DIVISION.

82. Division is the process of finding how many times one quantity is contained in another.

83. The **Dividend** is the quantity to be divided.

84. The **Divisor** is the quantity by which we divide.

85. The **Quotient** is the result obtained by the division.

86. The **Remainder** is the quantity which is sometimes left after dividing.

NOTE.—The symbol of division, $\div$, was introduced by *Dr. John Pell*, an English mathematician, born in 1610.

PRINCIPLES.

1. *Taking a factor out of a quantity divides the quantity by that factor.*

Thus, taking the factor a out of $4ab$, we have $4b$, which is the quotient of $4ab$ divided by a, since $4b$ multiplied by a is $4ab$.

2. *The exponent of a quantity in the quotient equals its exponent in the dividend diminished by its exponent in the divisor.*

Thus, a^6 divided by a^4 equals a^{6-4}, or a^2; since a^4 multiplied by a^2 equals a^{4+2}, or a^6.

3. *The quotient is* POSITIVE *when the dividend and divisor have* LIKE *signs, and* NEGATIVE *when they have* UNLIKE *signs.*

Thus, $+ab \div +b = +a$, since $+a \times +b = +ab$;
$-ab \div -b = +a$, since $+a \times -b = -ab$;
$+ab \div -b = -a$, since $-a \times -b = +ab$;
$-ab \div +b = -a$, since $-a \times +b = -ab$.

87. This principle, and the corresponding one in multiplication, may be briefly stated thus:

LIKE *signs give* PLUS, and UNLIKE *signs give* MINUS.

CASE I.

88. To divide a monomial by a monomial.

1. Divide $12ab$ by $4a$.

SOLUTION. To divide $12ab$ by $4a$, we divide by 4 and a. Dividing $12ab$ by 4 and a, by taking out the factors 4 and a (Prin. 1), we have $3b$. Hence, $12ab$ divided by $4a$ equals $3b$.

OPERATION.
$12ab \div 4a = 3b$

2. Divide $20a^5b$ by $5a^3$.

SOLUTION. To divide $20a^5b$ by $5a^3$, we divide by 5 and a^3. Dividing 20 by 5, we have 4; dividing a^5 by a^3, we have a^2 (Prin. 2). Hence the quotient is $4a^2b$.

OPERATION.
$20a^5b \div 5a^3 = 4a^2b$

Rule.—I. *Divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient.*

II. *Write the letters of the dividend in the quotient, giving each an exponent equal to its exponent in the dividend minus its exponent in the divisor.*

III. *Make the quotient positive when the two terms have like signs, and negative when they have unlike signs.*

NOTE.—An equal literal factor in dividend and divisor is suppressed in the quotient, since it is canceled by the division.

EXAMPLES.

3. Divide $12x^3$ by $4x^2$. *Ans.* $3x$.
4. Divide $20ab^3$ by $5b^2$. *Ans.* $4ab$.
5. Divide $4abc^2$ by $2ac$. *Ans.* $2bc$.
6. Divide $8a^3b^2$ by $4ab^2$. *Ans.* $2a^2$.
7. Divide $9m^2n^5$ by $3mn^2$. *Ans.* $3mn^3$.
8. Divide $-15ay^3$ by $3ay$. *Ans.* $-5y^2$.
9. Divide $-ab^5c^3$ by ab^2c. *Ans.* $-b^3c^2$.
10. Divide $16a^7b^5$ by $-8a^3b^4$. *Ans.* $-2a^4b$.
11. Divide $-24x^6z$ by $-6x^3$. *Ans.* $4x^3z$.

12. Divide $-15c^7d^8$ by $-5c^2d^5$. *Ans.* $3c^5d^3$.

13 Divide $14a^3b^5c$ by $7b^3c$. *Ans.* $2a^3b^2$.

14. Divide a^m by a^n. *Ans.* a^{m-n}.

15. Divide a^{m+n} by a^n. *Ans.* a^m.

16. Divide a^{m-n} by a^n. *Ans.* a^{m-2n}.

17. Divide a^{2m} by a^n. *Ans.* a^{2m-n}.

18. Divide $14a^{m+n}$ by $-2a^{m-n}$. *Ans.* $-7a^{2n}$.

19. Divide $-24a^{p+1}$ by $6a^{p-1}$. *Ans.* $-4a^2$.

20. Divide $a^{3m}b^{4n}$ by a^mb^n. *Ans.* $a^{2m}b^{3n}$.

21. Divide $84a^5c^6$ by $7a^nc^z$. *Ans.* $12a^{5-n}c^{6-z}$.

22. Divide $(a+b)^5$ by $(a+b)^3$: *Ans.* $(a+b)^2$.

23. Divide $(a-c)^{2n}$ by $(a-c)^n$. *Ans.* $(a-c)^n$.

24. Divide $12a^2b^3(a-b)^{n+2}$ by $4b^5(a-b)^{n-2}$.
Ans. $3a^2b^{-2}(a-b)^4$.

CASE II.

