5. Prove that $a^{-n} = \frac{1}{a^n}$

6. Find the value of a^2c^{-3}. *Ans.* $\frac{a^3}{c^3}$

7. Find the value of $\frac{ab}{cx^{-4}}$ *Ans.* $\frac{abx^4}{c}$

8. Reduce to an integer $\frac{1}{a^{n-1}}$. *Ans.* a^{1-n}.

9. Reduce to an integer $\frac{1}{a^{n-m}}$ *Ans.* a^{m-n}.

10. Multiply $a^{-3}b^4c^{-5}$ by $a^2b^{-2}c^3$. *Ans.* $\frac{b^2}{ac^2}$.

11. Divide $18a^{-6}b^3c^2$ by $3a^{-4}b^{-2}$. *Ans.* $6\frac{b^5c^2}{a^2}$.

12. Divide $a^{-3n} - b^{6n}$ by $a^{-n} - b^{2n}$. *Ans.* $a^{-2n} + a^{-n}b^{2n} + b^{4n}$.

NOTE.—The pupil will readily infer that *negative exponents* may be used in operations in the *same way as positive exponents*; a generalization which is rigidly demonstrated in the *Theory of Exponents*.

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Addition. Sum. State the principles. The cases. The rule for each. Is the sum of two quantities ever less than the greater? When?

Define Subtraction. Minuend. Subtrahend. Remainder. State the principles of Subtraction. The cases. The rule for each. The principles of the parenthesis. Is the difference ever greater than the minuend? When?

Define Multiplication. Multiplicand. Multiplier. Product. State the principles of Multiplication. The cases. The rule for each. What is meant by *expanding expressions?* Why do we add exponents in multiplying? Why does plus into minus give minus? Why does minus into minus give plus?

Define Division. Dividend. Divisor. Remainder. State the principles. The cases. The rules. Why do we subtract exponents? Why does plus divided by minus give minus? Why does minus divided by minus give plus?

State the principles of the *Changes of Terms*. Of Zero exponent. Of Negative exponent. Origin of a Zero exponent. Of a Negative exponent.

SECTION III.

COMPOSITION AND FACTORING.

95. Composition is the process of forming *composite quantities*.

96. A **Composite Quantity** is one that is formed by the product of two or more quantities.

97. The **Square** of a quantity is the product obtained by using the quantity twice as a factor.

Composite quantities may be formed by actual multiplication, or, in several cases, by means of the following theorems.

NOTE.—In the fundamental operations each synthetic process has its corresponding analytic process. Thus, Addition is synthetic; Subtraction is analytic; Multiplication is synthetic; Division is analytic. It follows, therefore, that there should be a synthetic process corresponding to the analytic process of Factoring. This process I have called *Composition*. This new generalization, and the term applied to it, will, I trust, meet the approval of teachers and mathematicians.

THEOREM I.

The square of the sum of two quantities equals the square of the first, plus twice the product of the first and second, plus the square of the second.

Let a represent one of the quantities, and b the other; then $a+b$ will represent their sum. Now, $(a+b)^2$ equals $(a+b)(a+b)$, which we find by multiplying equals $a^2 + 2ab + b^2$. a^2 is the square of the first; $2ab$ is twice the product of the first and second; and b^2 is the square of the third. Therefore, etc.

OPERATION.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

EXAMPLES.

1. Square $x+y$. *Ans.* $x^2 + 2xy + y^2$.
2. Square $m+n$. *Ans.* $m^2 + 2mn + n^2$.
3. Square $2a+3b$. *Ans.* $4a^2 + 12ab + 9b^2$.

4. Square $3x+4y$. *Ans.* $9x^2+24xy+16y^2$.
5. Square $4p+6q$. *Ans.* $16p^2+48pq+36q^2$.
6. Square $A+B$. *Ans.* $A^2+2AB+B^2$.
7. Square x^2+y^2. *Ans.* $x^4+2x^2y^2+y^4$.
8. Square a^m+b^m. *Ans.* $a^{2m}+2a^mb^m+b^{2m}$.

THEOREM II.

The square of the difference of two quantities equals the square of the first, minus twice the product of the first and second, plus the square of the second.

Let $a-b$ represent the difference of two quantities; then $(a-b)^2$ will equal $(a-b)(a-b)$, which, by multiplying, we find is equal to $a^2-2ab+b^2$. a^2 is the square of the first; $2ab$ is twice the product of the first and second; and b^2 is the square of the second. Therefore, etc.

OPERATION.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

EXAMPLES.

1. Square $a-x$. *Ans.* $a^2-2ax+x^2$.
2. Square $c-d$. *Ans.* $c^2-2cd+d^2$.
3. Square $1-c$. *Ans.* $1-2c+c^2$.
4. Square a^2-3c. *Ans.* $a^4-6a^2c+9c^2$.
5. Square $3b^2-5d^3$. *Ans.* $9b^4-30b^2d^3+25d^6$.
6. Square $A-B$. *Ans.* $A^2-2AB+B^2$.
7. Square $4ax-2b^2$. *Ans.* $16a^2x^2-16ab^2x+4b^4$.
8 Square $7a^n-9b^n$. *Ans.* $49a^{2n}-126a^nb^n+81b^{2n}$.

THEOREM III.

The product of the sum and difference of two quantities equal the difference of their squares.

