stration of a theorem is a course of reasoning employed in establishing its truth.

55. An **Axiom** is a self-evident truth. *Axioms* are the laws which control the reasoning processes.

AXIOMS.

1. If equals be added to equals, the sums will be equal.
2. If equals be subtracted from equals, the remainders will be equal.
3. If equals be multiplied by equals, the products will be equal
4. If equals be divided by equals, the quotients will be equal.
5. If a quantity be both increased and diminished by another, the value of the former will not be changed.
6. If a quantity be both multiplied and divided by another, the value of the former will not be changed.
7. Quantities which are equal to the same quantity are equal to each other.
8. Like powers of equal quantities are equal. Like roots of equal quantities are equal.

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Mathematics. Quantity. Arithmetic. Geometry. Algebra. Symbols of Algebra. State the classes of symbols.

Define a Symbol of Quantity. Name Symbols of Quantity. Of Known Quantities. Of Unknown Quantities. Use of 0; of ∞. Of Accents. Of Subscript figures. The sign of continuation.

Define a Symbol of Operation. Explain the sign of Addition, Subtraction, etc. Define Coefficient. Power. Exponent. Index.

Define a Symbol of Relation. Explain the sign of Equality, etc.

Define an Algebraic Expression, the Terms, etc.

Define Algebraic language. Numeration. Notation. Numerical Value. State principles of positive and negative quantities. Define Reasoning. A Problem. A Theorem. An Axiom. Enunciate the Axioms.

SECTION II.

FUNDAMENTAL OPERATIONS.

ADDITION.

56. **Addition** is the process of finding the sum of two or more algebraic quantities.

57. The **Sum** of several algebraic quantities is a single quantity equal in value to the several quantities united.

NOTE.—The symbol + was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

CASE I.

58. To add when the terms are similar.

CLASS I. When the terms have the same sign.

1. Find the sum of $2a$, $3a$ and $5a$.

OPERATION.

$$\begin{array}{r} 2a \\ 3a \\ 5a \\ \hline 10a \end{array}$$

SOLUTION. $5a$, plus $3a$, are $8a$; and $8a$, plus $2a$, are $10a$. Hence the sum of $2a$, $3a$ and $5a$ is $10a$.

Rule.—*Add the coefficients, and prefix the sum with its proper sign to the common literal part.*

EXAMPLES.

(2.)	(3.)	(4.)	(5.)	(6.)
$3a$	$5x$	$-5ax$	$4a^2c$	$-9a^2b^3c$
$4a$	$3x$	$-7ax$	$5a^2c$	$-18a^2b^3c$
$5a$	$7x$	$-6ax$	$12a^2c$	$-a^2b^3c$
$7a$	$8x$	$-8ax$	$15a^2c$	$-6a^2b^3c$
$19a$	$23x$	$-26ax$	$36a^2c$	$-34a^2b^3c$

7. Find the sum of $4a$, $6a$ and $7a$. *Ans.* $17a$.
8. Find the sum of $-2a$, $-3a$ and $-5a$. *Ans.* $-10a$.
9. Find the sum of $3ab$, $5ab$, $6ab$ and $8ab$. *Ans.* $22ab$.
10. Find the sum of $-3ac$, $-4ac$, $-7ac$ and $-9ac$. *Ans.* $-23ac$.

11. Find the sum of $5a^2b^3$, $7a^2b^3$, $9a^2b^3$ and $10a^2b^3$. *Ans.* $31a^2b^3$.

12. Find the sum of $6x^3y^5$, $7x^3y^5$, $9x^3y^5$ and $12x^3y^5$. *Ans.* $34x^3y^5$.

13. Find the sum of $-5abc$, $-4abc$, $-7abc$ and $-9abc$. *Ans.* $-25abc$.

59. CLASS II. When the terms have different signs.

1. Find the sum of $7a$ and $-4a$.

SOLUTION. $7a$ is equal to $3a+4a$. Now, $-4a$ united with $+4a$, a part of $7a$ is equal to nothing, Prin. 3, Art. 50; therefore $-4a$ added to $3a+4a$ equals $3a$. Hence, $-4a$ added to $7a$ equals $3a$.

OPERATION
$7a$
$-4a$
$3a$

SOLUTION 2D. Plus $7a$ may indicate some quantity *increased* by $7a$, and $-4a$ may indicate some quantity *diminished* by $4a$. A quantity *increased* by $7a$ and then *diminished* by $4a$ is evidently *increased* by $3a$; hence the sum of $7a$ and $-4a$ is *plus* $3a$.

2. Find the sum of $-8a$, $4a$, $-7a$ and $9a$.

SOLUTION. The sum of the positive quantities, $9a$ and $4a$, is $13a$; and the sum of the negative quantities, $-7a$ and $-8a$, is $-15a$. Now, $-15a=-13a-2a$; $+13a$ united to $-13a$ is equal to nothing, and there remains $-2a$. Hence the sum is $-2a$.

OPERATION.
$-8a$
$4a$
$-7a$
$9a$
$-2a$

SOLUTION 2D. The latter part may be given thus: Any quantity *increased* by $13a$ and then *diminished* by $15a$ is evidently *diminished* by $2a$; hence the sum is *minus* $2a$.

