## ERRATA.

Page 12, line 12, first column, for "these" put "their."
" 42, " 16, " " " $(Mi)$ " $(M_1i)$.
" 51, " 4, " " " $ab$ " $aa'$
" 51, " 4, " " " $a'b'$ " $bb'$
" 55, " 26, " " " $a$ " $a_1$
" 66, " 11, second " " B. Cremona put L. Cremona.

NEW CONSTRUCTIONS

IN

# GRAPHICAL STATICS.

# NEW CONSTRUCTIONS

IN

# GRAPHICAL STATICS.

## CHAPTER I.

It is the object of this work to fully discuss the stability of all forms of the arch, flexible or rigid, by means of the equilibrium polygon—the now well recognized instrument for graphical investigation. One or two other constructions of interest may also be added in the sequel. The discussion will pre-suppose an elementary knowledge of the properties of the equilibrium polygon, and its accompanying force polygon, for parallel forces.

As ordinarily used in the discussion of the simple or continuous girder, the equilibrium polygon has an entirely artificial relation to the problem in hand, and the particular horizontal stress assumed is a matter of no consequence; but not so with respect to the arch. As will be seen, there is a special equilibrium polygon appertaining to a given arch and load, and in this particular polygon the horizontal stress is the actual horizontal thrust of the arch. When this thrust has been found in any given case, it permits an immediate determination of all other questions respecting the stresses. This thrust has to be determined differently in arches of different kinds, the method being dependent upon the number, kind, and position of the joints in the arch.

The methods we shall use depend upon our ability to separate the stresses induced by the loading into two parts; one part being sustained in virtue of the reaction of the arch in the same manner as an inverted suspension cable (*i.e.*, as an equilibrated linear arch), and the remainder in virtue of its reaction as a girder. These two ways in which the loading is sustained are to be considered somewhat apart from each other. To this end it appears necessary to restate and discuss, in certain aspects, the well-known equations applicable to elastic girders acted on by vertical pressures due to the load and the resistances of the supports.

Let $P$ represent any one of the various pressures, $P_1$, $P_2$, $P_3$, applied to the girder.

Consider an ideal cross section of the girder at any point $O$.

Let $x$=the horizontal distance from $O$ to the force $P$.

Let $R$=the radius of curvature of the girder at $O$.

At the cross section $O$, the equations just mentioned become:—

Shearing stress, $S=\Sigma\,(P)$

Moment of flexure, $M=\Sigma\,(Px)$

Curvature, $P'=\dfrac{1}{R}=\dfrac{M}{EI}$

Total bending, $B=\Sigma(P')=\Sigma\left(\dfrac{M}{EI}\right)$

Deflection, $D=\Sigma\,(P'x)=\Sigma\left(\dfrac{Mx}{EI}\right)$

in which $E$ is the modulus of elasticity of the material, and $I$ is the moment of inertia of the girder; and as is well known, the summation is to be extended from the point $O$ to a free end of the girder, or, if not to a free end, the summation expresses the effect only of the quantities included in the summation.

Let a number of points be taken at equal distances along the girder, and let the values of $P$, $S$, $M$, $B$, $D$ be computed for these points by taking $O$ at these points successively, and also erect ordinates at these points whose lengths are proportional to the quantities computed. First, suppose $I$ is the same at each of the points chosen, then the values of these ordinates may be expressed as follows, if $a$, $b$, $c$, etc., are any real constants whatever:

$$y_p = a\,.\,P \quad \ldots \quad (1)$$

$$y_s = b\,.\,\Sigma(P) \quad \ldots \quad (2)$$

$$y_m = c\,.\,\Sigma(Px) = c\,.\,M.. \quad (3)$$

$$y_b = d\,.\,\Sigma(M) \quad \ldots \quad (4)$$

$$y_d = e\,.\,\Sigma(Mx) \quad \ldots \quad (5)$$

If $I$ is not the same at the different cross sections, let $P = M \div I$; then the last three equations must be replaced by the following:

$$y_m' = f\,.\,P' \quad \ldots \quad (3')$$

$$y_b' = g\,.\,\Sigma(P') \quad \ldots \quad (4')$$

$$y_d' = h\,.\,\Sigma(P'x)\,. \quad \ldots \quad (5')$$

The ordinates $y_m$ and $y_m'$ are not equal, but can be obtained one from the other when we know the ratio of the moments of inertia at the different cross sections.

