our last experiment. The forks are now so adjusted that they differ by a semitone, — that is, their rates of vibration are as 24 : 25. The result is the beautiful curve (Fig. 144)

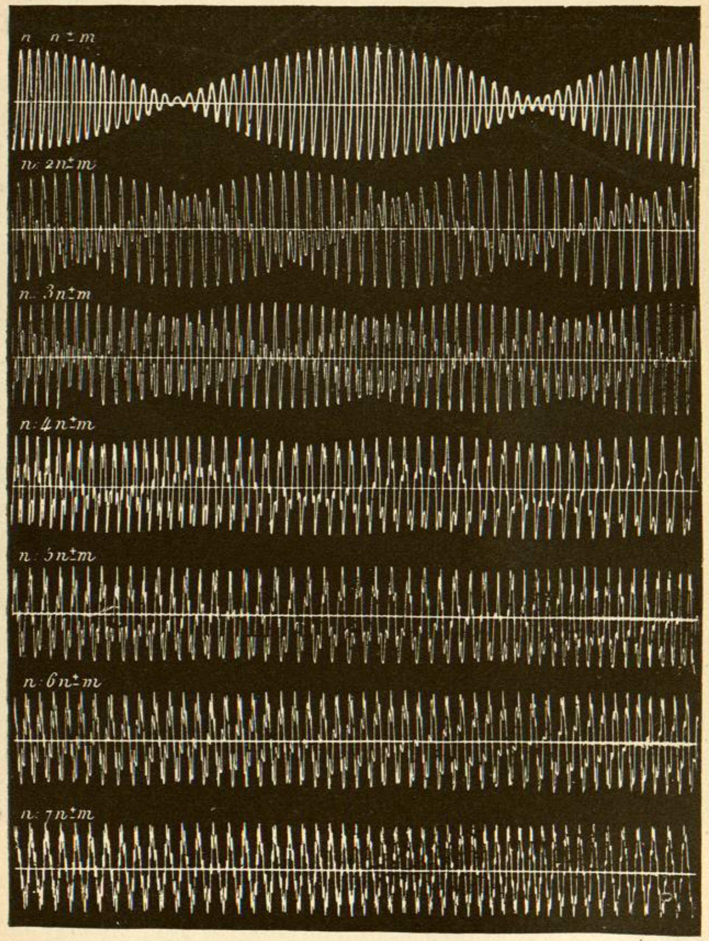

Fig. 145.

now on the screen. If we diminish the interval so that the relative frequencies of the two forks are as 80 : 81, we get a smaller interval, known to musicians as a comma. As seen on the screen, — see adjoining figure, — its image resembles that corresponding to the interval of a semi-

tone, except that the number of sinuosities is greater for the comma, while their amplitude is less. Fig. 144 also shows a number of curves corresponding to simpler intervals. In Fig. 145 are several beautiful and complex curves obtained by Koenig from the altered harmonic intervals indicated at the left-hand side of the figure.

Nothing can give us a better idea of the nature of beats than the figures we have just been studying. The swellings and contractions of the indentations seen on the screen are the exact counterparts of the condition of the atmosphere in this hall. The reinforcements and interferences of sound so beautifully depicted in the different figures we have seen, tell us how the atmosphere that surrounds us is alternately agitated and quiescent, and why it is that we perceive in rhythmic order the varying periods of resonance and silence.

These experiments are made specially to appeal to the eye. I shall now make an experiment that will appeal with equally telling effect to the ear.

Before you (Fig. 146) is a superb instrument designed by Dr. Koenig, by means of which we are enabled to study the phenomena of beats with more satisfaction than with any other instrument we have yet seen. It is a powerful C_2 tuning-fork, actuated by electricity, and fixed before a large adjustable copper resonator. Both branches of the fork are hollow, having been bored by a drill of small diameter. These borings unite with each other in the stem of the fork, where they communicate with a small reservoir of mercury. The mercury can be made to move up and down the branches of the fork by means of a small piston that works in a cylindrical piece of steel, which serves as the reservoir. By raising the mercury, the number of vibrations of the fork is lessened, and its note lowered in proportion. By lowering the mercury, the note is correspondingly raised. We thus have the means of readily changing the frequency of the fork within a comparatively wide limit, and of having a note whose intensity remains unchanged. Dr. Koenig appropriately calls this a fork of variable pitch.