Rule.—I. *Find the sum of the coefficients of the positive and negative terms separately.*

II. *Take the difference of these sums, and prefix it, with the sign of the greater, to the common literal part.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)	(7.)
$+7ax$	$-5a^2c^3$	$-7z^2$	$+27xy$	$-2x^2y^3z$
$-9ax$	$+3a^2c^3$	$-z^2$	$-34xy$	$-12x^2y^3z$
$+8ax$	$-9a^2c^3$	$+9z^2$	$-150xy$	$+x^2y^3z$
$-3ax$	$+4a^2c^3$	$-5z^2$	$+27xy$	$+28x^2y^3z$
$+3ax$	$-7a^2c^3$	$-4z^2$		

Find the sum—

8. Of $6ab$, $-5ab+8ab$ and $-3ab$. *Ans.* $6ab$.

9. Of $3cd$, $-6cd$, $-7cd$, $+8cd$ and $-4cd$. *Ans.* $-6cd$.

10. Of $7xy$, $+8xy$, $-9xy$, $+3xy$ and $-4xy$. *Ans.* $5xy$.

11. Of $5an$, $+7an$, $-12an$, $+15an$, $-19an$. *Ans.* $-4an$.

12. Of $7a^2b$, $-9a^2b$, $+10a^2b$, $+12a^2b$, $-30a^2b$. *Ans.* $-10a^2b$.

13. Of $12a^2c^3$, $-6a^2c^3$, $+a^2c^3$, $-15a^2c^3$, $+7a^2c^3$. *Ans.* $-a^2c^3$.

14. Of $15xy^2z$, $-19xy^2z$, $+12xy^2z$, $-10xy^2z$, $-15xy^2z$ and $+22xy^2z$. *Ans.* $5xy^2z$.

15. Of $5ac^3b^5$, $+6ac^3b^5$, $-7ac^3b^5$, $+8ac^3b^5$, $-17ac^3b^5$, $-4ac^3b^5$, $+5ac^3b^5$. *Ans.* $-4ac^3b^5$.

16. Of $21am^2nx^3$, $-19am^2nx^3$, $-21am^2nx^3$, $+25am^2nx^3$, $+19am^2nx^3$, $-25am^2nx^3$. *Ans.* 0.

CASE II.

60. To add when the terms are dissimilar.

1. Find the sum of $3a$, $4b$ and $-ab$.

SOLUTION. Since the quantities are dissimilar, we cannot unite them into one sum by adding their coefficients; we therefore indicate the addition by writing them one after another with their respective signs. We thus have $3a+4b-ab$.

OPERATION.
$3a$
$4b$
$-ab$
$3a+4b-ab$

2. Find the sum of $2a+3ab$, $3a-4ab+5b$ and $5ab-7b$.

SOLUTION. We write the similar terms in the same column for convenience in adding, and begin at the left to add: $3a$ and $2a$ are $5a$, which we write under the column added; $5ab$, $-4ab$, $+3ab$ are $+4ab$, which we write under the column added; $-7b$, $+5b$ equals $-2b$, which we write under the column added. Hence the sum is $5a+4ab-2b$.

OPERATION.
$2a+3ab$
$3a-4ab+5b$
$5ab-7b$
$5a+4ab-2b$

Rule.—I. *Write similar terms, with their proper signs, in the same column.*

II. *Add each column separately, and connect the results with their proper signs.*

EXAMPLES.

(3.)	(4.)	(5.)
$2a+3b$	$3x-5xy$	$12ab^2+28cx^3$
$5a-7b$	$7x+8xy$	$-ab^2+25cx^3$
$a+9b$	$a-9x-6xy$	$24ab^2-23cx^3$
$3a-8b$	$4a-5x+7xy$	$-35ab^2-17cx^3$
$11a-3b$	$5a-4x+4xy$	$+13cx^3$

6. Find the sum of $3ac-5ax$, $7ac+6ax$, $5ac-12ax$ and $9ac+15ax$. *Ans.* $24ac+4ax$.

7. Find the sum of $5ab+12bc-7cd$, $9ab-18bc+11cd$ and $17ab-15bc+13cd$. *Ans.* $31ab-21bc+17cd$.

8. Find the sum of $3ax-2b^2c$, $5ax+7c^3$, $9b^2c-12c^3$, $8ax+15c^3$ and $14b^2c-18c^3$. *Ans.* $16ax+21b^2c-8c^3$.

9. Find the sum of $m+3n^2-5mn$, $3m-8n^2$, $7n^2-8mn$, $19m+27mn$ and $16n^2-17mn$. *Ans.* $23m-3mn+18n^2$.

10. Find the sum of $a+2b+3c$, $2a-b-2c$, $b-a-c$ and $c-a-b$. *Ans.* $a+b+c$.

11. Find the sum of $3a-4p+q$, $7p+3q-6$, $9a-7+3p$ and $9q-12+11p$. *Ans.* $12a+17p+13q-25$.

12. Find the sum of $a+b-c$, $a-b+c$, $a+c+b$ and $b-a+c$. *Ans.* $2a+2b+2c$.

13. Find the sum of $4a+7a^2c-8m^3$, $7a+16m^3$, $15a^2c-20m^3+17$ and $12m^3-5-22a^2c$. *Ans.* $11a+12$.

14. Add $34ax^3-16ay^2$, $-25ax^3-13ay^3+14ay^2$, $16+15ay^3$, $15ay^2-16$ and $22ax^3+7ay^2-11ay^3$. *Ans.* $31ax^3+20ay^2-9ay^3$.