Equation (1) expresses the loading, and $y_p$ may be considered to be the depth of some uniform material as earth, shot or masonry constituting the load. Lines joining the extremities of these ordinates will form a polygon, or approximately a curve which is the upper surface of such a load. When the load is uniform the surface is a horizontal line.

For the purposes of our investigation, a distributed load whose upper surface is the polygon or curve, above described, is considered to have the same effect as a series of concentrated loads proportional to the ordinates $y_p$ acting at the assumed points of division. If the points of division be assumed sufficiently near to each other, the assumption is sufficiently accurate.

If a polygon be drawn in a similar manner by joining the extremities of the ordinates $y_m$ computed from equation (3), it is known that this polygon is an equilibrium polygon for the applied weights $P$, and it can also be constructed directly without computation by the help of a force polygon having some assumed horizontal stress.

Now, it is seen by inspection that equations (3) and (5), or (3') and (5'), have the same relationship to each other that equations (1) and (3) have. The relationship may be stated thus:—If the ordinates $y_m$ (or $y_m'$) be regarded as the depth of some species of loading, so that the polygonal part of the equilibrium polygon is the surface of such load, then a second equilibrium polygon constructed for this loading will have for its ordinates proportional to $y_d$. But these last are proportional to the actual deflections of the girder.

Hence a second equilibrium polygon, so constructed, might be called the deflection polygon, as it shows on an exaggerated scale the shape of the neutral axis of the deflected girder.

The first equilibrium polygon having the ordinates $y_m$ may be called the moment polygon.

It may be useful to consider the physical significance of equations (3), (4), (5), or (3'), (4'), (5').

According to the accepted theory of perfectly elastic material, the sharpness of the curvature of a uniform girder is directly proportional to the moment of the applied forces, and for different girders or different portions of the same girder, it is inversely proportional to the resistance which the girder can afford. Now this resistance varies directly as $I$ varies, hence curvature varies as $M \div I$, which is equation (3) or (3').

Now curvature, or bending at a point, is expressed by the acute angle between two tangents to the curve at the distance of a unit from each other; and the total



bending, *i.e.* the angle between the tangent at $O$, and that at some distant point $A$ is the sum of all such angles between $O$ and the point $A$. Hence the total bending is proportional to $\Sigma(M \div I)$, the summation being extended from $O$ to the point $A$, which is equation (4) or (4').

Again, if bending occurs at a point distant from $O$, as $A$, and the tangent at $A$ be considered as fixed, then $O$ is deflected from this tangent, and the amount of such deflection depends both upon the amount of the bending at $A$, and upon its distance from $O$. Hence the deflection from the tangent at $A$ is proportional to $\Sigma\,(Mx \div I)$ which is equation (5) or (5').

It will be useful to state explicitly several propositions, some of which are implied in the foregoing equations. The importance and applicability of some of them has not, perhaps, been sufficiently recognized in this connection.

Prop. I. Any girder (straight or otherwise) to which vertical forces alone are applied (*i.e.*, there is no horizontal thrust) sustains at any cross-section the stress due to the load, solely by developing one internal resistance equal and opposed to the shearing, and another equal and opposed to the moment of the applied forces.

Prop. II. But any flexible cable or arch with hinge joints can offer no resistance at these joints to the moment of the applied forces, and their moment is sustained by the horizontal thrust developed at the supports and by the tension or compression directly along the cable or arch.

It is well known that the equilibrium polygon receives its name from its being the shape which such a flexible cable, or equilibrated arch, assumes under the action of the forces. In this case we may say for brevity, that the forces are sustained by the cable or arch in virtue of its being an equilibrium polygon.

Prop. III. If an arch not entirely flexible is supported by abutments against which it can exert a thrust having a horizontal component, then the moment due to the forces applied to the arch will be sustained at those points which are not flexible, partly in virtue of its being approximately an equilibrium polygon, and partly in virtue of its resistance as a girder.

