always be reduced to the case in which one of the two quantities is unity.

Thus $(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n=a^n(1+y)^n$, if $y=\frac{x}{a}$. We may then expand $(1+y)^n$, and multiply each term by a^n, and thus obtain the expansion of $(a+x)^n$.

471. The *sum of the coefficients* of the terms in the expansion of $(1+x)^n$ is 2^n.

For $(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\ \ldots\ nx^{n-1}+x^n$. Now, since this is true for all values of x, it is true when $x=1$; whence

$$(1+1)^n=2^n=1+n+\frac{n(n-1)}{1\cdot 2}+\ \ldots\ n+1;$$

that is, the sum of the coefficients is 2^n.

472. The *sum of the coefficients* of the *odd terms* in the expansion $(1+x)^n$ *is equal* to the *sum of the coefficients* of the *even terms.*

If we let $x=-1$, the expansion of $(1+x)^n$ becomes

$$0=1-n+\frac{n(n-1)}{1\cdot 2}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\text{etc.},$$

in which we have the sum of the *odd coefficients, minus* the sum of the *even coefficients* equal to zero; hence the two sums are equal.

473. Since the two sums are equal, each is one-half of 2^n (Art. 471), or $2^n\div 2=2^{n-1}$.

NOTES.—1. It may also be shown that in the expansion of $(a+x)^n$ the *middle term* will have the *greatest coefficient* when n is *even;* and the *two middle terms* will have *equal coefficients* when n is *odd*, and be the *greatest terms.*

2. This demonstration of the binomial theorem is restricted to n being a *positive integer.* The theorem is also true when n is *negative* or *fractional;* but the demonstration is too difficult for a work on elementary algebra. We shall assume that n is general, and give examples showing its application.

EXAMPLES.

1. Expand $(b+y)^{-2}$.

SOLUTION. In the general formula (Art. 465) substitute b for a, y for x, and -2 for n, and we have

$$(b+y)^{-2}=b^{-2}+-2b^{-2-1}y+\frac{-2(-2-1)}{1\cdot 2}b^{-2-2}y^2$$
$$+\frac{-2(-2-1)(-2-2)}{1\cdot 2\cdot 3}b^{-2-3}y^3+\text{etc.}$$

Reducing, we have $(b+y)^{-2}=\frac{1}{b^2}-\frac{2y}{b^3}+\frac{3y^2}{b^4}-\frac{4y^3}{b^5}+\text{etc.}$

2. Expand $(1+y)^{\frac{1}{2}}$.

SOLUTION. In the general formula put 1 for a, y for x, and $\frac{1}{2}$ for n; substitute and reduce, and we have

$$(1+y)^{\frac{1}{2}}=1+\tfrac{1}{2}y-\tfrac{1}{8}y^2+\tfrac{1}{16}y^3-\tfrac{5}{128}y^4+\text{etc.}$$

3. Expand $(1+2x-x^2)^{\frac{1}{2}}$.

SOLUTION. Put y for $2x-x^2$; then $(1+2x-x^2)^{\frac{1}{2}}=(1+y)^{\frac{1}{2}}$.

$$(1+y)^{\frac{1}{2}}=1+\tfrac{1}{2}y-\tfrac{1}{8}y^2+\tfrac{1}{16}y^3-\tfrac{5}{128}y^4+\text{etc.}$$
$$=1+\tfrac{1}{2}(2x-x^2)-\tfrac{1}{8}(2x-x^2)^2+\tfrac{1}{16}(2x-x^2)^3-\tfrac{5}{128}(2x-x^2)^4+\text{etc.}$$

Reducing, $(1+2x-x^2)^{\frac{1}{2}}=1+x-x^2+x^3-\frac{3}{2}x^4+\text{etc.}$

4. Expand $(1-x)^{-1}$. *Ans.* $1+x+x^2+x^3+x^4+x^5+\text{etc.}$

5. Expand $(1+a)^{-\frac{1}{2}}$. *Ans.* $1-\frac{1}{2}a+\frac{3}{8}a^2-\frac{5}{16}a^3+\frac{35}{128}a^4-\text{etc.}$

6. Expand $(1-x)^{\frac{1}{3}}$.

Ans. $1-\dfrac{x}{3}-\dfrac{2x^2}{3\,.\,6}-\dfrac{2\,.\,5x^3}{3\,.\,6\,.\,9}-\dfrac{2\,.\,5\,.\,8x^4}{3\,.\,6\,.\,9\,.\,12}-\text{etc.}$

7. Expand $\dfrac{a}{(1-x)^2}$. *Ans.* $a+2ax+3ax^2+4ax^3+5ax^4+\text{etc.}$

8. Expand $\sqrt{a^2-x^2}$. *Ans.* $a-\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}-\dfrac{x^6}{16a^5}-\dfrac{5x^8}{128a^7}-\text{etc.}$

9. Write the coefficient of x^r in $(1-x)^{-2}$, and coefficient of x^r in $(1-x)^{-4}$. *Ans.* $r+1$; $\dfrac{(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3}$.

10. Write 4th term of $(x+2y)^n$, and 6th term of $(3x-y)^{-\frac{3}{4}}$.

Ans. $\dfrac{4n(n-1)(n-2)}{3}x^{n-3}y^3$; $-\dfrac{3\,.\,7\,.\,11\,.\,15\,.\,19}{4^5\,\underline{|5}}(3x)^{-\frac{3}{4}-5}y^5$.

11. Write the $(r+1)$th term of $(1-x)^{-3}$, and the 5th term of $(3x^{\frac{1}{2}}-4y^{\frac{1}{2}})^9$. *Ans.* $\dfrac{(r+1)(r+2)}{2}x^r$; $\dfrac{9\,.\,8\,.\,7\,.\,6}{\underline{|4}}3^5x^{\frac{5}{2}}4^4y^2$.

12. Write the middle term of $(a+x)^{10}$, and two middle terms of $(a+x)^9$. *Ans.* $\dfrac{\underline{|10}}{\underline{|5}\;\underline{|5}}a^5x^5$; $\dfrac{\underline{|9}}{\underline{|4}\;\underline{|5}}(a^5x^4+a^4x^5)$.

13. $(1+x-x^2)^4=1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.

14. $(1+x+x^2)^{\frac{1}{2}}=1+\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{3x^3}{16}$, etc.

15. If the 6th, 7th, and 8th terms of $(x+y)^n$ are respectively 112, 7, and $\frac{1}{4}$, find x, y, and n. *Ans.* $x=4$, $y=\frac{1}{2}$, $n=8$.