15. Add $12x+9y-6z$, $5a-12y+13x$, $7y-16x+10z$ and $10x-5a+12z$. *Ans.* $19x+4y+16z$.

16. Add x^n-ax^2+3b, $3ax^2-2b+y^{2n}$, $5x^n+4b-3y^{2n}$, $7b-4x^n+y^{2n}-3ax^2$. *Ans.* $2x^n-ax^2-y^{2n}+12b$.

17. Add $5a-9b+5c+3-d$, $a-3b-8-d$, $3a+2b-3c+4+5d$, $2a+5c-6-3d$. *Ans.* $11a-10b+7c-7$.

18. Add $x^4-4x^3y+6x^2y^2-4xy^3+y^4$, $4x^3y-12x^2y^2+12xy^3-4y^4$, $6x^2y^2-12xy^3+6y^4$ and $4xy^3-4y^4$. *Ans.* x^4-y^4.

19. Add $a^3+ab^2+ac^2-a^2b-abc-a^2c$, $a^2b+b^3+bc^2-ab^2-b^2c-abc$ and $a^2c+b^2c+c^3-abc-bc^2-ac^2$. *Ans.* $a^3+b^3+c^3-3abc$.

20. Add $4ab-3mn+10am-6an$, $7mn-7am+4an$, $3ab+7an+9$, $6-4mn-3am-5n^2$ and $4n^2-15-2m^2$.

Ans. $7ab-2m^2-n^2+5an$.

FACTORED FORMS.

61. Similar quantities in any form may be added by taking the algebraic sum of their coefficients.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$6\sqrt{7}$	$8(a-b)$	$-12\sqrt{a+b}$	$5(m-n+2)$
$4\sqrt{7}$	$-5(a-b)$	$15\sqrt{a+b}$	$7(m-n+2)$
$5\sqrt{7}$	$6(a-b)$	$-18\sqrt{a+b}$	$-9(m-n+2)$
$15\sqrt{7}$	$9(a-b)$	$-15\sqrt{a+b}$	$3(m-n+2)$

5. What is the sum of $5(x-y)$, $-12(x-y)$, $3(x-y)$, $10(x-y)$ and $-14(x-y)$? *Ans.* $-8(x-y)$.

6. What is the sum of $7(a-b)^3$, $-9(a-b)^3$, $+12(a-b)^3$, $+16(a-b)^3$, $-18(a-b)^3$? *Ans.* $8(a-b)^3$.

7. Find the sum of $3\sqrt{a+x}$, $5\sqrt{a+x}$, $-7\sqrt{a+x}$, $+8\sqrt{a+x}$, $-5\sqrt{a+x}$ and $12\sqrt{a+x}$. *Ans.* $16\sqrt{a+x}$.

8. Add $4ax+7(a^2-b^2)$, $6ax-5(a^2-b^2)$, $+3(a^2-b^2)-5ax$, $12(a^2-b^2)-7ax$, $16(a^2-b^2)+9ax$, $-33(a^2-b^2)$. *Ans.* $7ax$.

9. Add $2a^2-3(a+x)$, $5a^2+6(x-y)^2$, $4a^2-7(x-y)^2$, $9(a+x)-6a^2$, $3(a+x)-9(x-y)^2$, $a^2-(a+x)+(x-y)^2$.

Ans. $6a^2+8(a+x)-9(x-y)^2$.

62. Dissimilar Terms having a common factor may be added by taking the algebraic sum of the dissimilar parts, enclosing it in a parenthesis, and affixing the common factor.

1. Find the sum of $ax+bx-cx$.

OPERATION.

SOLUTION. a times x, $+b$ times x, $-c$ times x, equals $(a+b-c)$ times x; hence the sum is $(a+b-c)x$.

$$\begin{array}{r} ax \\ bx \\ -cx \\ \hline (a+b-c)x. \end{array}$$

EXAMPLES.

2. Find the sum of $ax^3-bx^3+cx^3$. *Ans.* $(a-b+c)x^3$.

3. Find the sum of $az^5-mz^5+nz^5-qz^5$.

Ans. $(a-m+n-q)z^5$.

4. Find the sum of $2ax-2bx+(a-b)x$. *Ans.* $3(a-b)x$.

5. Find the sum of $4ax+3x+2ax-5x+bx-5ax+2x-2bx$.

Ans. $(a-b)x$.

6. Find the sum of $3ay-2by+(a+2b+c)y$. *Ans.* $(4a+c)y$.

7. Find the sum of $3an-5am+2an-3bn+3am-5m+6bn+2am-3bn+5m+cn$. *Ans.* $(5a+c)n$.

SUBTRACTION.

63. Subtraction is the process of finding the difference of two algebraic quantities.

64. The **Subtrahend** is the quantity to be subtracted.

65. The **Minuend** is the quantity from which the subtrahend is to be subtracted.

66. The **Difference** or **Remainder** is a quantity which, added to the subtrahend, will equal the minuend.

NOTE.—The symbol $-$ was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

CASE I.

67. To subtract when all the terms are positive.

1. Subtract $4a$ from $7a$.

SOLUTION. 4 times a quantity subtracted from 7 times the quantity equals 3 times the quantity; hence, $4a$ subtracted from $7a$ equals $3a$.

OPERATION

$$\begin{array}{r} 7a \\ 4a \\ \hline 3a \end{array}$$

2. Subtract $7a$ from $4a$.

SOLUTION. $4a$ equals $7a-3a$; $7a$ subtracted from $7a-3a$ leaves $-3a$; hence $7a$ subtracted from $4a$ equals $-3a$.

OPERATION.

$$\begin{array}{rl} 4a = & 7a-3a \\ 7a = & 7a \\ \hline -3a & \quad -3a \end{array}$$

SOLUTION 2D. Plus $4a$ may indicate some quantity *increased* by $4a$, and $+7a$ may indicate some quantity increased by $7a$. A quantity *increased by* $4a$ is evidently $3a$ *less* than the quantity *increased by* $7a$; hence, $7a$ subtracted from $4a$ equals *minus* $3a$.