It is evident from the nature of the equilibrium polygon that it is possible with any given system of loading to make an arch of such form (viz., that of an equilibrium polygon) as to require no bracing whatever, since in that case there will be no tendency to bend at any point. Also it is evident that any deviation of part of the arch from this equilibrium polygon would need to be braced. As, for example, in case two distant points be joined by a straight girder, it must be braced to take the place of part of the arch. Furthermore, the greater the deviation the greater the bending moment to be sustained in this manner. Hence appears the general truth stated in the proposition.

It will be noticed that the moment called into action, at any point of a straight girder, depends not only on the applied forces which furnish the polygonal part of the equilibrium polygon, but also on the resistance which the girder is capable of sustaining at joints or supports, or the like. For example, if the girder rests freely on its end-supports, the moment of resistance vanishes at the ends, and the "closing line" of the polygon joins the extremities of the polygonal part. If however the ends are fixed horizontally and there are two free (hinge) joints at other points of the girder, the polygonal part will be as before, but the closing line would be drawn so that the moments at those two points vanish. Similarly in every case (though the conditions may be more complicated than in the examples used for illustration) the position of the closing line is fixed by the joints or manner of support of the girders, for these furnish the conditions which the moments (*i. e.*, the ordinates of the equilibrium polygon) must fulfill. For example, in a straight uniform girder without joints and fixed horizontally at the ends, the conditions are evidently these; the total bending vanishes when taken from end to end, and the deflection of one end below the tangent at the other end also vanishes.

Prop. IV. If in any arch that equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from consideration of the conditions imposed by the supports, etc.; and if furthermore the curve of the arch itself be regarded as another equilibrium polygon due to some system of loading not given, and its closing line be also found from the same considerations respecting supports, etc., then, when these two polygons are placed so that these closing lines coincide and their areas partially cover each other, the ordinates intercepted between these two polygons are proportional to the real bending moments acting in the arch.

Suppose that an equilibrium polygon due to the weights be drawn having the same horizontal thrust as the arch. We are in fact unable to do this at the outset as the horizontal thrust is unknown. We only suppose it drawn for the purpose of discussing its properties. Let also the closing line be drawn, which may be done, as will be seen hereafter. Call the area between the closing line and the polygon, $A$. Draw the closing line of the curve of the arch itself (regarded as an equilibrium polygon) according to the same law, and call the area between this closing line and its curve $A''$. Further let $A'$ be the area of a polygon whose ordinates represent the actual moments bending the arch, and drawn on the same scale as $A$ and $A''$. Since the supports etc., must influence the position of the closing line of this polygon in the same manner as that of $A$, we have by Prop. III not only

$$A=A'+A''$$

which applies to the entire areas, but also

$$y=y'+y''$$

as the relation between the ordinates of these polygons at any of the points of division before mentioned, from which the truth of the proposition appears.

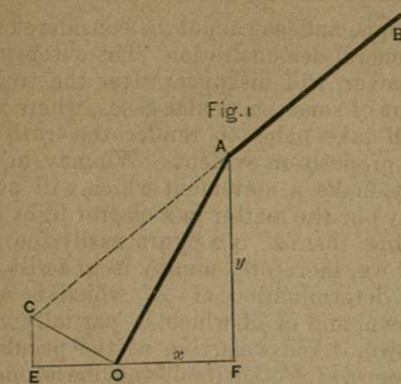
This demonstration in its general form may seem obscure since the conditions imposed by the supports, etc., are quite various, and so cannot be considered in a general demonstration. The obscurity, however, will disappear after the treatment of some particular cases, where we shall take pains to render the truth of the proposition evident. We may, however, make a statement which will possibly put the matter in a clearer light by saying that $A''$ is a figure easily found, and we, therefore, employ it to assist in the determination of $A'$ which is unknown, and of $A$ which is partially unknown. And we arrive at the peculiar property of $A''$, that its closing line is found in the same manner as that of $A$, by noticing that the positions of the closing lines of $A$ and $A'$ are both determined in the same manner by the supports, etc.; for the same law would hold when the rise of the arch is nothing as when it has any other value. But $A''$ is the difference of $A$ and $A'$. Hence what is true of $A$ and $A'$ separately is true of their difference $A''$, the law spoken of being a mere matter of summation.

From this proposition it is also seen that the curve of the arch itself may be regarded as the curved closing line of the polygon whose ordinates are the actual bending moments, and the polygon itself is the polygonal part of the equilibrium polygon due to the weights.

