had 4 less she would have $\frac{1}{3}$ as many as Jane; how many has each? *Ans.* Mary, 5; Jane, 3.

7. A bill of £120 was paid in guineas (21s.) and moidores (27s.), and the number of pieces of both kinds was just 100; required the number of pieces of each kind. *Ans.* 50.

8. A boy expends 30 cents for apples and pears, buying his apples at 4 and his pears at 5 for a cent, and afterward accommodates his friend with half his apples and one-third of his pears for 13 cents at the price paid; how many did he buy of each? *Ans.* Apples, 72; pears, 60.

9. A person rents 25 acres of land for £7 12s. per annum: for the better part he receives 8s. an acre, and for the poorer part, 5s. an acre; required the number of acres of each sort. *Ans.* 9; 16.

10. If 9 be added to a number consisting of two digits, the two digits will change places, and the sum of the two numbers will be 33; what is the number? *Ans.* 12.

11. Said A to B, "If you will give me $20 of your money, I will have twice as much as you have left;" but says B to A, "If you will give me $20 of your money, I will have thrice as much as you have left;" how much had each? *Ans.* A, $44; B, $52.

12. A and B laid a wager of $20: if A loses, he will have as much as B will then have; if B loses, he will have half of what A will then have; find the money of each. *Ans.* A, $140; B, $100.

13. There is a number consisting of two figures which is double the sum of its digits; and if 9 be subtracted from 5 times the number, the digits will be inverted; what is the number? *Ans.* 18.

14. Several persons engaged a boat for sailing: if there had been 3 more, they would have paid $1 each less than they did; if there had been 2 less, they would have paid $1 each more than they did; required the number of persons and what each paid. *Ans.* 12 persons; $5 each.

15. A and B ran a race which lasted five minutes: B had a start of 20 yards, but A ran 3 yards while B was running 2, and won by 30 yards; find the length of the course and the speed of each. *Ans.* 150 yds.; A 30 yds. and B 20 yds. a min.

16. If a certain rectangular field were 10 feet longer and 5 feet broader, it would contain 400 square feet more; but if it were 5 feet longer and 10 feet broader, it would contain 450 square feet more; required its length and breadth. *Ans.* 30 ft.; 20 ft.

17. A market-woman bought eggs—some at 3 for 7 cents, and some at two for 5 cents, paying $2.62 for the whole; she afterward sold them at 36 cents a dozen, clearing 62 cents; how many of each kind did she buy? *Ans.* 48; 60.

18. A and B ran a mile, A giving B a start of 20 yards at the first heat, and beating him 30 seconds; at the second heat A gives B a start of 32 seconds, and beats him $9\frac{5}{11}$ yards; at what rate per hour does A run? *Ans.* 12 miles.

SIMPLE EQUATIONS,

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

190. Equations containing three or more unknown quantities may be solved by either of the methods of elimination already explained.

1. Given $\left\{\begin{array}{l} x+y+z=6 \\ x+2y+3z=14 \\ 2x+3y+z=11 \end{array}\right\}$ to find x, y and z.

OPERATION.

$$\begin{array}{rr} x+y+z=6 & (1) \\ x+2y+3z=14 & (2) \\ 2x+3y+z=11 & (3) \\ \hline y+2z=8 & (4) \\ y-z=-1 & (5) \\ \hline 3z=9 & (6) \\ z=3 & (7) \\ y+6=8 & (8) \\ y=2 & (9) \\ x+2+3=6 & (10) \\ x=1 & (11) \end{array}$$

SOLUTION. Subtracting equation (1) from equation (2), we have equation (4). Multiplying (1) by 2, and subtracting from (3), we have (5); subtracting (5) from (4), we have (6); dividing (6) by 3, we have $z=3$; substituting the value of z in (4), we have 8. Transposing, we have $y=2$; substituting the values of z and y in (1), we have (10): from which we have $x=1$.

Rule.—I. *Eliminate successively the same unknown quantity from each of the equations; this will give a number of equations and of unknown quantities one less than the given number.*

II. *In the same way eliminate one of the unknown quantities from each of the derived equations, and thus continue until an equation is found containing one unknown quantity, and then find the value of the unknown quantity in this last equation.*

III. *Substitute this value in one of the equations containing two unknown quantities, and find the value of a second; substitute these values in an equation containing three unknown quantities, and find the value of a third, and thus continue until all are found.*

NOTES.—1. The pupil will exercise his judgment which quantity to eliminate first, and also what method of elimination to use.

2. When one of the equations contains but one or two of the unknown quantities, the method of substitution will often be found the shortest.

EXAMPLES.