3. Subtract $b+c$ from a.

SOLUTION. Subtracting b from a, we have the remainder $a-b$; but we wish to subtract b *increased* by c from a, hence the true remainder will be $a-b$ diminished by c, or $a-b-c$.

OPERATION.

$$\begin{array}{l} a \\ b+c \\ \hline a-b-c \end{array}$$

Rule.—*Change the signs of the subtrahend and proceed as in addition.*

NOTE.—Signs of terms are said to be changed when, being *plus*, they are changed to *minus*, or being $-$, they are changed to $+$.

EXAMPLES.

(4.)	(5.)	(6.)	(7.)	(8.)
$12a$	$15x^2y$	$21m^2n^2$	$5a^2+3b$	$3a+4b$
$9a$	$9x^2y$	$28m^2n^2$	$3a^2+7b$	$9b+2c$
$3a$	$6x^2y$	$-7m^2n^2$	$2a^2-4b$	$3a-5b-2c$

9. From $19ab$ take $12ab$. *Ans.* $7ab$.
10. From $21ac^2$ take $16ac^2$. *Ans.* $5ac^2$.
11. From $10axy$ take $17axy$. *Ans.* $-7axy$.
12. From $12m^2n^3$ take $18m^2n^3$. *Ans.* $-6m^2n^3$.
13. From $4a^2+6b$ take $12b$. *Ans.* $4a^2-6b$.
14. From $7a+5c$ take $10c$. *Ans.* $7a-5c$.
15. From $3x^2+2y^2$ take $4x^2+y^2$. *Ans.* $-x^2+y^2$.
16. From $2a+3b$ take $a+2b$. *Ans.* $a+b$.
17. From $4a+2b$ take $2a+3b$. *Ans.* $2a-b$.
18. From $a^2+4ab+b^2$ subtract $a^2+2ab+b^2$. *Ans.* $2ab$.
19. From a^2+b^2 subtract $a^2+2ab+b^2$. *Ans.* $-2ab$.
20. From $4a+2b$ subtract $3a+4b+2c$. *Ans.* $a-2b-2c$.
21. From $7a^2b+3ac$ subtract $5ac+4a^2b$. *Ans.* $3a^2b-2ac$.
22. From $ab+bc+cd$ subtract $bc+2cd+c$. *Ans.* $ab-cd-c$.

CASE II.

68. To subtract when one or more terms are negative.

1. Subtract $-c$ from $+a$.

SOLUTION. a equals $a+c-c$, since increasing and diminishing a quantity by the same quantity does not change its value. Now, $-c$ subtracted from $a+c-c$, leaves $a+c$. Hence, $-c$ subtracted from $+a$ leaves $a+c$.

OPERATION.

$$\begin{array}{rl} +a= & a+c-c \\ -c= & \quad -c \\ \hline a+c & \quad a+c \end{array}$$

SOLUTION 2D. The difference between any quantity *increased* by a and *diminished* by c is evidently the *sum* of a and c; hence, $-c$ subtracted from $+a$ equals $a+c$.

OPERATION.

$$\begin{array}{c} +a \\ -c \\ \hline a+c \end{array}$$

2. Subtract $b-c$ from a.

SOLUTION. Subtracting b from a, we have the remainder $a-b$; but we wish to subtract b diminished by c from a; we have therefore subtracted c too much, consequently the remainder, $a-b$, is c too small; hence the true remainder is $a-b$ increased by c, or $a-b+c$.

OPERATION.

$$\begin{array}{c} a \\ b-c \\ \hline a-b+c \end{array}$$

Rule.—I. *Write the subtrahend under the minuend, placing similar terms one under another.*

II. *Conceive the signs of the subtrahend to be changed, and then proceed as in addition.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$9a^2b$	$5a^2$	$-2a$	$7m^2-3n$
$-6a^2b$	$3a^2-2b$	$a-b$	$-4m^2-6n+c$
$15a^2b$	$2a^2+2b$	$-3a+b$	$11m^2+3n-c$

(7.)	(8.)	(9.)
$2a^2-5a+6b$	$c-5m+4$	$ax^2-2ac+3\frac{1}{2}$
$-a^2-3a+4b$	$-3c-7m-2$	$ax^2-5ac+2\frac{1}{4}-z^2$
$3a^2-2a+2b$	$4c+2m+6$	$3ac+1\frac{1}{4}+z^2$

10. From $a+b$ take $a-b$. *Ans.* $2b$.
11. From $10a$ take $-10a$. *Ans.* $20a$.
12. From $a-b$ take $b-a$. *Ans.* $2a-2b$.
13. From $a+2b$ take $a-b$. *Ans.* $3b$.
14. From $5m-5n$ take $4m+6n$. *Ans.* $m-11n$.
15. From $1+a^2x^2$ take $1-a^2x^2$. *Ans.* $2a^2x^2$.
16. From $4a^m-3b^n$ take $2a^m-5b^n$. *Ans.* $2a^m+2b^n$.
17. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$. *Ans.* $4ab$.
18. From a^2-b^2 take $a^2-2ab+b^2$. *Ans.* $2ab-2b^2$.
19. From $3a+c+d-f-8$ take $c+3a-d$. *Ans.* $2d-f-8$.
20. From $4ab+3b^2-2c$ take $4ab-2b^2-3d$. *Ans.* $5b^2-2c+3d$
21. From $7am-3bc-c^2$ take $5am-2c^2-3bc-5x^3$. *Ans.* $2am+c^2+5x^3$.
22. From $2a+2b-3c-8$ take $3c+4b-3a-5$. *Ans.* $5a-2b-6c-3$.
23. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$. *Ans.* $6a^2b+2b^3$.
24. From $a^2-3ab-b^2+bc-2c^2$ take $a^2-5ab+5bc-3b^2-2c^2$. *Ans.* $2ab+2b^2-4bc$.