It is believed that Prop. IV contains an important addition to our previous knowledge as to the bending moments in an arch, and that it supplies the basis for the heretofore missing method of obtaining graphically the true equilibrium polygon for the various kinds of arches.

Prop. V. If bending moments $M$ act on a uniform inclined girder at horizontal distances $x$ from $O$, the amount of the vertical deflection $y_d$ will be the same as that of a horizontal girder of the same cross section, and having the same horizontal span, upon which the same moments $M$ act at the same horizontal distances $x$ from $O$. Also, if bending moments $M$ act as before, the amount of the horizontal deflection, say $x_d$, will be the same as that of a vertical girder of the same cross section, and having the same height, upon which the same moments $M$ act at the same heights.






Let the moment $M$ act at $A$, producing according to equation (5) the deflection

$$OC=e.M.AO$$

whose vertical and horizontal components are

$$y_d = CE \quad \text{and} \quad x_d = OE$$

For the small deflections occurring in a girder or arch, $AOC=90°$

$$\therefore AO : OF :: OC : CE$$

$$\therefore CE=\frac{OC}{AO}.OF=e.M.OF$$

$$\therefore y_d = e.Mx$$

$$\text{Also, } AO : AF :: OC : OE$$

$$\therefore OE=\frac{OC}{AO}.AF=e.M.AF$$

$$\therefore x_d = e.My$$

The same may be proved of any other moments at other points; hence a similar result is true of their sum; which proves the proposition.

It may be thought that the demonstration is deficient in rigor by reason of the assumption that $AOC=90°$.

Such, however, is not the fact as appears from the analytic investigation of this question by Wm. Bell in his attempted graphical discussion of the arch in Vol. VIII of this Magazine, in which the only approximation employed is that admitted by all authors in assuming that the curvature is exactly proportional to the bending moment.

We might in this proposition substitute $f.M \div I$ for $e.M$, and prove a similar but more general proposition respecting deflections, which the reader can easily enunciate for himself.

Before entering upon the particular discussions and constructions we have in view, a word or two on the general question as to the manner in which the problem of the arch presents itself, will perhaps render apparent the relations between this and certain previous investigations. The problem proposed by Rankine, Yvon-Villarceaux, and other analytic investigators of the arch, has been this:—Given the vertical loading, what must be the form of an arch, and what must be the resistances of the spandrils and abutments, when the weights produce no bending moments whatever? By the solution of this question they obtain the equation and properties of the particular equilibrium polygon which would sustain the given weights. Our graphical process completely solves this question by at once constructing this equilibrium polygon. It may be remarked in this connection, that the analytic process is of too complicated a nature to be effected in any, except a few, of the more simple cases, while the graphical process treats all cases with equal ease.

But the kind of solution just noticed, is a very incomplete solution of the problem presented in actual practice; for, any moving load disturbs the distribution of load for which the arch is the equilibrium polygon, and introduces bending moments. For similar reasons it is necessary to stiffen a suspension bridge. The arch must then be proportioned to resist these moments. Since this is the case, it is of no particular consequence that the form adopted for the arch in any given case, should be such as to entirely avoid bending moments when not under the action of the moving load.

So far as is known to us, it is the universal practice of engineers to assume the form and dimensions, as well as the loading of any arch projected, and next to determine whether the assumed dimensions are consistent with the needful strength and stability. If the assumption is unsuited to the case in hand, the fact will appear by the introduction of excessive bending moments at certain points. The considerations set forth furnish a guide to a new assumption which shall be more suitable, it being necessary to make the form of the arch conform more closely to that of the equilibrium polygon for the given loading.

The question may be regarded as one of economy of material, and ease of construction, analogous to that of the truss bridge. In this latter case, constructors have long since abandoned any idea of making bridges in which the inclination of the ties and posts should be such as to require theoretically the minimum amount of material. Indeed, the amount of material in the case of a theoretic minimum, differs by such an inconsiderable quantity from that in cases in which the ties and posts have a very different inclination, that the attainment of the minimum is of no practical consequence.