Let $a+b$ represent the sum and $a-b$ the difference of two quantities; then by multiplying we find their product to be a^2-b^2. a^2 is the square of the first, and b^2 is the square of the second. Therefore, etc.

OPERATION

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \quad -b^2 \end{array}$$

EXAMPLES.

1. Expand $(c+d)(c-d)$. *Ans.* c^2-d^2.
2 Expand $(m+n)(m-n)$. *Ans.* m^2-n^2.
3 Expand $(3a+2b)(3a-2b)$. *Ans.* $9a^2-4b^2$.
4. Expand $(5x-6y)(5x+6y)$. *Ans.* $25x^2-36y^2$.
5. Expand $(A+B)(A-B)$. *Ans.* A^2-B^2.
6. Expand $(1+3x)(1-3x)$. *Ans.* $1-9x^2$.
7. Expand $(4a^2-\frac{1}{3}c^3)(4a^2+\frac{1}{3}c^3)$. *Ans.* $16a^4-\frac{1}{9}c^6$.
8. Expand $(a^m+b^n)(a^m-b^n)$. *Ans.* $a^{2m}-b^{2n}$.
9. Expand $(a^{2m}+b^{3n})(a^{2m}-b^{3n})$. *Ans.* $a^{4m}-b^{6n}$.

THEOREM IV.

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the other two terms into the common term, and the product of the unlike terms.

Let $x+a$ and $x+b$ represent two such quantities; their product is $x^2+(a+b)x+ab$. x^2 is the square of the common term, $(a+b)x$ is the sum of the other two terms into the common term, and ab is the product of the unlike terms.

OPERATION.

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

EXAMPLES.

1. Expand $(x+4)(x+3)$. *Ans.* $x^2+7x+12$.
2. Expand $(x+5)(x-9)$. *Ans.* $x^2-4x-45$.
3. Expand $(a+m)(a+n)$. *Ans.* $a^2+a(m+n)+mn$.
4. Expand $(a-c)(a-b)$. *Ans.* $a^2-a(b+c)+bc$.
5. Expand $(2x-7)(2x-1)$. *Ans.* $4x^2-16x+7$.
6. Expand $(5ac+a)(5ac-c)$. *Ans.* $25a^2c^2+(a-c)5ac-ac$.
7. Expand $(3a^n-5)(3a^n+12)$. *Ans.* $9a^{2n}+21a^n-60$.
8. Expand $(2x^n+3a)(2x^n-5a)$. *Ans.* $4x^{2n}-4ax^n-15a^2$.
9. Expand $(2a^{\frac{n}{2}}-3a)(2a^{\frac{n}{2}}+7a)$. *Ans.* $4a^n+8a^{\frac{n+2}{2}}-21a^2$.

APPLICATIONS.

98. These principles may be applied with great convenience in finding the product of three or more binomials.

1. What is the value of $(a+2)(a+2)(a+2)$?

OPERATION.

$$\begin{array}{r} a^2+4a+4 \\ a+2 \\ \hline a^3+4a^2+4a \\ 2a^2+8a+8 \\ \hline a^3+6a^2+12a+8 \end{array}$$

SOLUTION. By Theorem I., $(a+2)(a+2)=a^2+4a+4$, which we can write without multiplying. Then, multiplying by $a+2$, we have $a^3+6a^2+12a+8$.

EXAMPLES.

2. Expand $(a+3)(a+3)(a+3)$. *Ans.* $a^3+9a^2+27a+27$.
3. Expand $(a+c)(a-c)(a+c)$. *Ans.* $a^3+a^2c-ac^2-c^3$.
4. Expand $(x-3)(x-3)(x-4)$. *Ans.* $x^3-10x^2+33x-36$.
5. Expand $(1-x)(1-x)(1+x)$. *Ans.* $1-x-x^2+x^3$.
6. Expand $(1+x)(1-x)(1-x^2)$. *Ans.* $1-2x^2+x^4$.
7. Expand $(x+y)(x-y)(x^2+y^2)$. *Ans.* x^4-y^4.
8. Expand $(3ac-4a^2b)(3ac+4a^2b)$. *Ans.* $9a^2c^2-16a^4b^2$.
9. Expand $2^3(a^2-x^2)(a^2+x^2)$. *Ans.* $8(a^4-x^4)$.
10. Expand $(3+\overline{m-n})(3-\overline{m-n})$. *Ans.* $9-(m-n)^2$.
11. Expand $(a+\overline{b-c})(a-\overline{b-c})$. *Ans.* $a^2-b^2+2bc-c^2$.
12. Expand $(5xz-6yz)(5xz+6yz)$. *Ans.* $25x^2z^2-36y^2z^2$.
13. Expand $(a+c)(a-c)(a^2-c^2)$. *Ans.* $a^4-2a^2c^2+c^4$.
14. Expand $(x-3)(x-4)(x+5)$. *Ans.* $x^3-2x^2-23x+60$.
15. Expand $(a+1)(a-1)(a-2)(a+2)$. *Ans.* a^4-5a^2+4.
16. Expand $(a^n-b^m)(a^n+b^m)(a^{2n}+b^{2m})$. *Ans.* $a^{4n}-b^{4m}$.

FACTORING.