2. Given $\left\{\begin{array}{l} x+2y+z=13 \\ x+y+3z=15 \\ 2x+3y+2z=22 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=4, \\ z=3. \end{array}\right.$

3. Given $\left\{\begin{array}{l} 2x+4y+3z=35 \\ x+3y+2z=23 \\ 3x-5y+4z=17 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=3, \\ z=5. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=7, \\ z=4. \end{array}\right.$

5. Given $\left\{\begin{array}{l} 2x+3y-z=27 \\ 3x-4y+3z=12 \\ 4x+2y-5z=15 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=7, \\ y=6, \\ z=5. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=12, \\ y=20, \\ z=30. \end{array}\right.$

7. Given $\left\{\begin{array}{l} x+y+z=24 \\ x-y+z=8 \\ x+y-z=6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=7, \\ y=8, \\ z=9 \end{array}\right.$

8. Given $\left\{\begin{array}{l} x+y+z=a \\ x+y-z=b \\ x-y+z=c \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=\frac{1}{2}(b+c), \\ y=\frac{1}{2}(a-c), \\ z=\frac{1}{2}(a-b). \end{array}\right.$

9. Given $\left\{\begin{array}{l} x+y=a \\ x+z=b \\ y+z=c \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{1}{2}(a+b-c), \\ y=\frac{1}{2}(a+c-b), \\ z=\frac{1}{2}(b+c-a). \end{array}\right.$

10. Given $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x}+\frac{1}{z}=6 \\ \frac{1}{y}+\frac{1}{z}=7 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{1}{2}, \\ y=\frac{1}{3}, \\ z=\frac{1}{4}. \end{array}\right.$

11. Given $\left\{\begin{array}{l} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x}{a}+\frac{z}{c}=1 \\ \frac{y}{b}+\frac{z}{c}=1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{a}{2}, \\ y=\frac{b}{2} \\ z=\frac{c}{2}. \end{array}\right.$

12. Given $\left\{\begin{array}{l} x+y-z=c \\ x+z-y=b \\ y+z-x=a \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{1}{2}(b+c), \\ y=\frac{1}{2}(a+c), \\ z=\frac{1}{2}(a+b). \end{array}\right.$

13. Given $\left\{\begin{array}{l} x+y+z=12 \\ x+y+u=13 \\ x+z+u=14 \\ y+z+u=15 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=4, \\ z=5, \\ u=6. \end{array}\right.$

14. Given $\left\{\begin{array}{l} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3 \\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}=1 \\ \frac{a}{x}+\frac{2b}{y}-\frac{c}{z}=2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a, \\ y=b, \\ z=c. \end{array}\right.$

NOTE.—There are certain artifices which may be employed to simplify the solution of several of these problems. In the 9th, take the sum of the three equations, and subtract twice each equation from it. In the 10th, eliminate without clearing of fractions.

In the 13th, let the sum of the four quantities be represented by s; then we shall have $s-u=12$; $s-z=13$; $s-y=14$; $s-x=15$; from which we can find s, and then readily find the other quantities.

PROBLEMS

PRODUCING SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

1. A bin contains 47 bushels of wheat, rye and oats; there are 7 bushels less of oats than of wheat and rye, and 17 bushels less of rye than of wheat and oats; required the quantity of each. *Ans.* Wheat, 12 bu.; rye, 15 bu.; oats, 20 bu.

2. A, B and C have $1800: the sum of $\frac{1}{2}$ of A's, $\frac{1}{3}$ of B's and $\frac{1}{4}$ of C's equals $600, and the sum of 4 times A's, 3 times B's and twice C's is $5000; required the fortune of each.
Ans. A's, $400; B's, $600; C's, $800.

3. A drover bought 230 animals: the number of horses added to $\frac{1}{2}$ of the number of sheep and cows, equals 145; the number of cows, plus $\frac{1}{5}$ of the number of horses and sheep, equals 110; how many were there of each?
Ans. Horses, 60; cows, 80; sheep, 90.

4. Divide the number 150 into 3 such parts that twice the first part increased by 35, 3 times the second part increased by 5, and 4 times the third part divided by 5, may all be equal to each other. *Ans.* $23\frac{1}{10}$; $25\frac{2}{5}$; $101\frac{1}{2}$.

5. Three men, A, B and C, were discussing their ages, when it appeared that the sum of A's and B's was 94 years, the sum of B's and C's was 98 years, and the sum of A's and C's was 96 years; what was the age of each?
Ans. A's, 46 yrs.; B's, 48 yrs.; C's, 50 yrs.

6. A man has $82 in one, five and ten dollar bills; his ones, plus $\frac{1}{2}$ of his fives and tens, amount to $47; and his fives, plus $\frac{1}{3}$ of his ones and tens, amount to $43; how many has he of each? *Ans.* 12 ones; 6 fives; 4 tens.