Beside it stands another fork exactly similar, except that it is not provided with the arrangement for changing its frequency. The pitch of this second fork is constant. By properly adjusting the height of the mercury in the fork of variable pitch, we can bring it into unison with the fork of constant pitch. If we now connect each fork with a single Grenet cell, the forks are at once set in vibration.

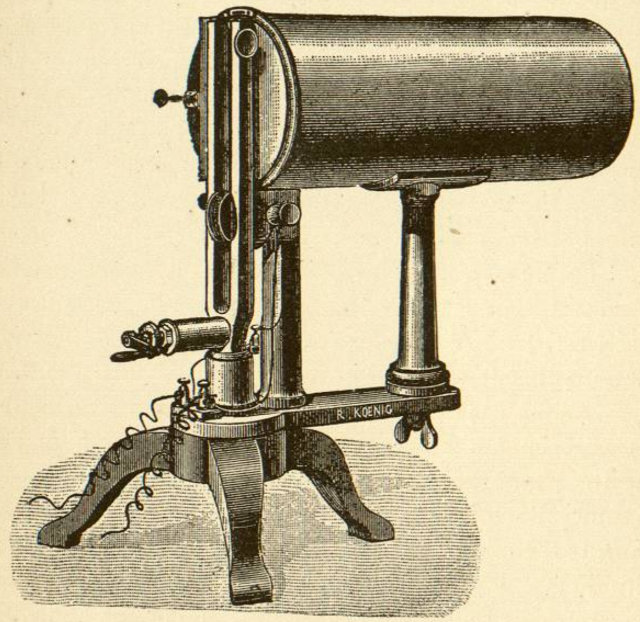

FIG. 146.

You hear little or no sound as yet, because the resonators are closed. One is now opened, and then the other, when you hear notes of exceeding purity and volume. At present the two forks are in unison, and the notes produced seem to proceed from a single source of sound. By raising or lowering the mercury, we can disturb the unison. The slightest movement of the piston is sufficient to change the relative frequencies of the two forks, and to induce beats.

The beats are now very slow, because the piston has

been moved but slightly, and the rate of vibration of the fork has been affected but little. You can, however, perceive a rising and falling of the sound, although each beat persists for several seconds, because of the extraordinary power of the sounds that engender these beats. The notes emitted are as nearly simple tones as may be, for the forks have been so constructed that all the upper partials have been quenched. We have, then, nothing to consider except the primes, and the beats to which, by virtue of their different rates of vibrations, they give rise.

Changing the position of the mercury in the fork still more, we can hear several beats a second. Watch in hand, I count the number now heard in ten seconds. There are thirty; that is, three for each second. I have all along been raising the mercury in the bores of the fork. This has been equivalent to weighting its prongs, and causing them to move more slowly. The number of vibrations has consequently been reduced, and the note correspondingly lowered. The number of beats tells us exactly the extent of this change. C_2 executes 128 vibrations per second, and the fork of variable pitch makes, therefore, three less; namely, 125 per second.

Raising the mercury still higher in the fork, the beats become so rapid that it is difficult, if not impossible, to count them. It is, indeed, difficult to count beats when there are more than four per second; and their enumeration is almost equally difficult when the number falls below one a second. It requires practice to count them with any degree of accuracy when there are two each second. Their enumeration is easiest when there are three or four per second.

The tuning of musical instruments, as now used, is effected solely by counting, or at least by estimating the beats generated by various notes that are successively sounded two by two. Proficiency in tuning requires not only an accurate ear, but also long years of practice during several hours each day. Without the aid of beats, any approximation to correct tuning with the unassisted ear would be impossible. Without beats, it is hardly possible for even the most accurate ear to tune just intervals, not to speak of the more complicated intervals employed in our so-called tempered instruments.

Mr. Ellis, in referring to this matter, says: "But few ears could be trusted to tune a succession of perfect fifths and fourths. Herr G. Appunn" — the brother of Anton Appunn, who made some of the high-pitch forks used in our second lecture — "told me that it cost him an immense labor to tune thirty-six notes, forming perfect fifths and fourths, upon an experimental harmonium, and he had the finest ear for the appreciation of intervals that I ever heard of. The accumulation of almost insensible into intolerable errors besets all attempts to tune by a long series of similar intervals. Even octaves are rarely tuned accurately through the compass of a grand pianoforte."