99. Factoring is the process of resolving a composite quantity into its factors.

100. The **Factors** of a composite quantity are the quantities which multiplied together will produce it.

101. A **Prime Quantity** is one that cannot be produced by the multiplication of other quantities; as, 11, a^2+b^2.

102. Quantities are *prime to each other* when they have no common factor; as 5 and 9, $6a^2b$ and $11cd^2$.

103. The **Square Root** of a quantity is *one* of its *two equal* factors; thus, $2ab$ is the square root of $4a^2b^2$.

NOTE.—*Composition* and *Factoring* are the converse of each other; composition is *synthetic;* factoring is *analytic;* one is the putting together of the factors to make the number, and the other is the separating of the number into its factors.

CASE I.

104. To resolve a monomial into its prime factors.

1. Find the prime factors of $6a^2b^3$.

OPERATION.

$$\begin{array}{l} 6=2\times3 \\ a^2=a\times a \\ b^3=b\times b\times b \\ \hline 6a^2b^3=2\times3aabbb \end{array}$$

SOLUTION. The factors of 6 are 2 and 3; $a^2=aa$ and $b^3=bbb$; hence, $6a^2b^3$ equals $2\times3aabbb$.

Hence we have the following rule:

Rule.—*Resolve the numerical coefficient into its prime factors, and annex to it each letter written as many times as there are units in the exponent.*

EXAMPLES.

What are the prime factors—

2. Of $12a^3b$. *Ans.* $2\times2\times3aaab$.
3. Of $15a^2b^2c$. *Ans.* $3\times5aabbc$.
4. Of $18x^3y^2z$. *Ans.* $2\times3\times3xxxyyz$.
5. Of $27a^{2n}b^{3n}c$. *Ans.* $3\times3\times3a^na^nb^nb^nb^nc$.
6. Of $105a^{n+2}b^{n-2}c$. *Ans.* $3\times5\times7a^naab^nb^{-1}b^{-1}c$.
7. Of $72a^nb^{2n}c^{3n}$, when $n=1$. *Ans.* $2\times2\times2\times3\times3abbccc$.
8. Of $84a^{2n}c^{3n}d^{4n}$, when $n=2$. *Ans.* $2\times2\times3\times7aaaaccccccdddddddd$.

CASE II.

105. To find one of the two equal factors of a monomial.

1. Find one of the two equal factors of $25a^2b^4$.

SOLUTION. Resolving $25a^2b^4$ into its factors, we have $5\times 5aabbbb$. Since there are *two* 5's, one of the equal factors will contain *one* 5; since there are *two* a's, the equal factor will contain *one* a; since there are *four* b's, the equal factor will contain *two* b's; hence the factor is $5abb$, or $5ab^2$. This, we see, is equivalent to extracting the square root of the coefficient, and dividing the exponents of the letters by 2.

OPERATION.

$25a^2b^4=$
$5\times 5aabbbb$
$5abb=5ab^2$

Rule.—*Extract the square root of the coefficient and divide the exponents of the letters by 2.*

EXAMPLES.

Find one of the two equal factors—

2. Of $4a^4b^2$. *Ans.* $2a^2b$.
3. Of $9a^4b^6$. *Ans.* $3a^2b^3$.
4. Of $16x^6y^2$. *Ans.* $4x^3y$.
5. Of $36c^8d^4$. *Ans.* $6c^4d^2$.
6. Of $\frac{1}{16}x^6y^8z^{10}$. *Ans.* $\frac{1}{4}x^3y^4z^5$.
7. Of $25a^{4n}b^{6n}$. *Ans.* $5a^{2n}b^{3n}$.
8. Of $\frac{1}{64}x^{8a}z^{12c}$. *Ans.* $\frac{1}{8}x^{4a}z^{6c}$.
9. Of $81(a+b)^4$. *Ans.* $9(a+b)^2$.
10. Of $1764(a-x)^6$. *Ans.* $42(a-x)^3$.

CASE III.

106. To resolve a polynomial into two factors one of which is a monomial.

1. Find the factors of $2ac-4ab$.

SOLUTION. We see that $2a$ is a factor common to all the terms; hence, dividing $2ac-4ab$ by $2a$ we find the other factor to be $c-2b$; hence, $2ac-4ab=2a(c-2b)$.

OPERATION.

$2a)\underline{2ac-4ab}$
$c-2b$
$2a(c-2b)$

Rule.—*Divide the polynomial by the greatest factor common to all the terms, enclose the quotient in a parenthesis, and prefix the divisor as a coefficient.*

EXAMPLES.

Find the factors—

2. Of $6a^2b+9a^3c$. *Ans.* $3a^2(2b+3ac)$.
3. Of $8a^3b^2-12ab^4$. *Ans.* $4ab^2(2a^2-3b^2)$.
4. Of $14ax^2z+56abx^2$. *Ans.* $7ax^2(2z+8b)$.
5. Of $ac^2-ba^2c^3+adc^4$. *Ans.* $ac^2(1-abc+dc^2)$.
6. Of $ax^2y-a^2xy^3+dx^3y$. *Ans.* $xy(ax-a^2y^2+dx^2)$.

CASE IV.

107. To resolve a trinomial into two equal binomial factors.

1. Factor $a^2+2ab+b^2$.

SOLUTION. a^2 is the square of a, and b^2 is the square of b, and since $2ab$ is twice the product of a and b, the trinomial is the square of $(a+b)$, (Theo. I.); hence, $a+b$ is one of the two equal factors of $a^2+2ab+b^2$.