7. A boy bought at one time 3 apples and 4 peaches for 11 cents; at another time 4 apples and 5 pears for 19 cents; at another, 4 pears and 5 oranges for 32 cents; and at another, 6 apples and 7 oranges for 34 cents; what was the price of each?
Ans. Apples, 1 ct.; peaches, 2 cts.; pears, 3 cts.; oranges, 4 cts.

8. Divide the number 27 into 4 such parts that if the first part is increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the results will be equal. *Ans.* 4; 8; 3; 12.

9. A and B can do a piece of work in 12 days, A and C in 15 days, and B and C in 20 days; how many days will it take each person to perform the same work alone?
Ans. A, 20; B, 30; C, 60.

10. The sum of 3 fractions is $2\frac{1}{4}$: the sum of the first and third equals twice the second fraction, and the difference between the first and third is $\frac{1}{3}$ of the third fraction; what are the fractions? *Ans.* $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$.

11. In a naval engagement the number of ships captured was 7 more, and the number burned was 2 less, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many ships did the fleet consist?
Ans. 32.

12. A cistern has 3 pipes opening into it. If the first be closed, the cistern may be filled in 20 minutes; if the second be closed, in 25 minutes; if the third be closed, in 30 minutes. How long would it take each pipe alone to fill it?
Ans. 1st, $85\frac{5}{7}$ min.; 2d, $46\frac{2}{13}$ min.; 3d, $35\frac{5}{17}$ min.

13. I have 3 watches, and a chain which is worth $60. The first watch and chain are worth $\frac{1}{3}$ as much as the second and third watches; the second watch and chain are worth $\frac{3}{5}$ as much as the first and third watches; and the third watch and chain are worth 3 times as much as the first and second watches. What is the value of each watch?
Ans. 1st, $20; 2d, $60; 3d, $180.

14. There is a number consisting of 3 digits: the sum of the digits is 9; the digit in the tens' place is half the sum of the other 2 digits; and if 198 be added to the number, the result will be expressed by the figures of the number reversed; required the number. *Ans.* 234.

15. A sum of money consists of quarter dollars, dimes and half dimes. It is worth as many dimes as there are pieces of money; it is worth as many quarters as there are dimes; and

the number of half dimes is one more than the number of dimes. What is the number of each?

Ans. 3 quarters; 8 dimes; 9 half dimes.

16. Three boys, A, B and C, were playing marbles. First A loses to B and C as many as each of them has; next, B loses to A and C as many as each of them now has; lastly, C loses to A and B as many as each of them now has; and it is then found that each of them has 16 marbles. How many had each at first? *Ans.* A, 26; B, 14; C, 8.

17. Some smugglers discovered a cave which would exactly hold the cargo of their boat, which consisted of 13 bales of cotton and 33 casks of liquor. While they were unloading, a custom-house cutter hove in sight, and they sailed away with 9 casks and 5 bales, leaving the cave $\frac{1}{3}$ full; how many bales or casks would it hold? *Ans.* 24 bales; 72 casks.

18. A person has 2 horses and 2 saddles: the better saddle cost \$50, and the other \$15. If he puts the better saddle upon the first horse, and the worse saddle upon the second, then the latter is worth \$50 more than the former; but if he puts the worse saddle upon the first, and the better saddle upon the second horse, the latter is worth $1\frac{2}{3}$ times as much as the former; what is the value of each horse? *Ans.* 1st, \$165; 2d, \$250.

REVIEW QUESTIONS.

Define Independent Equations. Give an example of them. Define Simultaneous Equations. Give an example of them. Define Elimination. What is the use of Elimination? How many methods are there? State the methods of elimination.

Define Elimination by Substitution. State the rule for it. Define Elimination by Comparison. State the rule for it. Define Elimination by Addition and Subtraction. State the rule for it. Is there any other method of elimination? Which method of elimination is preferred?

What effect has transposition upon the signs of terms? Why does transposition change the sign of a term? Why does changing the signs of all the terms of an equation not affect the equality of the members? In solving problems when should a separate symbol be used for each unknown quantity?

SUPPLEMENT

TO SIMPLE EQUATIONS.

191. This **Supplement** to Simple Equations embraces the following subjects: *Zero and Infinity, Generalization, Negative Solutions, Discussion of Problems,* and *Indeterminate Problems.*

NOTE.—Teachers desiring a short course may omit this Supplement to simple equations.

ZERO AND INFINITY.

192. Zero and **Infinity** often occur in algebraic expressions. Such expressions may be interpreted by the following principles:

PRIN. 1. $0 \times A = 0$; that is, *if zero be multiplied by a finite quantity the product is zero.*

For, 0 multiplied by 2 is 0, 0 multiplied by 3 is 0, etc.; hence, 0 multiplied by *any number* is 0, or $0 \times A$ is 0.