89. To divide a polynomial by a monomial,

1. Divide $8a^4-16a^3b+12a^2c^5$ by $4a^2$.

SOLUTION. $4a^2$ is contained in $8a^4$, $2a^2$ times; $4a^2$ is contained in $-16a^3b$, $-4ab$ times; $4q^2$ is contained in $12a^2c^5$, $3c^5$ times. Hence, the quotient is $2a^2-4ab+3c^5$.

OPERATION.

$$4a^2\overline{)8a^4-16a^3b+12a^2c^5}$$
$$2a^2-4ab+3c^5$$

Rule.—*Divide each term of the dividend by the divisor and connect the results by their proper signs.*

EXAMPLES.

2. Divide $6a^3b-9ab^4$ by $3ab$. *Ans.* $2a^2-3b^3$.

3. Divide $8a^4c-12a^2c^5$ by $4ac$. *Ans.* $2a^3-3ac^4$.

4. Divide $9ab^5-15ab^3c^2$ by $3ab^3$. *Ans.* $3b^2-5c^2$.

5. Divide $6a^5c^2-12a^8c$ by $3a^4c$. *Ans.* $2ac-4a^4$.

6. Divide $abc-5a^2b^3c^4$ by abc. *Ans.* $1-5ab^2c^3$.

7. Divide $4a^{2n}-8a^{3n}$ by $2a^n$. *Ans.* $2a^n-4a^{2n}$.

8. Divide $16ab^3x^3-20b^2x^5z$ by $4b^2x^3$. *Ans.* $4ab-5x^2z$.

9. Divide $18a^5x^6-27a^6x^8-9a^3x^6$ by $9a^3x^6$.
Ans. $2a^2-3a^3x^2-1$.

10. Divide $-16x^3+24z^5-48$ by -8. *Ans.* $2x^3-3z^5+6$.

11. Divide $12a^3b-18a^2b^3+6a^2b$ by $6a^{-2}b$.
Ans. $2a^5-3a^4b^2+a^4$.

12. Divide $a^{2n}b^3-a^nb^4+a^{n+2}b^3$ by a^nb^3. *Ans.* a^n-b+a^2.

13. Divide $a^nb^m-a^{n+3}b^{m-2}$ by a^nb^m. *Ans.* $1-a^3b^{-2}$.

14. Divide $16(a-x)-24(c-z)$ by 8.
Ans. $2(a-x)-3(c-z)$.

15. Divide $7(m+n)-14a(m+n)$ by $(m+n.)$ *Ans.* $7-14a$.

16. Divide $(r-s)^2-(r-s)^4$ by $(r-s)$. *Ans.* $(r-s)-(r-s)^3$.

17. Divide $a(c-d)-b(c-d)$ by $c-d$. *Ans.* $a-b$.

18. Divide $4a(a-c)+(a-c)^2$ by $a-c$.
Ans. $4a+(a-c)$, or $5a-c$.

19. Divide $5x(x-y)^2-2(x-y)^3$ by $(x-y)^2$.
Ans. $5x-2(x-y)$ or $3x+2y$.

20. Divide $2a(1+c)^3-2ac(1+c)^2$ by $(1+c)^2$. *Ans.* $2a$.

CASE III.

90. To divide a polynomial by a polynomial.

1. Divide $a^2+2ab+b^2$ by $a+b$.

SOLUTION. We write the divisor at the right of the dividend, both being arranged according to the powers of a, and commence at the left to divide. Since the first term of the dividend must equal the product of the first terms of the divisor and quotient, we divide the first term of the dividend by the first term of the divisor for the first term of the quotient. a is contained in a^2, a times; a times $a+b$ equals a^2+ab. Subtracting and bringing down the next term of the dividend, we have $ab+b^2$.

OPERATION.

$$\begin{array}{l|l} a^2+2ab+b^2 & a+b \\ \cline{2-2} \underline{a^2+\ ab} & a+b \\ \quad\ \ ab+b^2 & \\ \quad\ \ \underline{ab+b^2} & \end{array}$$

Since the first term of this new dividend must be the product of the first term of the divisor by the second term of the quotient, we divide it by the first term of the divisor. a is contained in ab, b times; b times $a+b$ is $ab+b^2$. Subtracting, nothing remains. Hence the quotient is $a+b$.