OPERATION.

$a^2+2ab+b^2=$
$(a+b)(a+b)$

Rule.—*Extract the square root of the terms which are squares, and if twice the product of these roots equals the other term, these roots, connected by the sign of this other term, will be one of the equal factors.*

EXAMPLES.

Find one of the two equal factors—

2. Of $a^2-2ab+b^2$. *Ans.* $a-b$.
3. Of $x^2+2xy+y^2$. *Ans.* $x+y$.
4. Of $A^2-2AB+B^2$. *Ans.* $A-B$.
5. Of $4a^2-12ac+9c^2$. *Ans.* $2a-3c$.
6. Of $9m^2+12mn+4n^2$. *Ans.* $3m+2n$.
7. Of $1-2c^2+c^4$. *Ans.* $1-c^2$.
8. Of $16a^{2n}+40a^nc^n+25c^{2n}$. *Ans.* $4a^n+5c^n$.

CASE V.

108. To resolve a binomial consisting of the difference of two squares into its binomial factors.

1. Find the factors of a^2-b^2.

SOLUTION. The difference of the squares of two quantities equals the product of their sum and difference, (Theo. III.); hence, $a^2-b^2=(a+b)$ multiplied by $(a-b)$.

OPERATION.

$a^2-b^2=$
$(a+b)(a-b)$

Rule.—*Take the square root of each term, and make their sum one factor and their difference the other factor.*

EXAMPLES.

2. Factor a^2-c^2. *Ans.* $(a+c)(a-c)$.
3. Factor c^2-4d^2. *Ans.* $(c+2d)(c-2d)$.
4. Factor $4x^2-9y^2$. *Ans.* $(2x+3y)(2x-3y)$.
5. Factor $a^2x^2-b^2z^2$. *Ans.* $(ax+bz)(ax-bz)$.
6. Factor $\frac{1}{4}x^2-\frac{1}{9}y^2z^4$. *Ans.* $(\frac{1}{2}x+\frac{1}{3}yz^2)(\frac{1}{2}x-\frac{1}{3}yz^2)$.
7. Factor $9a^{2n}-16c^{4n}$. *Ans.* $(3a^n+4c^{2n})(3a^n-4c^{2n})$.
8. Factor a^4-c^4. *Ans.* $(a^2+c^2)(a+c)(a-c)$.
9. Factor $x^2y^2-y^2$. *Ans.* $y^2(x+1)(x-1)$.
10. Factor a^8-b^8. *Ans.* $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.

CASE VI.

109. To resolve a trinomial into two unequal binomial factors.

1. Resolve $a^2+5ac+6c^2$ into its factors.

SOLUTION. The first term of each factor is evidently a; the second terms must be $2c$ and $3c$, since their product will be $6c^2$ and their sum $5c$. (Theo. IV.)

OPERATION.

$a^2+5ac+6c^2=$
$(a+2c)(a+3c)$

Rule.—*Take the square root of one term for the first term of each factor, and for the second term take such quantities that their product will equal the third term of the trinomial, and their sum into the first term of the factor will equal the second term of the trinomial.*

EXAMPLES.

2. Factor x^2+3x+2. *Ans.* $(x+1)(x+2)$.
3. Factor a^2+5a+6. *Ans.* $(a+2)(a+3)$.
4. Factor x^2-x-2. *Ans.* $(x+1)(x-2)$.
5. Factor a^2-a-2. *Ans.* $(a-2)(a+1)$.
6. Factor $x^2+7x+12$. *Ans.* $(x+3)(x+4)$.
7. Factor $a^2-3a-10$. *Ans.* $(a-5)(a+2)$.
8. Factor x^4-9x^2-36. *Ans.* $(x^2+3)(x^2-12)$.
9. Factor $4x^2-6x-40$. *Ans.* $(2x+5)(2x-8)$.
10. Factor $a^2+4ac-21c^2$. *Ans.* $(a-3c)(a+7c)$.
11. Factor $a^{2n}+5a^n-84$. *Ans.* $(a^n-7)(a^n+12)$.

CASE VII.

110. To resolve a quadrinomial into two binomial factors.

111. When two binomials contain a common term their product will be a trinomial; when the terms are dissimilar the product will be a quadrinomial.

1. Factor $ac+bc+ad+bd$.

SOLUTION. $ac+bc$ equals $(a+b)c$, and $ad+bd$ equals $(a+b)d$; now, c times $(a+b)$ plus d times $(a+b)$ equals $(c+d)$ times $(a+b)$, or $(a+b)(c+d)$.

OPERATION.

$ac+bc+ad+bd=$
$(a+b)c+(a+b)d=$
$(a+b)(c+d)$

Rule.—*Factor each two terms which will give a common binomial factor, and then enclose the sum of the monomial factors in a parenthesis, and write it as the coefficient of the common binomial factor.*

EXAMPLES.

2. Factor $ab+ay+bx+xy$. *Ans.* $(a+x)(b+y)$.
3. Factor $ac-bc+ad-bd$. *Ans.* $(a-b)(c+d)$.
4. Factor $ax+bx-ay-by$. *Ans.* $(a+b)(x-y)$.
5. Factor $ab+b-2a-2$. *Ans.* $(a+1)(b-2)$.