PRIN. 2. $\frac{0}{A} = 0$; that is, *if zero be divided by a finite quantity the quotient is zero.*

For, 0 divided by 2 is 0, 0 divided by 3 is 0, etc.; hence, 0 divided by *any number* is 0, or $0 \div A$ is 0.

PRIN. 3 $\frac{0}{0} = A$; that is, *if zero be divided by zero the quotient is any finite quantity, or is indeterminate.*

For, if in Prin. 1 we divide both members by 0, we have $\frac{0}{0} = A$, in which A is any finite quantity.

PRIN. 4. $\frac{A}{0} = \infty$; that is, *if a finite quantity be divided by zero the quotient is infinity.*

To prove this, suppose we have the fraction $\frac{a}{b}$. Now, if a remains constant, the smaller b is, the greater will be the quotient, hence, if b becomes *infinitely small*, the quotient will become *infinitely large*; hence, when b is *zero*, the quotient is *infinity*.

PRIN. 5. $0 \times \infty = A$; that is, *if zero be multiplied by infinity, the product is a finite quantity.*

For, clearing the equation in Prin. 4 of fractions, we have $0 \times \infty = A$ n which A is any finite quantity.

PRIN. 6. $\frac{A}{\infty} = 0$; that is, *if a finite quantity be divided by infinity, the quotient is zero.*

For, dividing both members of the equation in Prin. 5 by ∞, we have 0 equals A divided by infinity, which proves the principle.

GENERALIZATION.

193. Generalization is the process of solving general problems, and interpreting the results.

194. A General Problem is one in which the quantities are represented by letters.

195. A Formula is a general expression for the solution of a problem. A formula expressed in ordinary language gives a *rule* by which all the problems of a class may be solved.

CASE I.

1. The difference between a times a number and b times the number is c; required the number.

Let $x =$ the number.

Then, $ax - bx = c$;

whence, $x = \frac{c}{a-b}$.

Expressing this formula in ordinary language, we have the following rule:

Rule.—*Divide the difference of the products by the difference of the multipliers.*

Apply this formula to the following problems:

2. The difference between 5 times a number and 3 times the number is 12; required the number.

3. The difference between 9 times a number and 6 times a number is 162; what is the number?

CASE II.

1. Find a number which being divided by two given numbers, a and b, the sum of the quotients may be c.

Let $x =$ the number.

Then, $\frac{x}{a} + \frac{x}{b} = c$;

whence, $x = \frac{abc}{a+b}$.

This formula, expressed in ordinary language, gives the following rule:

Rule.—*Divide the product of the three given quantities by the sum of the divisors when the sum of the quotients is given, and by the difference of the divisors when the difference of the quotients is given.*

Apply this formula to the following problems:

2. Find a number which being divided by 4 and by 6, the difference of the quotients is 4.

3. Find a number which being divided by 3 and by 7, the difference of the quotients is 16.

CASE III.

1. A can do a piece of work in a days, and B in b days; in what time can they both do it?

Let $x =$ the number of days in which both can do it.

Then, $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$.

whence, $x = \frac{ab}{a+b}$.

This formula, expressed in ordinary language, gives the following rule:

Rule.—*Divide the product of the numbers expressing the time in which each can perform the work by their sum.*

Apply this formula to the following problems:

2. A can do a piece of work in 4 days, and B in 8 days; in what time will they together do it?

3. A can reap a field in 6 days, and B in 9 days; in what time can they together reap it?

4. A pound of tea would last a man 12 months, and his wife 6 months; how long would it last them both?

CASE IV.

1. The sum of two numbers is a, and their difference is b; what are the numbers?

Let x = the greater number,
and y = the smaller number.
Then, $x+y=a$,
and $x-y=b$.

Whence $x=\frac{a+b}{2}$,

and $y=\frac{a-b}{2}$.

These formulas, expressed in ordinary language, will give the following rules:

Rule.—I. *To find the greater number, add half the difference to half the sum.*

II. *To find the less number, subtract half the difference from half the sum.*

Apply the formulas to the following problems;

Required the numbers whose—

2. Sum is 20; difference is 4.
3. Sum is 62; difference is 14.
4. Sum is 221; difference is 29.

MISCELLANEOUS PROBLEMS.

196. The pupils will obtain the formulas in the following problems, and derive rules from them:

1. Divide the number a into two parts, so that one part shall be n times the other part. *Ans.* $\frac{a}{n+1}$; $\frac{na}{n+1}$.

2. The sum of two numbers is a, and n times one number equals m times the other; what are the numbers?
Ans. $\frac{ma}{m+n}$; $\frac{na}{m+n}$.