Rule.—I. *Write the divisor at the right of the dividend, arranging both according to the powers of one of the letters.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result in the quotient; multiply the divisor by it, and subtract the product from the dividend.*

III. *Regard the remainder as a new dividend and proceed as before, and thus continue until the first term of the divisor is not contained in the first term of the dividend.*

NOTES.—1. When the first term of the arranged dividend is not divisible by the first term of the divisor, the division will not be exact.

2. Bring down no more terms of the dividend each time than are needed for use.

EXAMPLES.

2. Divide $a^3+2a^2b+2ab^2+b^3$ by $a+b$.

SOLUTION. We first arrange the dividend with reference to the powers of a, and then proceed as before. a is contained in a^3, a^2 times; a^2 times $a+b$ is a^3+a^2b. Subtracting and bringing down the next term, we have a^2b+2ab^2 for the next dividend, etc.

OPERATION

$$\begin{array}{l|l} a^3+2a^2b+2ab^2+b^3 & a+b \\ \underline{a^3+\ a^2b} & a^2+ab+b^2 \\ \quad a^2b+2ab^2 & \\ \quad \underline{a^2b+\ ab^2} & \\ \qquad ab^2+b^3 & \\ \qquad \underline{ab^2+b^3} & \end{array}$$

(3.)

$$\begin{array}{l|l} a^3+x^3 & a+x \\ \underline{a^3+a^2x} & a^2-ax+x^2 \\ \quad -a^2x+x^3 & \\ \quad \underline{-a^2-ax^2} & \\ \qquad ax^2+x^3 & \\ \qquad \underline{ax^2+x^3} & \end{array}$$

(4.)

$$\begin{array}{l|l} x^3-y^3 & x-y \\ \underline{x^3-x^2y} & x^2+xy+y^2 \\ \quad x^2y-y^3 & \\ \quad \underline{x^2y-xy^2} & \\ \qquad xy^2-y^3 & \\ \qquad \underline{xy^2-y^3} & \end{array}$$

5. Divide $a^2-2ax+x^2$ by $a-x$. *Ans.* $a-x$.
6. Divide $a^2-ax-6x^2$ by $a+2x$. *Ans.* $a-3x$.
7. Divide $a^3-ax^2+ax+x^2$ by $a+x$. *Ans.* a^2-ax+x.
8. Divide $m^3+2m^2n-mn^2-2n^3$ by $m+2n$. *Ans.* m^2-n^2.
9. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$. *Ans.* $a^2-2ab+b^2$.
10. Divide a^3-b^3 by $a-b$. *Ans.* a^2+ab+b^2.
11. Divide $x^3-9x^2+27x-27$ by $x-3$. *Ans.* x^2-6x+9.

12. Divide m^3-n^3 by m^2+mn+n^2. *Ans.* $m-n$.
13. Divide a^3-1 by $a-1$. *Ans.* a^2+a+1.
14. Divide $8x^3-27y^3$ by $2x-3y$. *Ans.* $4x^2+6xy+9y^2$.
15. Divide a^4-x^4 by $a-x$. *Ans.* $a^3+a^2x+ax^2+x^3$.
16. Divide $a^4+2a^2b^2+9b^4$ by $a^2-2ab+3b^2$. *Ans.* $a^2+2ab+3b^2$.
17. Divide $a^4+a^2c^2+c^4$ by a^2-ac+c^2. *Ans.* a^2+ac+c^2.
18. Divide $x^2+2xy+y^2-z^2$ by $x+y-z$. *Ans.* $x+y+z$.
19. Divide m^4-n^4 by m^2+n^2. *Ans.* m^2-n^2.
20. Divide a^4-1 by $a-1$. *Ans.* a^3+a^2+a+1.
21. Divide $a^{2n}-b^{2n}$ by a^n-b^n. *Ans.* a^n+b^n.
22. Divide m^5-n^5 by $m-n$. *Ans.* m^4+m^3n+, etc.
23. Divide s^4-t^4 by $s^3+s^2t+st^2+t^3$. *Ans.* $s-t$.
24. Divide $a^{2n}+2a^nb^n+b^{2n}$ by a^n+b^n. *Ans.* a^n+b^n.
25. Divide $27x^3-64y^3$ by $3x-4y$. *Ans.* $9x^2+12xy+16y^2$.
26. Divide x^5+1 by $x+1$. *Ans.* $x^4-x^3+x^2-x+1$.
27. Divide $1-z^5$ by $1-z$. *Ans.* $1+z+z^2+z^3+z^4$.
28. Divide $a^{3n}-b^{3n}$ by a^n-b^n. *Ans.* $a^{2n}+a^nb^n+b^{2n}$.
29. Divide a^6-b^6 by $a^3+2a^2b+2ab^2+b^3$. *Ans.* $a^3-2a^2b+2ab^2-b^3$.
30. Divide $(a-x)^2-(x-y)^2$ by $(a-x)-(x-y)$. *Ans.* $(a-x)+(x-y)$, or $a-y$.