In the early part of the lecture it was stated that the number of beats engendered when two slightly dissonant notes were sounded, is equal to the difference of the frequencies of the notes. This is true; but it would be misleading to have you left under the impression that beats are produced only when two notes are near unison. But this is just what is taught by most writers on sound and music, and by those especially who follow Helmholtz. Koenig, however, by a most exhaustive series of experiments over almost the entire compass of sound, has demonstrated, by means of the most perfect instruments that mechanical ingenuity could devise, that the generally accepted theory of beats must be materially modified to correspond with the results of his investigations. No one, I think, will question the accuracy of the statements of one who is known to be so careful as Koenig, and who is recognized as an expert of the greatest eminence in all that concerns the science of acoustics.

According to Koenig, beats are produced not only when the intervals are small, but also when the frequencies of the generators of sound are widely separated from each

other. In his experiments he was able to distinguish beats made by disturbed harmonic intervals up as far as the eighth and tenth partial. Thus, by taking C_1 of 64 vibrations as the prime, and C_4, making 512 vibrations, and three octaves above C_1, he was able, by slightly altering the frequency of either fork, to obtain beats. And although C_4 is the eighth partial of C_1, the beats were quite distinct, but not so loud as those yielded by two forks nearly in unison. More than this, under favorable circumstances he succeeded in obtaining beats with the intervals $C_1 : D_4$ — $64 : 576$, or $1 : 9$, — and $C_1 : E_4$ — $64 : 640$, or $1 : 10$.

Helmholtz and others have imagined that these beats were due to the upper partials of the forks used, or to the resultant tones, about which we shall see more presently; but Koenig took care to use forks that gave no upper partials whatever. Throughout his admirable investigation "On the Sounding of Two Tones at the Same Time,"[1] he studiously eschewed the use of forks that gave other than simple tones. Upper partials and resultant tones, therefore, cannot afford any explanation of the facts observed; namely, that two simple tones, of widely separated frequencies, give rise to beats as well as those which are only slightly removed from unison.

But this is not the only discovery made by Koenig concerning the production of beats. He has also demonstrated that two simple tones, called generators, are competent to excite two sets of beats that are quite different from each other. These beats he distinguishes as upper and lower beats. Their frequency for any given interval may be determined from the following law, which in all cases agrees with the results of experiment: —

The frequencies of the beats are equal to the differences between the number of vibrations of the upper generator and the vibration-numbers corresponding to the two multiples of the lower generator, between which the vibration-number of the upper generator is found.

Thus, according to this law, for the interval $2 : 5$, with the

[1] Quelques Expériences d'Acoustique, pp. 87 *et seq.*

notes $C_2 : E_3$, giving respectively 128 and 320 vibrations per second, the upper number, 5, of the interval ratio lies between 4 and 6. But these numbers are the second and third multiples of the lower number, 2. The frequencies of the two sets of beats will, therefore, be found by subtracting 4 from 5, = 1, for the lower beat, and 5 from 6, = 1, for the upper beat. The ratio, then, of the frequency of the lower generator to that of the lower beats will be 2 : 1, or, taking the vibration-number, 128, of the generator, the beat-frequency, as compared with it, will be 64. Similarly, the ratio representing the frequency of the upper generator and that of the upper beats will be 5 : 1, or 320 : 64. The frequencies both of the lower and upper beats in this case are equal.

If, however, we take the interval 3 : 8 for the notes $C_2 : F_3$, with vibrations equal to 128 : 341.3, we shall find that the frequencies of the upper and lower beats are different. Thus, taking the interval ratio, 3 : 8, of the two generators named, we find that the number 8 of the upper generator lies between 6 and 9, the first and second multiples of the number expressing the relative frequency of the lower generator. The relative frequencies, accordingly, of the upper and lower beats will be 8 — 6 = 2, and 9 — 8 = 1. That is, the frequency of the lower beats will be to the lower generator as 2 : 3, or, as the number of vibrations of the lower generator is 128, the number of beats will be 85.3. In like manner, the ratio of the relative frequency of the upper generator and the upper beats being 8 : 1, their absolute frequencies will be 341.3 : 42.6. The frequency of the upper beats, in this instance, is just one half that of the lower beats.

But it would be a mistake to infer, from what has been said, that both upper and lower beats are heard in every instance in which beats are produced. Such is not the case. More frequently only one set of beats is audible.