Similar considerations applied to the arch, lead us to the conclusion that the form adopted can in every case be composed of segments of one or more circles, and that for the purpose of construction every requirement will then be met as fully as by the more complicated transcendental curves found by the writers previously mentioned. If considerations of an artistic nature render it desirable to adopt segments of parabolas, ellipses or other ovals, it will be a matter of no more consequence than is the particular style of truss adopted by rival bridge builders.

We can also readily treat the problem in an inverse manner, viz:—find the system of loading, of which the assumed curve of the arch is the equilibrium polygon. From this it will be known how to load a given arch so that there shall be no bending moments in it. This, as may be seen, is often a very useful item of information; for, by leaving open spaces in the masonry of the spandrils, or by properly loading the crown to a small extent, we may frequently render a desirable form entirely stable and practicable.

## CHAPTER II.

### THE ARCH RIB WITH FIXED ENDS.

Let us take, as the particular case to be treated, that of the St. Louis Bridge, which is a steel arch in the form of the arc of a circle; having a chord or span of 518 feet and a versed sine or rise of one-tenth the span, *i. e.* 51.8 feet. The arch rib is firmly inserted in the immense skew-backs which form part of the upper portion of the abutments. It will be assumed that the abutments do not yield to either the thrust or weight of the arch and its load, which was also assumed in the published computations upon which the arch was actually constructed. Further, we shall for the present assume the cross section of the rib to have the same moment of inertia, $I$, at all points, and shall here only consider the stresses induced by an assumed load. The stresses due to changes in the length of the arch itself, due to its being shortened by the loading, and to the variations of temperature, are readily treated by a method similar to the one which will be used in this article, and will be treated in a subsequent chapter.

Let $b_8\, a\, b_8'$ in Fig. 2, be the neutral axis of the arch of which the rise is one-tenth the span. Let $a\, x\, y\, z$ be the area representing the load on the left half of the arch, and $a\, x'\, y'\, z'$ that on the right, so that $yp=a\,.\,P=xy$ on the left, and $yp=x'y'$ on the right.

Divide the span into sixteen equal parts $bb_1$, $bb_1'$, etc, and consider that the load, which is really uniformly distributed, is applied to the arch at the points $a$, $a_1$, $a_1'$, etc., in the verticals through $b$, $b_1$, $b_1'$, etc.; so that the equal weights $P$ are applied at each of the points on the left of $a$ and the equal weights $\frac{1}{2}P$ at each point on the right of $a$, while $\frac{3}{4}P$ is applied at $a$.

Take $b$ as the pole of a force polygon for these weights, and lay off the weights which are applied at the left of $a$ on the vertical through $b_8$, viz., $b_8\, w_1=\frac{1}{2}P=$ the weight coming to $a$ from the left; $w_1\, w_2=P=$ the weight applied at $a_1$; $w_2\, w_3=P=$ the weight applied at $a_2$, etc. Using $b$ still as the pole, lay off $b_8'\, w_1'=\frac{1}{4}P=$ the weight coming to $a$ from the right; $w_1'\, w_2'=\frac{1}{2}P=$ the weight applied at $a_1'$, etc. This amounts to the same thing as if all the weights were laid off in the same vertical. Part are put at the left and part at the right for convenience of construction. Now draw $bw_1$ until it intersects the vertical 1 at $c_1$; then draw $c_1\, c_2 \parallel bw_2$; and $c_2\, c_3 \parallel bw_3$, etc. In the same manner draw $bw_1'$ to $c_1'$; then $c_1'\, c_2' \parallel bw_2'$, etc. Then the broken line $bc_1 \ldots c_8$ is the equilibrium polygon due to the weights on the left of $a$, and $bc_1' \ldots c_8'$ is that due to the weights on the right. Had the polygon been constructed for the uniformly distributed load (not considered as concentrated), on the left we should have a parabola passing through the points $bc_1 \ldots c_8$, and another parabola on the right through $bc_1' \ldots c_8'$. From the properties of this parabola it is easily seen that $c_8$ must bisect $w_4\, w_5$, as $c_8'$ must also bisect $w_4'\, w_5'$; which fact serves to test the accuracy of our construction. This test is not so simple in cases of more irregular loading.