FACTORED FORMS.

69. Similar quantities in any form may be subtracted by taking the algebraic difference of their coefficients.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$9\sqrt{6}$	$12(a-b)$	$15\sqrt{a+b}$	$-7(a-b+4)$
$5\sqrt{6}$	$7(a-b)$	$-7\sqrt{a+b}$	$-12(a-b+4)$
$4\sqrt{6}$	$5(a-b)$	$22\sqrt{a+b}$	$5(a-b+4)$

5. From $5(x^2-y^2)$ take $-7(x^2-y^2)$. *Ans.* $12(x^2-y^2)$.
6. From $-6(a^2-b^2)$ take $12(a^2-b^2)$. *Ans.* $-18(a^2-b^2)$.
7. From $6a^2(a-b)$ take $-4a^2(a-b)$. *Ans.* $10a^2(a-b)$.
8. Find the value of $5\sqrt{2}-7\sqrt{2}+6\sqrt{2}$. *Ans.* $4\sqrt{2}$.
9. From $-5x^2(c-d)$ take $-12x^2(c-d)$. *Ans.* $7x^2(c-d)$.
10. Find the value of $7c^2(m-n)-13c^2(m-n)+12c^2(m-n)$. *Ans.* $6c^2(m-n)$.

70. Dissimilar terms having a common factor may be subtracted by taking the algebraic difference of the dissimilar parts, enclosing it in a parenthesis and affixing the common part.

1. From ax subtract cx.

OPERATION

$$\begin{array}{c} ax \\ cx \\ \hline (a-c)x \end{array}$$

SOLUTION. a times x minus c times x is evidently equal to $(a-c)$ times x, which is expressed thus $(a-c)x$.

EXAMPLES.

(2.)	(3.)	(4.)	(5.)
ax^2	mz^2	axy	az
bx^2	$-nz^2$	cxy	z
$(a-b)x^2$	$(m+n)z^2$	$(a-c)xy$	$(a-1)z$

6. From $5az$ take baz. *Ans.* $(5-b)az$.
7. From cax take $-3ax$. *Ans.* $(c+3)ax$.

8 From az take $bz-3z$. *Ans.* $(a-b+3)z$.

9. From $4n^2c+3c$ take $7c-4ac$. *Ans.* $(n^2-1+a)4c$.

10. From $an+cn+dn$ take $n+an+dn$. *Ans.* $(c-1)n$.

11. From $(6a+2x)cd$ take $4acd+2cdx$. *Ans.* $2acd$.

12. From $5a^2+10b^2$ take $-3a^2+2b^2$. *Ans.* $8(a^2+b^2)$.

13 From $6ay-3my$ subtract $-5my+6cy$. *Ans.* $(3a-3c+m)2y$.

14. From $6\sqrt{c}-a\sqrt{c}+b\sqrt{c}$ subtract $2a\sqrt{c}+b\sqrt{c}-2\sqrt{c}-ax\sqrt{c}$. *Ans.* $(8-3a+ax)\sqrt{c}$.

USE OF THE PARENTHESIS.

71. The **Parenthesis** is frequently used in Algebra: we will therefore now explain its use in Addition and Subtraction.

The plus sign before a parenthesis indicates that the quantity within the parenthesis is to be *added*, and the *minus* sign indicates that it is to be *subtracted*.

PRIN. 1. *A parenthesis with the plus sign before it may be removed from a quantity without changing the signs of its terms.*

Thus, $a+(b-c+d)$ is equal to $a+b-c+d$.

PRIN. 2. *A quantity may be enclosed in a parenthesis preceded by a plus sign without changing the signs of its terms.*

Thus, $a+b-c+d-e$ is equal to $a+(b-c+d-e)$, or to $a+b+(-c+d-e)$, etc.

PRIN. 3. *A parenthesis preceded by the minus sign may be removed from a quantity if the signs of all its terms be changed.*

This is evident from the rule for subtraction. Thus, $a-(b-c+d)$ is equal to $a-b+c-d$.

PRIN. 4. *A quantity may be enclosed in a parenthesis preceded by the minus sign if the signs of all its terms be changed.*

This is evident from the principles of subtraction, and also from the previous principle. Thus, $a-b+c-d$ is equal to $a-(b-c+d)$; or to $a-b-(-c+d)$, etc.

EXAMPLES.

Find the value—

1. Of $-(-a^2)$ and $x-(a-b)$. *Ans.* a^2; $x-a+b$.

2. Of $+(-ab)$ and $a-(b-c+d)$. *Ans.* $-ab$; $a-b+c-d$.

3. Of $-(b^2-a^2)$ and $3c-(2c-5)$. *Ans.* a^2-b^2; $c+5$.

4. Of $4a-5b-(a-5b+3c)$. *Ans.* $3(a-c)$.

5. Of $5a-2b-3c-(-5c+2a-2b)$. *Ans.* $3a+2c$.

6. Put in a parenthesis preceded by a plus sign the last three terms of $a+2b-3c+d-4$. *Ans.* $a+2b+(-3c+d-4)$.

7. Put in a parenthesis preceded by a minus sign the last three terms of $3a-4b+5c-7d$. *Ans.* $3a-(4b-5c+7d.)$

8. Find the value of $2a-(b+c-d+e-f)$ plus $2b-(a-c+d-e+g)$. *Ans.* $a+b-g+f$.

72. Expressions sometimes occur containing more than one pair of brackets, as $a-\{b-(c-d)\}$.

Such brackets may be removed in succession, *beginning*, for convenience, *with the inside pair.*

NOTE.—Brackets may also be removed by beginning with the *outer pair*, or with *any pair.*

Find the value—

1. Of $a-\{b+(c-d)\}$.

SOLUTION. $a-\{b+(c-d)\}=a-\{b+c-d\}=a-b-c+d$.