3. The difference of two numbers is a, and n times one number equals m times the other; what are the numbers?
Ans. $\frac{ma}{m-n}$; $\frac{na}{m-n}$.

4. The sum of two numbers is a, and n times their sum equals m times their difference; what are the numbers?
Ans. $\frac{(m+n)a}{2m}$; $\frac{(m-n)a}{2m}$.

5. Divide the number a into two such parts that one part increased by b shall be equal to m times the other part.
Ans. $\frac{ma-b}{m+1}$; $\frac{a+b}{m+1}$.

6. A is m times as old as B, and in a years he will be n times as old; what is the age of each at present?
Ans. A's, $\frac{am(n-1)}{m-n}$; B's, $\frac{a(n-1)}{m-n}$.

7. A courier starts from a place and travels a miles a day; n days after he is followed by another, who travels b miles a day; in what time will the second overtake the first?
Ans. $\frac{na}{b-a}$ days.

8. I bought two kinds of sugar—one at a cents a pound, and the other at b cents a pound; how much of each kind must I take to make a mixture of m pounds worth c cents a pound?
Ans. $\frac{m(c-b)}{a-b}$; $\frac{m(a-c)}{a-b}$.

NEGATIVE SOLUTIONS.

197. In the solution of a problem the value of the unknown quantity is sometimes a *negative quantity.*

198. A solution of a problem which gives a negative quantity is called a *Negative Solution.*

1. What number must be added to the number 12 that the result shall be 8?

SOLUTION. Let x equal the number; then, $12+x=8$ or $x=-4$. Now, the result -4 is said to satisfy the question in an *algebraic sense,* since -4 added to 12 equals 8; but the problem is evidently impossible in an *arithmetical sense.* Since, however, adding -4 gives the same result as subtracting $+4$, the negative result indicates that the problem should be, What number must be *subtracted* from 12 that the result shall be 8?

OPERATION.

Let $x=$ the number.
Then, $12+x=8$
$x=-4$

2. A is 45 years old, and B is 15 years old; how many years hence will A be 4 times as old as B?

SOLUTION. Let $x=$ the number of years; then, solving the problem, we have $x=-5$. This result, -5, indicates a reckoning of time *backward* instead of *forward;* hence, it was 5 years *ago* instead of 5 years *hence.* The problem should, therefore, be modified to read, "How many years *since,*" instead of "How many years *hence.*"

OPERATION.

Let $x=$ the number of years.
Then, $(45+x)=4(15+x)$;
whence, $x=-5$

3. A is 45 years old, and B is 15 years old; at what time from the present is A 4 times as old as B?

SOLUTION. Solving this problem by supposing the date to be in the future, we find $x=-5$. Had the result been $+5$, it would have indicated 5 years *hence.* The problem is stated in general terms, so as to admit either result, and need not be modified in its statement. This result, -5, therefore indicates 5 years *since,* or 5 years *ago,* which, by examining the problem, we see is the correct time.

4. There is a fraction such that if 1 be added to its numerator the result is $\frac{1}{3}$, and if 1 be added to the denominator the result is $\frac{1}{2}$; what is the fraction?

SOLUTION. Solving this problem, we find the fraction to be $\frac{-4}{-9}$. This can be verified algebraically, but is absurd arithmetically considered. The problem can be made consistent, however, by changing the word "added" to "subtracted."

199. From these examples and illustrations we derive the following conclusions:

1. *The negative solution indicates some inconsistency or absurdity in the statement of the problem, or a wrong supposition respecting some condition of it.*

2. *A problem which gives a negative solution can usually be so modified that it will become consistent, and the result will be positive.*

3. *The negative solution sometimes merely indicates the direction in which the result is to be reckoned.*

200. The pupils will solve, interpret, and modify the enunciation of the following problems:

5. What number must be *added* to 18 that the result may be 15? *Ans.* -3.

6. What number must be *subtracted* from 12 that the result may be 15? *Ans.* -3.

7. Required a number such that $\frac{2}{3}$ of it shall exceed $\frac{3}{4}$ of it by 2. *Ans.* -24.

8. A man was born in 1825, and his son in 1855; find at what time the father's age is 4 times the son's age, dating from 1870. *Ans.* -5.

9. A man labored 10 days, his little son being with him 8 days, and received $18; at another time he labored 14 days, his son being with him 12 days, and received $25; required the wages of each. *Ans.* Father, $2; son, -25 cts.

DISCUSSION OF PROBLEMS.

201. The **Discussion** of a problem is the process of assigning different conditions to a problem, and interpreting the results which thus arise.

1. If we subtract b from a, by what number must the result be multiplied to give a product of c?

OPERATION.

Let x = the number.