In going from unison, 1 : 1, to the octave, 1 : 2, or from the octave, 1 : 2, to the twelfth, 1 : 3, we shall find that the lower beats extend a little over the lower half of each

interval, and the upper beats over a little more than the upper half. Over a short space near the middle of each interval, both sets of beats are heard with varying degrees of distinctness. In higher periods of intervals, as from 1 : 3 to 1 : 4, from 1 : 4 to 1 : 5, etc., the audibility of both upper and lower beats has a more limited range. This is explained by the fact that in each period the upper beats are more feeble than the lower beats. As a consequence, the intensity of both upper and lower beats diminishes from period to period in proportion as we ascend from lower to higher periods.

Illustrations showing the order of occurrence of lower and upper beats, and of their occurrence together, are easily found. Thus the interval, 8 : 9, according to what has been stated, should give only lower beats. Taking the notes $C_1 : D_1$, whose vibration-numbers are 64 : 72, we have only lower beats, whose frequency, 8, is equal to the difference between the frequencies of the two generators. With the interval 8 : 15, we have only upper beats, the relative frequency of which is 1 · 8   2 = 16. 16 — 15 = 1. Choosing the interval $C_1 : B_1$, whose vibration-ratio is 64 : 120, we obtain upper beats having a frequency of 8. According to the rule just given, we double the frequency of the lower generator, 64, which gives us 128, and subtract from this the vibration-number, 120, of the upper generator, whereby, as above, we have 8 as a remainder. With $C_2 : F_2$, whose vibration-ratio is 128 : 170.6, giving an interval 3 : 4, and near the middle of the octave, we have both upper and lower beats. 170.6 — 128 gives us 42.6 as the frequency of the lower beats; and 128 × 2 = 256, and 256 — 170.6 = 85.4, gives us the frequency of the upper beats. But when two generators separated by an interval of a fifth are employed, — that is, when their frequencies are as 2 : 3, — then the frequencies of the upper and lower beats are invariably equal.

The frequency of the beats, as we have seen, increases as the generators depart from unison. At first their frequency is very low, and can easily be counted. Gradually it becomes more and more rapid, the beats changing into a roll, and then into a confused rattle.

The question now arises, "Can beats link themselves together so as to give rise to a continuous sound?" Lagrange and Dr. Thomas Young, the latter of whom gave the subject much attention, thought they could. Helmholtz and his followers say No. Koenig takes up the subject, and by a long series of the most careful observations, with large tuning-forks especially constructed for the purpose, comes to the conclusion that beats can and do change into sounds when their number attains a certain limit. We shall take a hurried review of Koenig's investigations, when, I think, you will be content to accept his views as, in the main, correct.

In 1714 Tartini, the celebrated Italian violinist and musical composer, discovered that when two notes were simultaneously sounded on the violin with sufficient intensity, they gave rise to a third note distinct from both. He called them *terzi suoni*, — third sounds. They are often called, after their inventor, Tartini's tones. They are likewise variously denominated *differential*, *resultant*, and *combinational* tones. Koenig calls them *beat-notes*, or *beat-tones*. Tartini made his discovery the basis of a new system of music, — a system which he developed in his "Trattato di Musica, Secondo la Vera Scienza dell' Armonia," published in 1754, and in a second work on "Dei Principii dell' Armonia Musicale," published in 1767.

Helmholtz distinguishes two kinds of combinational tones, — viz., *differential* and *summational* tones. The former are called differential tones, because their frequencies are equal to the difference of the vibration-numbers of the generating tones. The latter are designated summational tones, because their frequencies are equal to the sum of the vibration-numbers of their generators. For reasons that will appear as we go along, I shall, after Koenig, call both of these tones beat-notes, or beat-tones.

For the ordinary harmonic intervals, — that is, those comprised between unison and a major sixth, — it is quite

true that we obtain tones whose frequencies are equal to the difference of their primaries. But it is only these few intervals that afford a basis for the name differential tones, and the various theories with which they have been associated. In the following table are exhibited the beat-tones — the so-called differential tones — of the more common musical intervals: —

Intervals.	Ratio of Frequencies.	Difference.	The Beat-tone is deeper than the Lower Generator by
Octave	1 : 2	1	0
Fifth	2 : 3	1	An octave
Fourth	3 : 4	1	A twelfth
Major third . .	4 : 5	1	Two octaves
Minor third . .	5 : 6	1	Two octaves and a major third
Major sixth . .	3 : 5	2	A fifth
Minor sixth . .	5 : 8	3	A major sixth