2. Of $a-\{b-(c-d)\}$. *Ans.* $a-b+c-d$.

3. Of $a-\{b-c-(d-e)\}$. *Ans.* $a-b+c+d-e$.

4. Of $2a-\{b-(a-2b)\}$. *Ans.* $3a-3b$.

5. Of $3a-\{b+(2a-b)-(a-b)\}$. *Ans.* $2a-b$.

6. $7a-[3a-\{4a-(5a-2a)\}]$. *Ans.* $5a$.

7. Of $6a-[4b-\{4a-(6a-4b)\}]$. *Ans.* $4a$.

8. Of $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$. *Ans.* $3a-2c$.

REMARKS UPON ADDITION AND SUBTRACTION.

1. Addition and Subtraction may also be explained by regarding the positive and negative quantities as representing, respectively, *gain* and *loss* in business, distance *north* and *south*, etc. But these illustrations, though they may aid the beginner, are not sufficiently general to be embodied in a solution.

2. The problem, subtract $7a$ from $4a$, may be explained by the following method: $7a$ equals $4a+3a$; subtracting $4a$ from $4a$, nothing remains, and there is still $3a$ to be subtracted, which we may represent by writing $-3a$. This method, however, is not general; it will not explain several cases, such as $-7a$ from $3a$, nor the general problem, subtract $-c$ from a.

3. Special attention is invited to the method of explaining Addition and Subtraction given in the "Solution 2d" of Articles 67 and 68. The peculiarity of the method consists in regarding a *positive* term as indicating that *some quantity* is *increased* by the term, and a *negative* term as indicating that *some quantity* is *diminished* by that term, or in using an *auxiliary quantity*.

Thus, to subtract $-2a$ from $+3a$, we regard $+3a$ as indicating that *some quantity* is to be *increased* by $3a$, and $-2a$ that *some quantity* is to be *diminished* by $2a$; then since *a quantity increased*, by $3a$ is *greater* than the quantity *diminished* by $2a$, by the sum of $3a$ and $2a$, or $5a$, we infer that $-2a$ taken from $+3a$ leaves $+5a$. Hence we use "*a quantity*" as *auxiliary*.

The same idea is presented in the following form of statement: The *difference* between a quantity *increased* by $3a$ and *diminished* by $2a$ is evidently the sum of $3a$ and $2a$, or $5a$; hence $-2a$ subtracted from $3a$ leaves $+5a$. The *plus* sign before the remainder will show that the minuend is *greater* than the subtrahend; the *minus* sign before the remainder will show that the *minuend* is *less* than the subtrahend.

4. This method enables us to give a simple explanation to each of the eight possible cases in the subtraction of monomials. It will be well to have the pupils explain each of the cases given below:

(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)
$7a$	$4a$	$-7a$	$-4a$	$-7a$	$7a$	$4a$	$-4a$
$4a$	$7a$	$-4a$	$-7a$	$4a$	$-4a$	$-7a$	$7a$
$3a$	$-3a$	$-3a$	$+3a$	$-11a$	$11a$	$11a$	$-11a$

MULTIPLICATION.

73. Multiplication is the process of taking one quantity as many times as there are units in another.

74. The **Multiplicand** is the quantity to be multiplied.

75. The **Multiplier** is the quantity by which we multiply.

76. The **Product** is the result obtained by multiplying.

77. The Multiplicand and Multiplier are called *factors* of the product.

NOTE.—The symbol × was introduced by *Wm. Oughtred*, an English mathematician, born in 1574.

PRINCIPLES.

1. *The product of two or more quantities is the same in whatever order the factors are arranged.*

Thus, a times b is the same as b times a, as may be seen by assigning special values to the letters; and the same is true of any number of quantities.

2. *Multiplying any factor of a quantity multiplies the quantity.*

Thus, 4 times the quantity 2×3 equals $4\times2\times3$, which is 8×3, or $2\times3\times4$, which is 2×12. Thus, also, 3 times $2a$ is $6a$; 4 times $3ab$ is $12ab$.

3. *The exponent of a quantity in the product is equal to the sum of its exponents in the two factors.*

Thus, $a^2\times a^3$ equals a^5, since a used as a factor *twice*, multiplied by a used as a factor *three* times, equals a used as a factor *five* times. It may also be seen thus: $a^2\times a^3=aa\times aaa$, which equals $aaaaa$, which equals a^5.

4. *The product of two factors having* LIKE *signs is positive, and the product of two factors having* UNLIKE *signs is negative.*

To prove this, multiply b by a; $-b$ by a; b by $-a$, and $-b$ by $-a$.

First, $+b$, taken any number of times, as a times, is evidently $+ab$.

Second, $-b$ taken once is $-b$; taken twice, is $-2b$, etc.; hence, $-b$, taken any number of times, as a times, is $-ab$.

OPERATION.

$+b$	$-b$	$+b$	$-b$
$+a$	$+a$	$-a$	$-a$
$+ab$	$-ab$	$-ab$	$-(-ab)$
			$=+ab$

Third, b multiplied by $-a$ means that b is to be taken *subtractively* a times; b taken a times is ab, and taken *subtractively* is $-ab$.