6. Factor $ac-2bc-3ad+6bd$. *Ans.* $(a-2b)(c-3d)$.

7. Factor $a^2c^2-b^2c^2+a^2d^2-b^2d^2$. *Ans.* $(a^2-b^2)(c^2+d^2)$.

8. Factor $a^nx^n-b^nx^n+a^ny^n-b^ny^n$. *Ans.* $(a^n-b^n)(x^n+y^n)$.

CASE VIII.

112. To factor any binomial consisting of two equal powers of two quantities.

This case relates to binomials of the form of a^2-b^2, a^3-b^3, a^4-b^4, etc.; also a^3+b^3, a^5+b^5, etc.

THEOREM I.

The difference of the same powers of two quantities is divisible by the difference of the two quantities.

Let a and b be any quantities, then $a-b$ will be their difference, and a^n-b^n will be the difference of the same power of the quantities; then will a^n-b^n be divisible by $a-b$. Divide a^n-b^n by $a-b$ until we obtain two remainders.

OPERATION.

$$\begin{array}{rl} & a^n-b^n \quad \lfloor a-b \\ & \underline{a^n-a^{n-1}b} \quad a^{n-1}+a^{n-2}b \\ \text{1st Rem.} = & a^{n-1}b-b^n \\ & \underline{a^{n-1}b-a^{n-2}b^2} \\ \text{2d Rem.} = & a^{n-2}b^2-b^n \\ n\text{th Rem.} = & a^{n-n}b^n-b^n \\ = & a^0b^n-b^n \\ = & 1b^n-b^n=0 \end{array}$$

Now, if the division terminates, some remainder will reduce to zero; and if we obtain an expression for the nth remainder, we will find that its value is 0. Let us then find the nth remainder.

The *last term* of the nth remainder is evidently $-b^n$. In the 1st remainder the *first term* is $a^{n-1}b$; in the 2d remainder it is $a^{n-2}b^2$; in the 3d remainder it is $a^{n-3}b^3$; hence, in the nth remainder it is $a^{n-n}b^n$; hence, the nth remainder is $a^{n-n}b^n-b^n$, which equals $a^0b^n-b^n$, or $1b^n-b^n$, or 0. Hence, the nth remainder is 0, and the division terminates.

NOTE.—The latter part may be given as follows: If $n=3$, the 3d remainder becomes $a^{3-3}b^3-b^3$, or $a^0b^3-b^3$, or $1b^3-b^3=0$. If $n=4$, the 4th remainder will also reduce to 0; hence, the nth remainder is always equal to 0, and since the remainder is zero, the division is exact.

THEOREM II.

The difference of the same even powers of two quantities is divisible by the sum of the quantities.

Let a^n-b^n be the difference of the same even powers of a and b, n being *even*. Now, a^n-b^n is divisible by $(a-b)$ (Theo. I.). Let $b=-c$; then, substituting, we shall have $[a^n-(-c)^n]\div[a-(-c)]$. But $(-c)^n=c^n$, since n is even, and $[a-(-c)]=a+c$; hence we have a^n-c^n is divisible by $a+c$. Therefore, etc.

OPERATION

$$(a^n-b^n)\div(a-b)$$
$$[a^n-(-c)^n]\div[a-(-c)]$$
$$(a^n-c^n)\div(a+c)$$

NOTE.—This theorem may also be demonstrated independently in a manner similar to Theo. I.

THEOREM III.

The sum of the same odd powers of two quantities is divisible by the sum of the quantities.

Let a^n-b^n be the difference of the same odd powers of a and b; n being odd. Now, a^n-b^n is divisible by $a-b$ (Theo. I.). Let $b=-c$; then, substituting, we shall have $a^n-b^n=a^n-(-c)^n$ and $a-b=a-(-c)$. Now, $(-c)^n$ equals $-c^n$, since n is odd; hence, $a^n-(-c)^n=a^n+c^n$; and $a-(-c)=a+c$; hence, a^n+c^n is divisible by $a+c$.

OPERATION.

$$(a^n-b^n)\div(a-b)=$$
$$[a^n-(-c^n)]\div[a-(-c)]$$
$$(a^n+c^n)\div(a+c)$$

NOTE.—This may also be demonstrated independently in a manner similar to Theo. I.

EXAMPLES.

1. Factor a^3-b^3. *Ans.* $(a-b)(a^2+ab+b^2)$.
2. Factor a^4-c^4. *Ans.* $(a^2+c^2)(a+c)(a-c)$.
3. Factor a^5-x^5. *Ans.* $(a-x)(a^4+a^3x+a^2x^2+ax^3+x^4)$.
4. Factor a^3+b^3. *Ans.* $(a+b)(a^2-ab+b^2)$.
5. Factor a^5+b^5. *Ans.* $(a+b)(a^4-a^3b$, etc.$)$.
6. Factor a^3-8x^3. *Ans.* $(a-2x)(a^2+2ax+4x^2)$.
7. Factor $8a^3+27c^3$. *Ans.* $(2a+3c)(4a^2-6ac+9c^2)$.
8. Factor a^6-b^6. *Ans.* $(a^3+b^3)(a^3-b^3)=$ $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

NOTE.—Other factors of the 8th problem may be found; thus, uniting the 1st and 3d, and the 2d and 4th of the last expression, we have $(a^2-b^2)(a^4+a^2b^2+b^4)$.

GREATEST COMMON DIVISOR.