Putting this result in musical notation, showing the generating tones as minims and the beat-tones as crotchets, we have, —

By means of the tuning-forks before you it is easy to render these beat-tones audible. Taking the two forks, C_4 and G_4, whose interval is 2 : 3, and whose frequencies are 512 and 768 respectively, we obtain, on exciting the two forks, a loud and distinct beat-tone, C_3, whose frequency is $768 - 512 = 256$. By sounding simultaneously C_4 and F_4, — interval-ratio 3 : 4, and frequencies 512 : 682.6, — we have a beat-note which, as above indicated, is a twelfth below the lower generator; namely, F_2 of 170.6 vibrations. In like manner, C_4 and E_4 — interval 4 : 5, frequencies 512 : 640 — give us C_2 of 128 vibrations as a beat-note. Proceeding in like manner, it would be easy to render audible the beat-notes due to all the other intervals of the above table.

The rule for determining what are the beat-tones for any two generators is precisely the same as that given for calculating the number of beats produced by two sources of sound. The law governing both beats and beat-tones is identical. This is what might be expected if Koenig's theory, that beats when sufficiently numerous change into beat-tones, is true. Beats are best observed with grave notes, where the difference in frequency is necessarily small. Beat-tones, on the contrary, are best studied with the higher notes, whose vibration-numbers give a correspondingly greater difference of frequency. According to Koenig, beats are best heard with tuning-forks below C_4 of 512 vibrations. Above C_4 all the intervals, except those very near unison, give rise to beat-tones of greater or less intensity.

"But," you will say, "if the law governing beats and beat-tones be the same, we should have upper and lower beat-tones as well as upper and lower beats?" And so we have. And it is precisely these upper beat-tones, whose existence is not explained by Helmholtz's theory, that, with many other stubborn facts, contribute to render his theory untenable. Thus, for the intervals above considered, $C_4 : G_4$; $C_4 : F_4$; $C_4 : E_4$, we have, in addition to the lower beat-notes, C_3, F_2, C_2, also the upper beat-notes, C_3, F_3, and G_3. When the two beat-notes coincide, as the lower and upper beat-notes C_3, they tend to reinforce each other, and generate a proportionally louder sound. When they differ by an octave, as F_2 and F_3, they give rise to a note in which each seems to predominate alternately. The effect of the beat-tones C_2 and G_3 sounding together is the same as would be produced by two weak primaries of the same interval sounding at the same time.

For the intervals given in the table on page 323, the theory of differential tones may apply; but there are many other intervals where the beat-tones are not equal to the difference between the frequencies of their primaries.

The same difficulty obtains with summational tones. We can show experimentally the existence of beat-tones which are entirely different from summational tones, and which Helmholtz's theory is incompetent to explain.

Koenig's law regarding beat-tones is best illustrated with heavy forks emitting acute sounds. On the table is a set of twelve such forks, ranging from C_5 to C_7. With these forks we are able to get beat-tones that are extraordinarily loud and pure. We take two of them and clamp them in a heavy iron support, specially constructed for the purpose (Fig. 147). When sounded, they yield the notes C_6 and B_6, — interval 8 : 15, frequencies 2048 : 3840, — whence we get, as an upper beat-tone, C_3, of 256 vibrations. Taking the frequencies of the notes in question, we have, according to Koenig's law, $2048 \times 2 = 4096$; $4096 - 3840 = 256$. But there is no differential tone here. The differential tone, if one existed, should, in this case, be a note having a frequency of 1792, — a number obtained by subtracting 2048 from 3840, — and would, consequently, have a pitch equal to the seventh partial of the note actually heard, $256 \times 7 = 1792$.

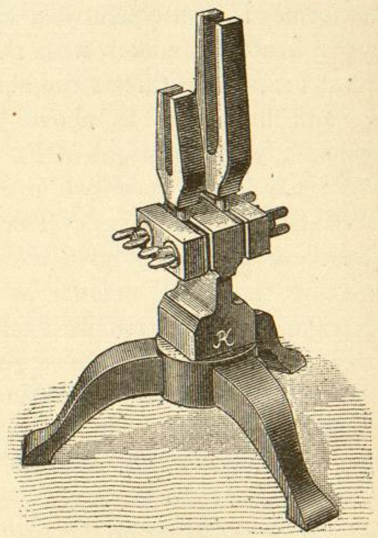

FIG. 147.