Fourth, $-b$ multiplied by $-a$ means that $-b$ is to be taken *subtractively* a times; $-b$ taken a times is $-ab$, and used *subtractively* is $-(-ab)$, which by the principles of subtraction is $+ab$.

Hence we infer that *the product of quantities having* LIKE *signs is* PLUS, *and having* UNLIKE *signs is* MINUS.

CASE I.

78. To multiply a monomial by a monomial.

1. Multiply $3b$ by $2a$.

SOLUTION. To multiply $3b$ by $2a$, we multiply by 2 and by a. $3b$ multiplied by 2 and by a equals $3b \times 2 \times a$, which, since the product is the same in whatever order the factors are placed, equals $3 \times 2 \times a \times b$, which equals $6ab$. Therefore, $3b$ multiplied by $2a$ is $6ab$.

OPERATION.

$$\begin{array}{r} 3b \\ 2a \\ \hline 6ab \end{array}$$

2. Multiply $4a^3$ by $3a^2$.

SOLUTION. To multiply $4a^3$ by $3a^2$, we may multiply one factor by 3 and the other factor by a^2 (Prin. 2). 3 times 4 are 12, and a^2 times a^3 is a^5 (Prin. 3). Therefore $4a^3$ multiplied by $3a^2$ equals $12a^5$.

OPERATION

$$\begin{array}{r} 4a^3 \\ 3a^2 \\ \hline 12a^5 \end{array}$$

Rule.—I. *Multiply the coefficients of the two factors together.*

II. *To this product annex all the letters of both factors, giving each letter an exponent equal to the sum of its exponents in the two factors.*

III. *Make the product positive when the factors have like signs, and negative when they have unlike signs.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$5a$	$-6a^2$	$5a^3$	$-6c^2b$
$2b$	$3a$	$4a^2$	$-3c$
$10ab$	$-18a^3$	$20a^5$	$+18c^3b$

(7.)	(8.)	(9.)	(10.)
$8ab^2$	$12ax$	$15m^2n$	$-5a^3c$
$3a^2b$	$4cx$	$-5an$	$6c^2d$
$24a^3b^3$	$48acx^2$	$-75am^2n^2$	$30a^3c^3d$

11. Multiply $7m^5n^3$ by $-5n^4x$. *Ans.* $-35m^5n^7x$.
12. Multiply $12a^2x^3y^2$ by $7a^3c^2xy$. *Ans.* $84a^5c^2x^4y^3$.
13. Multiply $-9a^3b^4c^5$ by $-7a^2b^3x^2$ *Ans.* $63a^5b^7c^5x^2$.
14. Multiply $4(a+b)^2$ by $2a$. *Ans.* $8a(a+b)^2$.
15. Multiply $-a(a-x)$ by b. *Ans.* $-ab(a-x)$.
16. Multiply $(a+b)^3$ by $(a+b)^2$. *Ans.* $(a+b)^5$.
17. Multiply $a(x-y)^2$ by $2(x-y)$. *Ans.* $2a(x-y)^3$.
18. Multiply $-5x(m-n)^3$ by $-3x(m-n)^5$. *Ans.* $15x^2(m-n)^8$.
19. Multiply a^m by a^n. *Ans.* a^{m+n}.
20. Multiply b^{2n} by b^n. *Ans.* b^{3n}.
21. Multiply c^m by c^2. *Ans.* c^{m+2}.
22. Multiply d^{3n} by d^{n+2}. *Ans.* d^{4n+2}.
23. Multiply $(a-x)^m$ by $(a-x)^{-n}$. *Ans.* $(a-x)^{m-n}$,
24. Multiply $-3a^2(l^2-m)^n$ by $-2a^3(l^2-m)^{-3}$. *Ans.* $6a^5(l^2-m)^{n-3}$.

CASE II.

79. To multiply a polynomial by a monomial.

1. Multiply $a-b$ by c.

SOLUTION. To multiply $a-b$ by c we must multiply each term by c. c times a is ac, and c times $-b$ is $-bc$. Hence, $a-b$ multiplied by c is $ac-bc$.

OPERATION.

$$\begin{array}{r} a-b \\ c \\ \hline ac-bc \end{array}$$

Rule.—*Multiply each term of the multiplicand by the multiplier, and connect the products by their proper signs.*

EXAMPLES.

(2.)	(3.)	(4.)
$7a^2-3b$	$6ax-5c^2y$	$3m^2-4n^3+7$
$3a$	$3ac$	$-2mn$
$21a^3-9ab$	$18a^2cx-15ac^3y$	$-6m^3n+8mn^4-14mn$

(5.)	(6.)	(7.)
$4a^n-3ab^n$	$3c^m-4bc+5d^n$	$4x^n-5xy^n$
$2a^n$	$3cd$	$3x^3y^{-2}$
$8a^{2n}-6a^{n+1}b^n$	$9c^{m+1}d-12bc^2d+15cd^{n+1}$	$12x^{n+3}y^{-2}-15x^4y^{n-2}$

8. Multiply $5ax^2-3x^3y$ by $-6a^2x$. *Ans.* $-30a^3x^3+18a^2x^4y$.
9. Multiply $11m^2-3$ by -5. *Ans.* $-55m^2+15$.
10. Multiply $a^2-2ab+b^2$ by ab. *Ans.* $a^3b-2a^2b^2+ab^3$.
11. Multiply $3a^n-4b^m$ by $a^{2n}b^{3m}$. *Ans.* $3a^{3n}b^{3m}-4a^{2n}b^{4m}$.
12. Multiply $a^{n-1}b-b^{n-2}c$ by ab^2. *Ans.* $a^nb^3-ab^nc$.
13. Multiply $5x^3-7m^3x+3\frac{1}{2}$ by $4mx$.
Ans. $20mx^4-28m^4x^2+14mx$.