$(a-b)x=c$

$$x=\frac{c}{a-b}$$

Discussion. Let x represent the number; then we shall have $(a-b)x=c$, from which we find the value of x equal to c divided by $a-b$.

202. This result may have five different forms depending on the values of a, b and c.

I. When a is greater than b.

In this case, $a-b$ is *positive;* and c being *positive*, the quotient is *positive;* hence, the required number is *positive.* This is as it should be, for the problem then is, *By what shall we multiply a positive quantity to produce a positive quantity?* Evidently the multiplier should be *positive.*

II. When a is less than b.

In this case $a-b$ is *negative*, and consequently c divided by $a-b$ is *negative;* hence the required number is *negative.* This is as it should be, for the problem then is, *By what shall we multiply a negative quantity to produce a positive quantity?* Evidently the multiplier should be *negative.*

III. When a is equal to b.

In this case $a-b=0$ and $x=\frac{c}{0}=\infty$ (Prin. 4, Art. 192); that is, *no finite quantity* will answer the conditions. This is correct, since the problem then becomes, *By what must we multiply 0, nothing, to produce c, something?* Evidently by *no finite quantity.*

IV. When c is 0, and a is either greater or less than b.

In this case we have $x=\frac{0}{a-b}$, which equals 0 (Prin. 2, Art. 192); that is, the multiplier is *zero.* This is correct, for the problem then is, *By what must we multiply $a-b$, something, to produce nothing?* Evidently the multiplier should be *zero.*

V. When $c=0$ and $a=b$.

In this case we have $x=\frac{0}{0}$, or *any quantity* whatever (Prin. 3, Art. 192). This is correct, for the problem then is, *By what shall we multiply 0 to produce 0?* Evidently the multiplier may be *any quantity.*

Note.—Let the pupils illustrate each form by using particular values for a, b and c.

PROBLEM OF THE COURIERS.

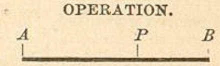
203. The **Problem of the Couriers** is a fine illustration of the subject under consideration. It was originally proposed by *Clairaut*, an eminent French mathematician.

1. Two couriers start at the same time from two places, A and B, a miles apart, the former traveling m miles an hour, and the latter n miles an hour; where will they meet?

There are evidently two cases of the problem.

Case I. When the couriers travel toward each other.

OPERATION.



$a=AB$

Let $x=AP$;

then $a-x=PB$.

$$\frac{x}{m}=\frac{a-x}{n}$$

$$x=\frac{am}{m+n}$$

$$a-x=\frac{an}{m+n}$$

Discussion. Let A and B represent the two places, and P the point at which they meet. $AB=a$. Let $x=AP$, the distance which the first travels; then $a-x=PB$, the distance which the second travels. Then $\frac{x}{m}$ = the number of hours the first travels, and $\frac{a-x}{n}$ = the number of hours the second travels. They both travel the same time; hence, we have $\frac{x}{m}=\frac{a-x}{n}$, from which we find the values of x and $a-x$.

1st. Suppose $m=n$; then, by substituting, we find $x=\frac{a}{2}$, and $a-x=\frac{a}{2}$; that is, if they travel at the *same rate*, each will travel *half the distance.* This is evident from the nature of the problem.

2d. Suppose $n=0$; then $x=a$ and $a-x=0$; that is, if the second does not travel, the first travels the *whole distance*, and the second *no distance.* This is evident from the conditions of the problem.

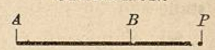
3d. Suppose $m=0$; then $x=0$ and $a-x=a$; that is, the first travels *no distance*, and the second travels the *whole distance*. This is evident from the conditions of the problem.

NOTE.—Let the pupil make the suppositions $m=2n$, $m=\frac{1}{2}n$, etc., and interpret the results.

CASE II. When the couriers travel in the same direction.

DISCUSSION. Let A and B represent the two places, and P the point where they meet, each traveling in the direction of P. $AB=a$. Let $x=AP$, the distance the first travels; then $x-a=BP$, the distance the second travels. The times they travel are equal; hence we have $\frac{x}{m}=\frac{x-a}{n}$, from which we find the values of x and $x-a$.

OPERATION.



$a=AB$.

Let $x=AP$;

then $x-a=BP$.

$$\frac{x}{m}=\frac{x-a}{n}$$

$$x=\frac{am}{m-n}$$

$$x-a=\frac{an}{m-n}$$

1st. Suppose $m>n$; then x and $x-a$ will be *positive*.

That is, they will meet at the *right* of B. This is as it should be by the conditions of the problem.

2d. Suppose $n>m$; then x and $x-a$ are both *negative*.