The equilibrium polygon $c_8\, b\, c_8'$ is that due to the applied weights, but if these weights act on a straight girder with fixed ends, this manner of support requires that the total bending be zero, when the sum is taken of the bending at the various points along the entire girder; for, the position of the ends does not change under the action of the weights, hence the positive must cancel the negative bending. To express this by our equations:

$$yb=e.\ \Sigma(M)=0\ \therefore\ \Sigma(M)=0.$$

This is one of two conditions which are to enable us to fix the position of the true closing line $h_8\, h_8'$ in this case. The other condition results from the fact that the algebraic sum of all the deflections of this straight girder must be zero if the ends are fixed horizontally.

This is evident from the fact that when one end of a girder is built in, if a tangent be drawn to its neutral axis at that end, the tangent is unmoved whatever deflections may be given to the girder; and if the other end be also fixed, its position with reference to this tangent is likewise unchanged by any deflections which may be given to the girder. To express this by our equations:

$$yd=f.\ \Sigma\,(Mx)=0\ \therefore\ \Sigma\,(Mx)=0$$

The method of introducing these conditions is due to Mohr. Consider the area included between the straight line $c_8\, c_8'$ and the polygon $c_8\, b\, c_8'$ as some species of plus loading; we wish to find what minus loading will fulfill the above two conditions. Evidently the whole negative loading must be equal numerically to the whole positive loading, if we are to have $\Sigma\,(M)=0$. Next, as the closing line is to be straight, the negative load $c_8\, c_8'\, h_8\, h_8'$ may be considered in two parts, viz., the two triangles, $c_8\, c_8'\, h_8$ and $c_8'\, h_8\, h_8'$. Let the whole span be trisected at $t$ and $t'$, then the total negative loading may be considered to be applied in the verticals through $t$ and $t'$, since the centers of gravity of the triangles fall in these verticals. Again, the positive loading we shall find it convenient to distribute in this manner: viz., the triangle $c_8\, b\, c_8'$ applied in the vertical through $b$, the parabolic area $b\, c_1 \ldots c_8$ in the vertical 4 which contains its center of gravity, and the parabolic area $bc_1' \ldots c_8'$ in 4'.

Now these areas must be reduced to equivalent triangles or rectangles, with a common base, in order that we may compare the loads they represent. Let the common base be half the span: then $bb_0=pp'$ is the positive load due to the triangle $c_8\, b\, c_8'$; and $\frac{2}{3}\, c_4\, c_0=pp_1$ and $\frac{2}{3}\, c_4'\, c_0'=p'p_1'$ are the positive loads due to the parabolic areas.

Now assume any point $q$ as a pole for the load line $p_1\, p_1'$ and find the center of gravity of the positive loading by drawing the equilibrium polygon, whose sides are parallel to the lines of this force polygon: viz., use $qp_1$ and $qp$ as the 1st and 2nd sides, and make $pq' \parallel qp'$, and $q'q_1 \parallel qp_1'$. The first and last sides intersect at $q_1$; therefore the center of gravity of the positive loads must lie in the vertical through $q_1$.

Now the negative loading must have its center of gravity in the same vertical, in order that the condition $\Sigma\,(Mx)=0$ may be satisfied, for it is the numerator of the general expression for finding the center of gravity of the loading. The question then assumes this form: what negative loads must be applied in the verticals through $t$ and $t'$ that their sum may be $p_1\, p_1'$, and that they may have their center of gravity in the vertical through $q_1$.

The shortest way to obtain these two segments of $p_1\, p_1'$ is to join $r$ and $r'$ which are in the horizontals through $p_1$ and $p_1'$, and draw an horizontal through $q_0$, which is the intersection of $r\, r'$ with the vertical through $q_1$; then $r\, r_1$ and $r'\, r_1'$ are the required segments of the negative load. For, let $r\,r_2=p_1'p_1$ and take $r'$ as the pole of the load $r\,r_2$; then, since $r_1 q_0 \parallel r_2 r'$ and $q_0 r' \parallel r\,r'$ we have the equilibrium polygon $r_1 q_0 r'$ fulfilling the required conditions.

Now these two negative loads $r_1 r_2 = r_1' r'$ and $r\,r_1$, are the required heights of the triangles $c_8\, h_8\, c_8'$ and $c_8\, c_8'\, h_8'$; therefore lay off $c_8\, h_8 = r' r_1'$ and $c_8' h_8' = r r_1$.