PRINCIPLES OF DIVISION.

91. The **Principles** of Division are the truths which relate to the process. They are of three kinds—*Changes of Terms*, *Zero Exponents* and *Negative Exponents*.

CHANGES OF TERMS.

92 The **Terms** may be changed both in *value* and in *sign*.

PRIN. 1. *Multiplying the dividend or dividing the divisor multiplies the quotient.*

Thus, $abcd \div ab = cd$. Multiplying the dividend by e, we have $abcde \div ab = cde$, which is the quotient, cd, multiplied by e. Dividing the divisor by b, we have $abcd \div a = bcd$, which is cd multiplied by b. Therefore, etc.

PRIN. 2. *Dividing the dividend or multiplying the divisor divides the quotient.*

Thus, $abcd \div ab = cd$. Dividing the dividend by d, we have $abc \div ab = c$, which equals the quotient, cd, divided by d. Multiplying the divisor by c, we have $abcd \div abc = d$, which also equals cd divided by c. Therefore, etc.

PRIN. 3. *Multiplying or dividing both dividend and divisor by the same quantity does not change the quotient.*

For, multiplying the dividend multiplies the quotient, and multiplying the divisor divides the quotient; hence, multiplying both dividend and divisor by the same quantity both multiplies and divides the quotient by that quantity, and hence does not change its value. Therefore, etc.

PRIN. 4. *Changing the sign of either dividend or divisor changes the sign of the quotient.*

If two terms have like signs, the quotient is *positive;* and if the sign of either term be changed, they will have *unlike* signs; hence the quotient will be changed from *plus* to *minus.*

If two terms have unlike signs, the quotient is *negative;* and if the sign of either term be changed, they will have *like* signs; hence the quotient will be changed from *minus* to *plus.* Therefore, etc.

PRIN. 5. *Changing the sign of both dividend and divisor does not change the sign of the quotient.*

For, if the signs of the terms are alike, when both are changed they will still be alike, and the quotient will remain *plus.* If the signs are unlike, when both are changed they will still be unlike, and the quotient will remain *minus.*

ZERO AND NEGATIVE EXPONENTS.

93. A Zero Exponent originates in dividing a quantity with any exponent by the same quantity with the same exponent. Thus, a^4 divided by a^4 equals a^{4-4}, or a^0.

PRIN. 1. *Any quantity whose exponent is zero is equal to unity.*

For, $a^4 \div a^4 = a^0$, by subtracting the exponents; but $a^4 \div a^4 = 1$, since any quantity divided by itself equals unity; hence, since a^0 and 1 are both equal to $a^4 \div a^4$, they are equal to each other. Therefore, $a^0 = 1$.

OPERATION.

$$a^4 \div a^4 = a^0$$
$$a^4 \div a^4 = 1$$
$$\therefore a^0 = 1$$

NOTE.—When a quantity whose exponent is zero is a factor of an algebraic expression, it may be omitted without changing the value of the expression, since its value is 1. It is sometimes retained to indicate the process by which the result was obtained.

94. A Negative Exponent originates in subtracting exponents when the exponent of the divisor is greater than the exponent of the dividend. Thus, $a^4 \div a^6 = a^{4-6}$, or a^{-2}.

PRIN. 2. *Any quantity with a negative exponent is equal to the reciprocal of the quantity with the sign of its exponent changed.*

For $a^4 \div a^6 = a^{-2}$; but $a^4 \div a^6 = \frac{a^4}{a^6}$, or, dividing both terms by a^4, is equal to $\frac{1}{a^2}$; and since a^{-2} and $\frac{1}{a^2}$ are both equal to $a^4 \div a^6$, they are equal to each other. Therefore, $a^{-2} = \frac{1}{a^2}$.

OPERATION.

$$a^4 \div a^6 = a^{-2}$$
$$a^4 \div a^6 = \frac{a^4}{a^6} = \frac{1}{a^2}$$
$$\therefore a^{-2} = \frac{1}{a^2}$$

PRIN. 3. *Any quantity is equal to the reciprocal of itself with the sign of its exponent changed.*

To prove this we must show that $a^n = \frac{1}{a^{-n}}$. By Prin. 2 we have $a^{-n} = \frac{1}{a^n}$; multiplying by a^n, we have $a^n \times a^{-n} = 1$; dividing by a^{-n}, we have $a^n = \frac{1}{a^{-n}}$.