113. A **Common Divisor** of two or more quantities is a quantity that will exactly divide each of them.

114. The **Greatest Common Divisor** of two or more quantities is the greatest quantity that will exactly divide each of them.

Principle.— *The greatest common divisor of two or more quantities equals the product of all their common prime factors.*

CASE I.

115. To find the greatest common divisor of quantities by factoring.

1. Find the greatest common divisor of $12a^3b^2c$ and $18a^2bd$.

Solution. Resolving the quantities into their factors, we perceive that 2, 3, a^2, and b are all of the common factors; hence the greatest common divisor is $2\times3\times a^2\times b=6a^2b$.

OPERATION.

$$12a^3b^2c=2\times2\times3a^3b^2c$$
$$18a^2bd=2\times3\times3a^2bd$$
$$\text{G. C. D.}=2\times3a^2b=6a^2b$$

Rule.—*Resolve the quantities into their prime factors, and take the product of all the common factors.*

EXAMPLES.

Find the greatest common divisor—

2. Of $18a^3x^2$ and $24a^4x^3c$. *Ans.* $6a^3x^2$.
3. Of $16a^5b^3z$ and $24a^2b^5z^4$. *Ans.* $8a^2b^3z$.
4. Of $15a^6b^7x^3z^2$ and $25a^2b^5xz^7$. *Ans.* $5a^2b^5xz^2$.
5. Of $28a^nc^{2m}z^5$, $35a^{3n}c^mx^2$, and $14a^{2n}c^{3m}x^2z^4$. *Ans.* $7a^nc^m$.
6. Of a^2-b^2 and $a^2-2ab+b^2$. *Ans.* $a-b$.
7. Of $a^2+2ab+b^2$ and a^2-b^2. *Ans.* $a+b$.
8. Of a^3-b^3 and $a^2-2ab+b^2$. *Ans.* $a-b$.
9. Of a^4-b^4 and a^6-b^6. *Ans.* a^2-b^2.
10. Of x^2-y^2 and $ax+ay+bx+by$. *Ans.* $x+y$.
11. Of $ac+bc+ad+bd$ and $ac-bd-ad+bc$. *Ans.* $a+b$

Note.—Young pupils may omit the next case of Greatest Common Divisor until review.

CASE II.

116. To find the greatest common divisor of polynomials when the common factors are not readily seen.

Prin. 1. *A divisor of a quantity is a divisor of any number of times that quantity.*

Thus, c^2, which is a divisor of bc^3, is a divisor of a times bc^3, or abc^3.

Prin. 2. *A common divisor of two quantities is a divisor of their sum and their difference.*

For, take the two quantities ac^2 and bc^2, in which the common divisor is c^2; their sum is ac^2+bc^2, or $(a+b)c^2$, which is divisible by c^2; their difference is ac^2-bc^2, or $(a-b)c^2$, which is also divisible by c^2. Therefore, etc.

Prin. 3. *Either of two quantities may be multiplied or divided by a factor not found in the other, without changing their greatest common divisor.*

For, it neither introduces nor omits a common factor, and hence cannot affect the greatest common divisor. Thus, the greatest common divisor of $2a^3c^2$ and $3bc^2$ is c^2. Multiply the first by a, or divide the second by b, and the greatest common divisor is still c^2.

1. Find the greatest common divisor of any two quantities A and B.

Solution. Divide A by B, indicating the quotient by Q, and the remainder by R; divide B by R, indicating the quotient by Q', and the remainder by R'; divide R by R', indicating the quotient by Q'', and thus continue. Then any remainder which exactly divides the preceding divisor will be the greatest common divisor of A and B. For,

OPERATION.

$$A \,\lfloor B \qquad A=QB+R$$
$$QB \quad Q \qquad B=Q'R+R'$$
$$A-QB=R$$
$$B \,\lfloor R$$
$$Q'R \quad Q'$$
$$B-Q'R=R'$$
$$R \,\lfloor R'$$
$$Q''$$

First. *Each remainder is a number of times the greatest common divisor.* A and B are each a number of times the G. C. D.; hence, $A-QB$ or R is a number of times the G. C. D. (Prin. 2); and since B and R are each a number of times the G. C. D., for the same reason $B-Q'R$ or R' is a number of times the G. C. D. Hence, each remainder is a number of times the G. C. D.

Second. *The last divisor will divide A and B.* Suppose that R' divides R, then it will divide $Q'R$ (Prin. 1), and also $Q'R+R'$ or B (Prin. 2);

and since it divides B, it will also divide QB (Prin. 1), and $QB+R$, or A (Prin. 2). Hence, the last divisor will divide A and B.

Therefore, since the last divisor divides A and B, and is a *number of times* the G. C. D., it must be *once* the G. C. D. Hence, the remainder which exactly divides the previous divisor is the G. C. D. of A and B.

NOTE.—The latter part of the solution may be given as follows: Since the last divisor divides A and B, it cannot be *greater* than the G. C. D., and since it is a number of times the G. C. D., it cannot be *less* than the G. C. D.; hence, since it is neither *greater* nor *less* than the G. C. D., it must be the G. C. D.

Rule.—*Divide the greater quantity by the less, the divisor by the remainder, and thus continue to divide the last divisor by the last remainder until there is no remainder; the last divisor will be the greatest common divisor.*

NOTES.—1. When the highest power of the leading letter is the same in both quantities, either quantity may be made the dividend.