The closing line $h_8\, h_8'$ can then be drawn, and the moments bending the straight girder will then be proportional to $h_1\, c_1$, $h_2\, c_2$, etc., the points of inflexion being where the closing line intersects the polygon. If the construction has been correctly made, the area above the closing line is equal to that below, a test easy to apply.

Let us now turn to the consideration of the curve of the arch itself, and treat it as an equilibrium polygon. Since the rise of the arch is such a small fraction of the span, the curve itself is rather flat for our purposes, and we shall therefore multiply its ordinates $ab$, $a_1\, b_1$, etc., by any number convenient for our purpose: in this case, say, by 3. We thereby get a polygon $d_8\, d\, d_8'$ such that $d\,b=3\,ab$, $d_1\, b_1 = 3\, d_1\, b_1$, etc. If a curve be described through $d_8 \ldots d \ldots d_8'$ it will be the arc of an ellipse, of which $d$ is the extremity of the major axis.

If we wish to find the closing line $k_8 k_8'$ of this curve, such that it shall make $\Sigma\,(M_d)=0$ and $\Sigma\,(M_d x)=0$, the same process we have just used is here applicable; but since the curve is symmetrical, the object can be effected more easily. By reason of the symmetry about the vertical through $b$, the center of gravity of the positive area above the horizontal through $b$ lies in the vertical through $b$. The center of gravity of the negative area lies there also; hence the negative area is symmetrical about the center vertical; the closing line must then be horizontal. It only remains then to find the height of a rectangle having the same area as the elliptical segment, and having the span for its base. This is done very approximately by taking (in this case where the span is divided into 16 equal segments) $\frac{1}{8}$ the sum of the ordinates $b_1\, d_1$, etc.

We thus find the height $bk$ and the horizontal through $k$ is the required closing line.

Before effecting the comparison which we intend to make between the polygons $c$ and $d$ (as we may briefly designate the polygons $c_8\, b\, c_8'$ and $d_8\, d\, d_8'$), let us notice the significance of certain operations which are of use in the construction before us. One of these is the multiplication of the ordinates of the polygon or curve $a$ to obtain those of $d$. If $a$ was inverted, certain weights might be hung at the points $a_1$, $a_2$, etc., such that the curve would be in stable equilibrium, even though there are flexible joints at these points. Equilibrium would still exist in the present upright position under these same applied weights, though it would be unstable. If now, radiating from any point, we draw lines, one parallel to each of the sides $aa_1$, $a_1\, a_2$, $aa_1'$, etc., of the polygon, then any vertical line intersecting this pencil of radiating lines will be cut by it in segments, which represent the relative weights needed to make $a$ their equilibrium polygon. By drawing the vertical line at a proper distance from the pole, its total length, *i.e.*, the total load on the arch can be made of any amount we please. The horizontal line from the pole to this vertical will be the actual horizontal thrust of the arch measured on the same scale as the load. If a like pencil of radiating lines be drawn parallel to the sides of the polygon $d$ and the load be the same as that we had supposed upon the polygon $a$, it is at once seen that the pole distance for $d$ is one-third of that for $a$; for, every line in $d$ has three times the rise of the corresponding one in $a$, and hence with the same rise, only one-third the horizontal span. The increase of ordinates, then, means a decrease of pole distance in the same ratio, and vice versa. As is well known, the product of the pole distance by the ordinate of the equilibrium polygon is the bending moment. This product is not changed by changing the pole distance.

Again, suppose the vertical load-line of a force polygon to remain in a given position, and the pole to be moved vertically to a new position. No vertical or horizontal dimension of the force polygon is affected by this change, neither will any such dimension of the equilibrium polygon corresponding to the new position of the pole be different from that in the polygon corresponding to the first position of the pole; the direction of the closing line, however, is changed. Thus we see that the closing line of any equilibrium polygon can be made to coincide with any line not vertical, and that its ordinates will be unchanged by the operation. It is unnecessary to draw the force polygon to effect this change.



Now to make clear the relationship between the polygons $c$ and $d$, let us suppose, for the instant, that the polygon $e$ has been drawn by some means as yet unknown, so that its ordinates from $d$, viz., $e_1 d_1 = y_1$, $e_2 d_2 = y_2$, etc., are proportional to the actual moments $M_e$ which tend to bend the arch.