With the forks $C_5 : D_6$, — intervals 4 : 9, frequencies 1024 : 2304, — we have only a lower beat-tone, C_3, — $1024 \times 2 = 2048$; $2304 - 2048 = 256 = C_3$. The differential tone in this case, if such existed, should be $2304 - 1024 = 1280 = E_5$, — a major third above the lower generator; but no such tone is audible.

The forks C_5 and F_6 — interval 3 : 8, frequencies 1024 : 2730.8 — give both upper and lower beat-tones; but, again, there is no differential tone. The beat-tones in this case are F_3 and F_4; the differential tone, if any existed, should be A_5. Here the upper generator, 8, lies between the second and third multiples of the lower generator, 3; that is, between 6 and 9. By subtraction we have the differences 2 and 1; $8-6=2$, and $9-8=1$. The lower beat-note thus forms with the lower generator the interval 2 : 3, and the former is, consequently, a fifth below the latter. The upper beat-note makes, with the upper generator, the interval 1 : 8, which throws the beat-note three octaves below F_6, and makes it, as above, F_3, of 341.3 vibrations, as against 2730.8 vibrations of F_6.

We have seen how two primary tones may give rise to beats and beat-tones. The question may now arise, Can these beat-tones give rise to other beats and beat-tones in the same manner as primary tones do? For the sake of distinction we shall call the beats and beat-tones produced by two given generators *primary beats* and *beat-tones*. Can, then, primary beat-tones, like their generators, give rise to beats and beat-tones also? They can. And the beats and beat-tones thus produced are called *secondary beats* and *beat-tones*. The great merit of Koenig's investigations is that he has been able to establish the law by which such beats and beat-tones are generated. I have not time to illustrate it in detail. It is sufficient to say that it is essentially the same as the law governing primary beats and beat-tones.

A little consideration will make it evident that secondary beats and beat-tones can be heard only when the sounds of the generators are very acute and very intense. I will pass by the secondary beats, and give you two examples of secondary beat-tones. For this purpose I shall employ for the first example two forks having the interval 8 : 11, and executing respectively 2048 and 2816 vibrations per second. The first fork corresponds to C_6, and the second emits a note between F_6 and G_6. The note it gives is, in

reality, the eleventh partial of C_3, — the frequency of C_3 multiplied by 11 giving 2816, the frequency of the fork in question.

When, therefore, these two forks are set in vibration, there are produced the primary lower beat-tone, G_4, of 768 vibrations, and the primary upper beat-tone, E_5, of 1280 vibrations. But besides these two beat-tones we may hear clearly a third note, C_4, of 512 vibrations. This is the secondary beat-tone, and is equal to the difference between the frequencies of the two primary beat-tones, G_4 and E_5, — $1280 - 768 = 512$.

I now take two similar forks, whose interval is 8 : 13, and whose frequencies are 2048 : 3228 vibrations. These forks answer to C_6, as in the preceding instance, and the thirteenth partial of C_3, — $256 \times 13 = 3328$. The lower primary beat-note yielded in this case is E_5. $3328 - 2048 = 1280 = E_5$. The upper primary beat-note is G_4, of 768 vibrations. $2048 \times 2 = 4096$. $4096 - 3328 = 768 = G_4$. The difference between the frequencies of these two beat-tones, as in the case of the first two forks, is 512, — $1280 - 768 = 512 = C_4$. You will observe that the secondary beat-tones for both the intervals assumed, — 8 : 11 and 8 : 13, — give rise to the same note, C_4. The primary beat-tones for both intervals are likewise the same, namely, G_4 and E_5. They, however, occur in an inverse order for the two intervals. In the interval 8 : 11, G_4 is the lower beat-tone, while E_5 is the upper; whereas in the interval 8 : 13, E_5 is the lower, and G_4 the upper beat-tone.

But we may go farther. If the primary beat-tones can originate beats and beat-tones, we might expect that the partials of any two prime tones would similarly produce beats and beat-tones. Koenig maintains that the partials of compound tones do produce such beats and beat-tones, and that they have the same frequencies as have Helmholtz's summational tones. For this reason, he concludes that there are no summational tones, as called for by Helmholtz's theory, and, for reasons based on facts adduced in the foregoing experiments in connection with