OPERATION.

$$a^{-n} = \frac{1}{a^n}$$
$$a^n \times a^{-n} = 1$$
$$a^n = \frac{1}{a^{-n}}$$

NOTE.—It has already been seen that a *positive quantity* signifies *addition,* and a *negative* quantity *subtraction.* From the above principles it is also seen that a *positive exponent* implies *multiplication,* and a *negative exponent* implies *division.* Hence, a *negative* sign always denotes the *opposite* of a *positive* sign.

EXAMPLES.

1. Show by dividing a^5 by a^5 that $a^0 = 1$.
2. Show by dividing a^n by a^n that $a^0 = 1$.
3. Prove that $a^{-3} = \frac{1}{a^3}$.
4. Prove that $\frac{1}{a^5} = a^{-5}$.

5. Prove that $a^{-n} = \frac{1}{a^n}$

6. Find the value of a^2c^{-3}. *Ans.* $\frac{a^2}{c^3}$.

7. Find the value of $\frac{ab}{cx^{-4}}$ *Ans.* $\frac{abx^4}{c}$

8. Reduce to an integer $\frac{1}{a^{n-1}}$. *Ans.* a^{1-n}.

9. Reduce to an integer $\frac{1}{a^{n-m}}$. *Ans.* a^{m-n}.

10. Multiply $a^{-3}b^4c^{-5}$ by $a^2b^{-2}c^3$. *Ans.* $\frac{b^2}{ac^2}$.

11. Divide $18a^{-6}b^3c^2$ by $3a^{-4}b^{-2}$. *Ans.* $6\frac{b^5c^2}{a^2}$.

12. Divide $a^{-3n}-b^{6n}$ by $a^{-n}-b^{2n}$. *Ans.* $a^{-2n}+a^{-n}b^{2n}+b^{4n}$.

NOTE.—The pupil will readily infer that *negative exponents* may be used in operations in the *same way as positive exponents;* a generalization which is rigidly demonstrated in the *Theory of Exponents.*

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Addition. Sum. State the principles. The cases. The rule for each. Is the sum of two quantities ever less than the greater? When?

Define Subtraction. Minuend. Subtrahend. Remainder. State the principles of Subtraction. The cases. The rule for each. The principles of the parenthesis. Is the difference ever greater than the minuend? When?

Define Multiplication. Multiplicand. Multiplier. Product. State the principles of Multiplication. The cases. The rule for each. What is meant by *expanding expressions?* Why do we add exponents in multiplying? Why does plus into minus give minus? Why does minus into minus give plus?

Define Division. Dividend. Divisor. Remainder. State the principles. The cases. The rules. Why do we subtract exponents? Why does plus divided by minus give minus? Why does minus divided by minus give plus?

State the principles of the *Changes of Terms.* Of Zero exponent. Of Negative exponent. Origin of a Zero exponent. Of a Negative exponent.

SECTION III.

COMPOSITION AND FACTORING.

95. Composition is the process of forming *composite quantities.*

96. A **Composite Quantity** is one that is formed by the product of two or more quantities.

97. The **Square** of a quantity is the product obtained by using the quantity twice as a factor.

Composite quantities may be formed by actual multiplication, or, in several cases, by means of the following theorems.

NOTE.—In the fundamental operations each synthetic process has its corresponding analytic process. Thus, Addition is synthetic; Subtraction is analytic; Multiplication is synthetic; Division is analytic. It follows, therefore, that there should be a synthetic process corresponding to the analytic process of Factoring. This process I have called *Composition.* This new generalization, and the term applied to it, will, I trust, meet the approval of teachers and mathematicians.

THEOREM I.

The square of the sum of two quantities equals the square of the first, plus twice the product of the first and second, plus the square of the second.

Let a represent one of the quantities, and b the other; then $a+b$ will represent their sum. Now, $(a+b)^2$ equals $(a+b)(a+b)$, which we find by multiplying equals $a^2+2ab+b^2$. a^2 is the square of the first; $2ab$ is twice the product of the first and second; and b^2 is the square of the third. Therefore, etc.

OPERATION.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

EXAMPLES.

1. Square $x+y$. *Ans.* $x^2+2xy+y^2$.
2. Square $m+n$. *Ans.* $m^2+2mn+n^2$.
3. Square $2a+3b$. *Ans.* $4a^2+12ab+9b^2$.

5F