2. If both quantities contain a common factor, it may be set aside, and afterward inserted in the greatest common divisor of the other parts.

3. If either quantity contains a factor not found in the other, it may be canceled before beginning the operation. (Prin. 3.)

4. When necessary, the dividend may be multiplied by any quantity not a factor of the divisor, which will render the first term divisible by the first term of the divisor. (Prin. 3.)

5. If we obtain a remainder which does not contain the leading letter, there is no common divisor.

6. When there are more than two quantities, find the G. C. D. of two, then of that divisor and the third quantity, etc. The last divisor will be the greatest common divisor.

2. Find the greatest common divisor of a^2-b^2 and $a^2-2ab+b^2$

SOLUTION. Dividing $a^2-2ab+b^2$ by a^2-b^2, we have a quotient of 1, and a remainder of $-2ab+2b^2$. Rejecting the factor $-2b$, which does not affect the greatest common divisor (Prin. 3), we have $a-b$. Dividing a^2-b^2 by $a-b$, we have a quotient $a+b$, with no remainder. Hence the last divisor, $a-b$, is the greatest common divisor of a^2-b^2 and $a^2-2ab+b^2$.

OPERATION.

$a^2-b^2)\ a^2-2ab+b^2\ (1$
a^2-b^2
$-2ab+2b^2$ Rejecting the factor $-2b$

$a-b)\ a^2-b^2\ (a+b$
a^2-ab
$ab-b^2$
$ab-b^2$

EXAMPLES.

3. Find the greatest common divisor of a^2-b^2 and $a^2+2ab+b^2$.
Ans. $a+b$.

4. Find the greatest common divisor of $2x^3-5x^2+3x$ and $4x^2-2x-2$.
Ans. $x-1$.

5. Find the greatest common divisor of a^3-x^3 and a^2-x^2.
Ans. $a-x$.

6. Find the greatest common divisor of $ab+bc+ad+dc$ and a^2-c^2.
Ans. $a+c$.

7. Find the greatest common divisor of a^3+x^3 and a^2-x^2.
Ans. $a+x$.

8. Find the greatest common divisor of $2ax^2-2ay^2$ and $4ax^3+4ay^3$.
Ans. $2a(x+y)$.

9. Find the greatest common divisor of a^4-b^4 and $a^3+a^2b-ab^2-b^3$.
Ans. a^2-b^2.

10. Find the greatest common divisor of x^3-x^2-12x and $x^2-4x-21$.
Ans. $x+3$.

THE LEAST COMMON MULTIPLE.

117. A **Multiple** of a quantity is any quantity of which it is a factor.

118. A **Common Multiple** of two or more quantities is a quantity which is a multiple of each of them.

119. The **Least Common Multiple** of two or more quantities is the least quantity which is a multiple of each of them.

NOTE.—The primary idea of a multiple is that of a number of times a quantity. The above definition is based upon a derivative truth. Another derivative definition is, a multiple of a quantity is any quantity which it will exactly divide.

PRIN. 1. *A multiple of a quantity contains all the factors of the quantity.*

Thus, it is evident that a multiple of ab, as $4a^2b$, must contain all the factors of ab.

PRIN. 2. *A common multiple of two or more quantities contains all the factors of each quantity.*

Thus, it is evident that $12a^2bc$, which is a common multiple of $2ab$ and $3ac$, contains all the factors of $2ab$ and $3ac$.

PRIN. 3. *The least common multiple of two or more quantities contains all the factors of each quantity, and no other factors.*

Thus, it is evident that the least common multiple of $2ab$ and $3ac$, which is $6abc$, must contain all the factors of $2ab$ and $3ac$, or it would not contain the quantities; and it must contain no other factors, or else it would not be the least common multiple.

CASE I.

120. To find the least common multiple by factoring.

1. Find the least common multiple of $12a^3bc^2$ and $18a^2bd^2$.

OPERATION.

$$12a^3bc^2 = 2\times2\times3a^3.b.c^2$$
$$18a^2bd^2 = 2\times3\times3a^2.b.d^2$$
$$\text{L. C. M} = 2\times2\times3\times3a^3.b.c^2.d^2 = 36a^3bc^2d^2$$

SOLUTION. We first resolve the quantities into their prime factors.

The least common multiple must contain all the different prime factors, and no others (Prin. 3). All the different prime factors are 2, 2, 3, 3, a^3, b, c^2, d^2, whose product equals $36a^3bc^2d^2$. Hence, the least common multiple of the given quantities is $36a^3bc^2d^2$.

Rule.—*Resolve the quantities into their prime factors, and take the product of all the different factors, using each factor the greatest number of times it appears in either quantity.*

EXAMPLES.

Find the least common multiple—

2. Of $18a^5b^3$ and $24a^3b^4c^3$. *Ans.* $72a^5b^4c^3$.
3. Of $24ax^3z^4$ and $42a^2x^2y^2z$. *Ans.* $168a^2x^3y^2z^4$.
4. Of $18a^2b$, $24ab^3c$ and $27a^2c^3z$. *Ans.* $216a^2b^3c^3z$.
5. Of $2(a+x)$ and (a^2-x^2). *Ans.* $2(a^2-x^2)$.
6. Of $a(b-c)$ and $b(b^2-c^2)$. *Ans.* $ab(b^2-c^2)$.
7. Of (a^2-b^2) and $a^2-2ab+b^2$. *Ans.* $a^3-a^2b-ab^2+b^3$.