That is, the point of meeting must be at the *left* of A, instead of at the right. Hence, if the second courier travel faster than the first, that they may meet, the *direction* in which they travel *must be changed*. This is evident from the conditions of the problem.

3d. Suppose $m=n$; then $x=\frac{am}{0}$, or ∞, and $x-a=\frac{an}{0}$, or ∞.

That is, if the couriers travel at the same rate, they can meet at no *finite* distance from A; in other words, the one can *never* overtake the other. This is evident from the circumstances of the problem.

4th. Suppose $a=0$; then $x=\frac{0}{m-n}$, or 0, and $x-a=\frac{0}{m-n}$, or 0.

That is, if the couriers are *no distance* apart, they will have to travel *no distance* to be together. This is evident from the circumstances of the problem.

5th. Suppose $m=n$ and $a=0$; then $x=\frac{0}{0}$ and $x-a=\frac{0}{0}$, or *anything*.

That is, if the couriers are *no* distance apart, and travel at the *same rate*, they will *always* be together. This is evident from the nature of the problem.

6th. Suppose $n=0$; then $x=\frac{am}{m}=a$ and $x-a=0$.

That is, if the rate at which the second travels is zero, the first courier travels the *whole* distance, and the second *no* distance. This is evident from the circumstances of the problem.

7th. Suppose $m=2n$; then $x=\frac{2am}{m}=2a$ and $x-a=a$.

That is, if the rate at which the first travels is twice the rate at which the second travels, the first courier travels *twice* the distance from A to B, overtaking the second a miles from B. This is evident from the circumstances of the problem.

8th. Suppose m is plus and n is minus; then

$$x=\frac{am}{m+n} \text{ and } x-a=\frac{-an}{m+n}.$$

Here the value of x is *positive*, and the value of $x-a$ is *negative*; hence, they meet at the *right* of A and at the *left* of B, or between A and B. This is evident, since *minus* n indicates that the second courier travels toward A; hence they must meet between A and B.

Changing the signs of the last expression, we have $a-x=\frac{an}{m+n}$, which is the same as the expression for the distance the second courier travels, obtained in the first case of this problem. Case I. is therefore but a special case of the general problem just discussed.

PROBLEMS FOR DISCUSSION.

1. Required a number that, being successively multiplied by m and n, the difference of the products shall equal a.

Ans. $\frac{a}{m-n}$.

When will the result be negative? When indeterminate? When infinite? Illustrate with numbers.

2. B is a years old, and A is m times as old; at what time will A be n times as old as B? *Ans.* $x=\frac{(m-n)a}{n-1}$.

Interpret the result when $m>n$; $m<n$; when $n<1$.

3. The hour- and minute-hands of a watch are a minute spaces apart between m and $m+1$ o'clock; what is the time?

Ans. $12\frac{(5m-a)}{11}$.

INDETERMINATE AND IMPOSSIBLE PROBLEMS.

204. An **Indeterminate Problem** is one whose conditions can be satisfied by different values of the unknown quantity.

I. *A problem giving only one equation containing two unknown quantities is indeterminate.*

Thus, the equation $x+y=12$ is indeterminate; for, by transposing, we have $x=12-y$; and this equation can be verified by any number of values of x and y.

II. *A problem which gives only two equations containing three unknown quantities is indeterminate.*

PROBLEM.—Given $x+3y-z=8$ and $x+2y-2z=2$, to find x and y.

SOLUTION. By elimination, we find $y+z=6$, which may be verified by any number of values of y and z, and is therefore indeterminate.

III. *A problem is sometimes indeterminate when it contains only one unknown quantity.*

PROBLEM.—What number is that whose $\frac{5}{6}$, diminished by its $\frac{3}{4}$, will equal its $\frac{1}{20}$ increased by its $\frac{1}{30}$?

SOLUTION.

Let $x=$ the number;

then $\frac{5x}{6}-\frac{3x}{4}=\frac{x}{20}+\frac{x}{30}$.

Clearing of fractions, we have $50x-45x=3x+2x$;

transposing, $5x-5x=0$;

factoring, $(5-5)x=0$;

whence, $0\times x=0$,

and $x=\frac{0}{0}$.

The value of x thus found is *indeterminate* (Prin. 3, Art. 192); the problem is therefore indeterminate.

205. An **Impossible Problem** is one whose conditions are impossible or contradictory.

I. *A problem is impossible when its conditions are contradictory*

PROBLEM.—Given the equations $x+y=7$, $x-y=1$ and $xy=16$, to find the values of x and y.

SOLUTION. Uniting equations first and second, we find $x=4$ and $y=3$. But the third equation requires their product to be 16, which is impossible for those values of x and y. Hence the problem is impossible.