CASE III.

80. To multiply a polynomial by a polynomial.

1. Multiply $2a-b$ by $a+2b$.

OPERATION.

$$\begin{array}{r} 2a-b \\ \underline{a+2b} \\ 2a^2-ab \\ \underline{+4ab-2b^2} \\ 2a^2+3ab-2b^2 \end{array}$$

SOLUTION. $a+2b$ times $2a-b$ equals a times $2a-b$ plus $2b$ times $2a-b$. a times $2a-b$ equals $2a^2-ab$: $2b$ times $2a-b$ equals $4ab-2b^2$. Adding the partial products, we have $2a^2+3ab-2b^2$. Therefore, etc.

Rule.—*Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

EXAMPLES.

(2.)
$$\begin{array}{r} 2a-3b \\ \underline{a-b} \\ 2a^2-3ab \\ \underline{-2ab+3b^2} \\ 2a^2-5ab+3b^2 \end{array}$$

(3.)
$$\begin{array}{r} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{+ab+b^2} \\ a^2+2ab+b^2 \end{array}$$

(4.)
$$\begin{array}{r} a-b \\ \underline{a+b} \\ a^2-ab \\ \underline{+ab-b^2} \\ a^2 \quad -b^2 \end{array}$$

(5.)
$$\begin{array}{r} a^2+ab+b^2 \\ \underline{a-b} \\ a^3+a^2b+ab^2 \\ \underline{-a^2b-ab^2-b^3} \\ a^3 \qquad -b^3 \end{array}$$

(6.)
$$\begin{array}{r} a^n-b^n \\ \underline{a^2-b^2} \\ a^{n+2}-a^2b^n \\ \underline{-a^nb^2+b^{n+2}} \\ a^{n+2}-a^nb^2-a^2b^n+b^{n+2} \end{array}$$

7. Multiply $3a-2b$ by $2a-3b$. *Ans.* $6a^2-13ab+6b^2$.
8. Multiply a^2-b^2 by a^2+b^2. *Ans.* a^4-b^4.

9. Multiply $3x-6y$ by $2x+4y$. *Ans.* $6x^2-24y^2$.
10. Multiply c^2+cd+d^2 by $c-d$. *Ans.* c^3-d^3.
11. Multiply x^3+y^3 by x^3-y^3. *Ans.* x^6-y^6.
12. Multiply $4x-3y$ by $4x+3y$. *Ans.* $16x^2-9y^2$.
13. Multiply $x^4-x^3z+x^2z^2-xz^3+z^4$ by $x+z$. *Ans.* x^5+z^5.
14. Multiply $a^{n-2}-b^{n-2}$ by a^2+b^2. *Ans.* $a^n-a^2b^{n-2}+a^{n-2}b^2-b^n$.
15. Multiply $a^2x^3+x^2y^3$ by $a^2x^3-x^2y^3$. *Ans.* $a^4x^6-x^4y^6$.
16. Multiply $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$. *Ans.* $x-y$.
17. Multiply $3\frac{1}{2}a^2+5\frac{1}{2}c^3$ by $2a^2+4c^3$. *Ans.* $7a^4+25a^2c^3+22c^6$.
18. Multiply c^2+cd-d^2 by $c-d$. *Ans.* $c^3-2cd^2+d^3$.
19. Multiply $a^2-3ab+4ab^2$ by $a^2+3ab-4ab^2$.
Ans. $a^4-9a^2b^2+24a^2b^3-16a^2b^4$.
20. Multiply n^2+np+p^2 by n^2-np+p^2. *Ans.* $n^4+n^2p^2+p^4$.
21. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$. *Ans.* $a^4-2a^2b^2+b^4$.
22. Multiply a^m+b^n by a^m-b^n. *Ans.* $a^{2m}-b^{2n}$.
23. Multiply a^n-b^m by a^n-b^m. *Ans.* $a^{2n}-2a^nb^m+b^{2m}$.
24. Multiply $m^3+m^2n+mn^2+n^3$ by $m-n$. *Ans.* m^4-n^4.
25. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.
26. Multiply $a^4-a^3+a^2-a+1$ by $a+1$. *Ans.* a^5+1.
27. Multiply $1+c$, $1-c$, $1+c+c^2$ and $1-c+c^2$. *Ans.* $1-c^6$.

EXPANDING EXPRESSIONS.

81. An algebraic expression is *expanded* when the multiplication indicated is performed.

1. Expand $(a-x)(a-x)$. *Ans.* $a^2-2ax+x^2$.
2. Expand $(2a^2-3b^n)(3a^2+4b^n)$. *Ans.* $6a^4-a^2b^n-12b^{2n}$.
3. Expand $(a^n+b^n)(a^m+b^m)$. *Ans.* $a^{m+n}+a^mb^n+a^nb^m+b^{m+n}$.
4. Expand $(a-2)(a-3)(a+2)(a+3)$. *Ans.* a^4-13a^2+36.
5. Expand $(a+b)(a-b)(a+b)(a-b)$. *Ans.* $a^4-2a^2b^2+b^4$.
6. Expand $(1+a)(1+a^4)(1-a+a^2-a^3)$. *Ans.* $1-a^8$.
7. Expand $(a^2+a+1)(a^2+a+1)(a-1)(a-1)$.
Ans. a^6-2a^3+1.