8. Of $a^2(a-z)$ and $x^2(a^2-z^2)$. *Ans.* $a^2x^2(a^2-z^2)$.
9. Of $3x^2(2a-1)$ and $4xy(4a^2-1)$. *Ans.* $12x^2y(4a^2-1)$.
10. Of x^2-y^2 and x^3-y^3. *Ans.* $x^4-xy^3+x^3y-y^4$.
11. Of $3a(a-b)$, $4ac(a^2-b^2)$ and $6c^2x(a+b)$. *Ans.* $12ac^2x(a^2-b^2)$.
12. Of $m^2+2mn+n^2$ and m^3+n^3. *Ans.* $(m^3+n^3)(m+n)$.

NOTE.—Young pupils can omit the next case of Least Common Multiple until review.

CASE II.

121. To find the least common multiple when the quantities are not readily factored.

The method will be readily understood from the following principle:

PRINCIPLE.—*The least common multiple of two quantities equals either quantity multiplied by the quotient of the other quantity divided by their greatest common divisor.*

OPERATION.

$$A = a\times c$$
$$B = b\times c$$
$$\text{L. C. M.} = a\times b\times c = \frac{A}{c}\times B$$

For, let A and B be any two quantities, and let their greatest common divisor be represented by c, and the other factors by a and b, respectively; then we shall have the L. C. M. $= a\times b\times c$, Case I.; but $b\times c = B$, and $a=\frac{A}{c}$; hence, L. C. M. $=\frac{A}{c}\times B$. Therefore, etc.

NOTE.—The least common multiple of two quantities equals their product divided by their greatest common divisor, for $\frac{A}{c}\times B = \frac{A\times B}{c}$.

1. Find the least common multiple of a^2-b^2 and $a^2-3ab+2b^2$.

OPERATION.

$$\text{G. C. D.} = a-b$$
$$\text{L. C. M.} = (a^2-b^2)\times\frac{a^2-3ab+2b^2}{a-b} = (a^2-b^2)(a-2b) = a^3-2a^2b-ab^2+2b^3$$

SOLUTION. We first find the greatest common divisor to be $a-b$. Then the L. C. M. equals a^2-b^2 multiplied by the quotient of $a^2-3ab+2b^2$ divided by $a-b$, or $(a^2-b^2)(a-2b)$, which equals $a^3-2a^2b-ab^2+2b^3$.

Rule.—I. *Find the greatest common divisor of the two quantities; divide one quantity by it, and multiply the other quantity by the quotient.*

II. *When there are more than two quantities, find the least common multiple of two of them, then of this multiple and the third quantity,* etc.

EXAMPLES.

2. Find the least common multiple of x^2-x-12 and $x^2-4x-21$.
Ans. $x^3-8x^2-5x+84$.

3. Find the least common multiple of x^2+5x+6 and x^2+6x+8.
Ans. $x^3+9x^2+26x+24$.

4. Find the least common multiple of $a^2+4ab+3b^2$ and a^2-b^2.
Ans. $a^3+3a^2b-ab^2-3b^3$.

5. Find the least common multiple of $a^2+3ab+2b^2$ and $a^2-ab-6b^2$.
Ans. $a^3-7ab^2-6b^3$.

6. Find the least common multiple of $x^2-ax+3x-3a$ and $x^2-3x-ax+3a$.
Ans. $x^3-ax^2-9x+9a$.

7. Find the least common multiple of x^2-x-2, x^2+3x+2 and x^2+5x+4.
Ans. $x^4+5x^3-20x-16$.

REVIEW QUESTIONS.

Define Composition. A Composite Quantity. State the relation of Composition to Factoring. State the four theorems of Composition.

Define Factoring. Factors. A Prime Quantity. Quantities prime to each other. State each case. Give the rule for each case. State the three theorems of Factoring.

Define Common Divisor. Greatest Common Divisor. State the cases. The principles. The rules. Define a Multiple. A Common Multiple. The Least Common Multiple. State the cases. The principles. The rules.

SECTION IV.

FRACTIONS.

122. A **Fraction** is a number of the equal parts of a unit.

123. A **Fraction** in Algebra is expressed by two quantities, one above the other, with a straight line between them; as, $\frac{a}{c}$ or $\frac{a-x}{a-z}$.

124. The **Denominator** of a fraction denotes the number of equal parts into which the unit is divided. It is written below the line.

125. The **Numerator** of a fraction denotes the number of equal parts taken. It is written above the line.

126. An **Entire Quantity** is one that has no fractional part; as, $2a^2$, $a+b$, etc.

127. A **Mixed Quantity** is one that has both an entire and a fractional part; as, $a+\frac{b}{c}$, $ax-\frac{2ac}{m+n}$.

128. An **Algebraic Fraction** is usually regarded as the expression of one quantity divided by another; thus, $\frac{a}{c}$ means a divided by c, etc.

PRINCIPLES OF FRACTIONS.

129. The **Principles of Fractions** are general laws showing the relation of the value of the fraction to its numerator and denominator.

PRIN. 1. *Multiplying the numerator or dividing the denominator of a fraction by any quantity multiplies the value of the fraction by that quantity.*