The conditions which then hold respecting these moments $M_e$, are three:—

$$\Sigma\,(M_e)=0,\quad \Sigma\,(M_e x)=0,\quad \Sigma\,(M_e y)=0.$$

The first condition exists because the total bending from end to end is zero when the ends are fixed. The second and third are true, because the total deflection is zero both vertically and horizontally, since the span is unvariable as well as the position of the tangents at the ends. These results are in accordance with Prop. V. Now by Prop. III these moments $M_e$ are the differences of the moments of a straight girder and of the arch itself; hence the polygon $e$ is simply the polygon $c$ in a new position and with a new pole distance. As moments are unchanged by such transformations, let us denote these moments by $M_c$. We have before seen that

$$\Sigma\,(M_c)=0, \text{ and } \Sigma\,(M_c x)=0$$

Subtract

$$\therefore\ \Sigma\,(M_c - M_e)=0, \text{ and } \Sigma(M_c - M_e)x=0$$

$$\therefore\ \Sigma\,(M_d)=0 \text{ and } \Sigma\,(M_d x)=0$$

From this it is seen that the polygon $d$ must have its closing line fulfill the same conditions as the polygon $c$. This is in accordance with Prop. IV.

$$\text{Again, } \Sigma\,(M_e y) = \Sigma\,(M_c - M_d)\,y = 0$$

$$\therefore\ \Sigma\,(M_c y) = \Sigma\,(M_d y).$$

This last condition we shall use for determining the pole distance of the polygon $e$, which is one-third of the actual thrust of the arch measured on the scale of the weights $w_1\, w_2$, etc. The physical significance of this condition may be stated according to Prop. V, thus: if the moments $M_d$ are applied to a uniform vertical girder $bd$ at the points $b$, $b_7''$, $b_6''$, $b_5''$, etc., at the same height with $b_8$, $d_7$, etc., they will cause the same total deflection $xd = e\,.\,\Sigma\,(M_d y)$ as will the moments $M_c$ when applied at the same points. Hence if $M_d$ are used as a species of loading, we can obtain the deflection by an equilibrium polygon. Suppose the load at $d_7$ is $d_7\, k_7$, and that at $d_6$ is $d_6\, k_6$, etc., then that at $b_8$ is $\frac{1}{2}\, b_8\, k_8$. This approximation is sufficiently accurate for our purposes.

Now lay off on $l_8\, l_8'$ as a load line $dm_8 = \frac{1}{2}\, b_8\, k_8$, $m_8\, m_7 = d_7\, k_7$, $m_7\, m_6 = d_6\, k_6$, etc. The direction of these loads must be changed when they fall on the other side of the line $k$; *e.g.*, $m_5\, m_4 = k_4\, d_4$. If this process be continued through the entire arch $m_5'$ (not drawn) will fall as far to the right of $d$ as $m_5$ does to the left, and the last load will just reach to $d$ again. This is a test of the correctness with which the position of the line $k_8\, k_8'$ has been found. Now using any point as $b$ for a pole, draw $bm_8$ to $f_7$, then draw $f_7 f_6 \parallel bm_7$, $f_6 f_5 \parallel bm_6$, etc. The curve $bf$ is then the exaggerated shape of a vertical girder $bd$, fixed at $b$, under the action of that part of moments $M_d$ which are in the left half of the arch. The moments $M_d$ on the right may act on another equal girder, having the same initial position $bd$, and it will then be equally deflected to the right of $bd$. This is not drawn.

Again, suppose these vertical girders fixed at $b$ are bent instead by the moments $M_c$. We do not know just how much these moments are, though we do know that they are proportional to the ordinates of the polygon $c$. Therefore make $dn_8 = \frac{1}{2}\, h_8\, c_8$, $n_8\, n_7 = h_7\, c_7$, $n_7\, n_6 = h_6\, c_6$, etc. When all these loads are laid off, the last one $n_8'\, d = \frac{1}{2}\, h_8'\, c_8'$ must just return to $d$. This tests the accuracy of the work in determining the position of $h_8\, h_8'$.

Now using $b$ as a pole as before, construct the deflection curves $bg$ and $bg'$. Since these two deflections, viz., $2\ df$ and $gg'$ ought to be the same, this fact