II. *A problem that contains only one unknown quantity is sometimes impossible.*

PROBLEM.—Required a number whose $\frac{1}{3}$, plus its $\frac{1}{4}$, diminished by 3, equals its $\frac{7}{12}$, increased by 5.

SOLUTION.

Let $x=$ the number.

Then $\frac{x}{3}+\frac{x}{4}-3=\frac{7x}{12}+5$;

whence, $7x-36=7x+60$,

and $0\times x=96$,

or $x=\frac{96}{0}=\infty$.

This value of x is *infinite*, which shows that no *finite number* will answer the conditions of the problem; it is therefore *impossible*.

NOTE.—When a problem contains more conditions than unknown quantities, the conditions which are unnecessary are said to be *redundant*.

EXAMPLES.

1. Find the value of x in the equation $\frac{3x+5}{x+2}=\frac{3x-6}{x-2}$.

OPERATION 1ST.

$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$

$$\frac{3x+5}{x+2}=3 \quad (2)$$

$$3x+5=3x+6 \quad (3)$$

$$(3-3)x=6-5 \quad (4)$$

$$0\times x=1$$

$$x=\frac{1}{0}=\infty \quad (5)$$

SOLUTION. Reducing the second member, we have equation (2); clearing of fractions, we have (3); transposing terms and factoring, we have (4); dividing by the coefficient of x, we have $x=\frac{1}{0}$, or ∞, which indicates that no finite value will answer the conditions.

SOLUTION 2D. Clearing the equation of fractions and reducing partly, we have (2); transposing and uniting, we have (3); whence we have (4). This apparent value of x cannot be verified, as the pupil may see by substitution.

OPERATION 2D.

$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$
$$3x^2-x-10=3x^2-12 \quad (2)$$
$$-x=-2 \quad (3)$$
$$x=2 \quad (4)$$

2. Required a number such that its $\frac{1}{4}$, increased by its $\frac{1}{6}$, is equal to its $\frac{11}{20}$, diminished by its $\frac{2}{15}$.

Ans. $\left(x=\frac{0}{0}\right)$. Indeterminate.

3 Required a number whose $\frac{5}{6}$, diminished by 4, is equal to the sum of its $\frac{1}{2}$ and $\frac{1}{3}$, diminished by 3.

Ans. ($x=\infty$). Impossible.

4. A and B dug a ditch for $20, A receiving $2, and B $3, a day; how many days did each labor, if they did not labor the same number of days? *Ans.* Indeterminate.

5. Twenty years ago, A was 40 years old, and his son was only $\frac{1}{4}$ as old; now the son is $\frac{1}{2}$ as old as the father; gaining thus, when will the son be as old as the father? *Ans.* ($x=\infty$).

6. Find a fraction such that if 2 be subtracted from the numerator, or if 3 be added to the denominator, the resulting fractions will equal $\frac{2}{3}$. *Ans.* Indeterminate.

7. Required a number such that 4 times the number, diminished by 12, divided by the number minus 3, may equal 4 times the number, plus 9, divided by the number plus 3.

Ans. Impossible.

NOTE.—The 6th reduces to the indeterminate form, although $\frac{4}{3}$, $\frac{8}{9}$, $1\frac{2}{15}$, etc., will answer the conditions of the problem.

REVIEW QUESTIONS.

State the principles of Zero and Infinity. Define Generalization. A General Problem. A Formula. A Negative Solution. State the principles of negative solutions. Define the Discussion of a Problem. An Indeterminate Problem. An Impossible Problem. When is a problem indeterminate? When in possible?

SECTION VI.

INVOLUTION, EVOLUTION AND RADICALS

INVOLUTION.

206. Involution is the process of raising a quantity to any given power.

207. A **Power** of a quantity is the product obtained by using the quantity as a factor any number of times.

208. An **Exponent** of a quantity is a number which indicates the power to which the quantity is to be raised.

Thus, let a represent any quantity;

then, $a=a^1$, is the *first* power of a.
$aa=a^2$, is the *second* power of a.
$aaa=a^3$, is the *third* power of a.
$aaaa=a^4$, is the *fourth* power of a.

When the exponent is n, as a^n, it indicates the nth power of a.

209. The **Exponent** (called also the *Index*) indicates how many times the quantity is used as a factor.

The FIRST POWER of a quantity is the quantity itself.

The SECOND POWER of a quantity is called its *square*.

The THIRD POWER of a quantity is called its *cube*.

PRINCIPLES.

1. *The square of a quantity is the product obtained by using the quantity as a factor twice.*
2. *The cube of a quantity is the product obtained by using the quantity as a factor three times.*
3. *All the powers of a positive quantity are positive.*

For, the square of a positive quantity is positive, since it is the product of two positive quantities; and its cube is positive, since it is also the product of two positive quantities, etc.