informs us that each of the ordinates $h_1 c_1$, $h_2 c_2$, must be increased in the ratio of $\frac{1}{2} gg'$ to df, in order that when they are considered as loads, they may produce a total deflection equal to $2\,df$. To effect this, lay off $bj = df$ and $bi = \frac{1}{2} gg'$, and draw the horizontals through i and j. At any convenient distance draw the vertical $i_0 j_0$, and draw bi_0 and bj_0. These last two lines enable us to effect the required proportions for any ordinates on the left, and these or two lines of the same slope on the right to do the same thing on the right. *E. g.* lay off the ordinate $bi_8' = h_8' c_8'$, then the required new ordinate is bj_8'. Then lay off $k_8' e_8' = bj_8'$. In the same manner find ke from hb, and $k_8 e_8$ from $h_8 c_8$. In the same manner can the other ordinates $k_1 e_1$, etc., be found; but this is not the best way to determine the rest of them, for we can now find the pole and pole distance of the polygon e.

As we have previously seen, the pole distance is decreased in the same ratio as the ordinates of the moment curve are increased, therefore prolong bi_0 to v_1, and draw a horizontal line through v_1 intersecting bj_0 at v_2 and the middle vertical at v_0; then is $v_2 v_0$ the pole distance decreased in the required ratio. Hence we move up the weight-line $w_1 w_8$ to the position $u_1 u_8$ vertically through v_2; and for convenience, lay off the weights $w_1' w_2'$ at $u_1' u_2'$, etc.

Furthermore, we know that the new closing-line is horizontal. To find the position of the pole o so that this shall occur, draw bv parallel to hh_8, and from v the horizontal vo. As is well known, v divides the total weight into the two segments, which are the vertical resistances of the abutments, and if the pole o is on the same horizontal with v, the closing line will be horizontal.

Now having determined the positions of the points e_8, e, e_8', starting from one of them, say e_8, draw $e_8 e_7 \parallel ou_8$, $e_7 e_6 \parallel ou_7$, etc.; then if the work be accurate, the polygon will pass through the other two points e and e_8'. The bending moments of the arch d or the arch a at a_1, a_2, etc., is the product of the pole distance $v_0 v_2 = v'o$ by the ordinates $d_1 e_1$, $d_2 e_2$, etc., respectively, and between these points a similar product gives the moment with sufficient accuracy. It would be useful for the sake of accuracy to multiply the ordinates of the arch by some number greater than 3.

As a final test of the accuracy of the work, let us see whether $\Sigma(M_e y)$ is actually zero, as should be. At d_7, for example, $y = d_7 l_7$, and M_e is proportional to $d_7 e_7$. Then $\overline{d_7 s_7}^2$ is proportional to $M_e y$ at that point if $e_7 s_7$ is the arc of a circle, of which $e_7 l_7$ is the diameter. Similarly find $d_7' s_7'$, etc. When e_4 for example falls above d_4, the circle must be described on the sum of $l_4 d_4$ and $d_4 e_4$ as a diameter, and $\overline{d_4 s_4}^2$ is proportional to a moment of different sign from that at d_7. We have distinguished the sign of the moments at the different points along the arch, by putting different signs before the letter s. It would have been slightly more accurate to have used only one-half the ordinates $b_8 e_8$ and $b_8' e_8'$, but as they nearly equal in this case and of opposite sign, we have introduced no appreciable error.

Now at any point s lay off $ss_7 = d_7 s_7$, and at right angles to it $s_7 s_8 = b_8 s_8$, then at right angles to the hypothenuse ss_8 make $s_8 s_5' = d_5' s_5'$, etc. Then the sum of the positive squares is ss_1', and similarly the sum of the negative squares is ss_1. If these are equal, then $\Sigma(M_e y)$ vanishes as it should, and the construction is correctly made.

It would have been equally correct to suppose the two vertical girders fixed at d, and bent by the moments acting. We could have determined the required ratio equally well from this construction. Further, in proving the correctness of the construction by taking the algebraic sum of the squares, we could have reckoned the ordinates, y, from any other horizontal line as well as from $l_8 l_8'$.

To find the resultant stress in the different portions of the arch, we must prolong $v'o$ to o', say, (not drawn) so that the pole distance $v'o' = 3\,v'o$; then if we join o' and u_8, $o'u_8$ will be the resultant stress in the segment $b_8 a_7$; $o'u_7$ will be the stress in $a_7 a_6$, etc., measured in the same scale as the weights $w_1 w_2$, etc. This resultant stress is not directly along the neutral axis of the arch.

The vertical shearing stress is constructed in the same manner as for a girder, by drawing one horizontal through w_8 between the verticals 7 and 8, another

through w_7 between 7 and 6, etc. (not drawn). Then the shear will be the vertical distance between vo and these horizontals through w_8, w_7, etc. It is seen that the shear will change sign on the vertical through b_1 with our present loading.

The actual position of the vertical through the center of gravity of the load may be found by prolonging the first and last sides of the polygon c. A weight $= \frac{1}{2} P = w_8 w_9$ ought, however, first to be applied at b_8, and another $= \frac{1}{4} P = w_8' w_9'$ at b_8'. The shearing stress under a distributed load will actually change sign on the vertical so found. It will not pass far however from b_1.

The resultant stress is the resultant of the horizontal thrust and the vertical shearing stress, and it can be resolved into a tangential thrust along the arch and a normal shearing stress. This resolution will be effected in Fig. 3 of the next chapter.

As to the position of the moving load which will produce the maximum bending moments, we may say that the position chosen, in which the moving load covers one-half the span, gives in general nearly this case. It is possible, however, to increase one or two of the moments slightly by covering a little more than half the span with the moving load.

The loading which produces maximum moments will be treated more fully in subsequent chapters.

The maximum resultant stress and maximum vertical shear occur in general when the moving load covers the whole span. The construction in this case is much simplified, as the polygon c is then the same on the right of b as it now is on the left, and the center of gravity of the area is in the center vertical; so that the closing line $h_8 h_8'$ is horizontal, and can be drawn with the same ease as $k_8 k_8'$ was drawn. We shall not, even in this case, be under the necessity of drawing the curves bg and bg', which would be both alike; for, as may be readily seen, the sum of the positive moments M_c on the left must be very approximately equal to the positive moments M_d on the left, and the same thing is true for the negative moments at the left. The same two equalities hold also on the right. From this we at once obtain the ratio by which the ordinates of the polygon c must be altered to obtain those of the polygon e.

This last approximation also shows us that for a total uniform load, the four points of inflection when the bending moment is zero, lie two above and two below the closing line. It is frequently a sufficiently close approximation in the case when the moving load covers only part of the span to derive the ratio needed by supposing that the sum of all the ordinates, both right and left, above the closing line in the polygon c must be increased, so that it shall equal the corresponding sum in the polygon d. If the sums taken below the closing lines give a slightly different result, take the mean value.

Thus the single construction we have given in Fig. 2, and one other much simpler than this, which can be obtained by adding a few lines to Fig. 2, give a pretty complete determination of the maximum stresses on the assumptions made at the commencement of the article.

One of these assumptions, viz., that of constant cross section (*i. e.* I=constant), deserves a single remark. In the St. Louis Arch I was increased one-half at each end for a distance of one-twelfth of the span. This very considerable change in the value of I slightly reduced the maximum moments computed for a constant cross section. From other elaborate calculations, particularly those of Heppel,* on the Britannia Tubular Bridge, it appears that the variation in the moments caused by the changes in cross section, which will adapt the rib to the stresses it must sustain, are relatively small, and in ordinary cases are less than five per cent. of the total stress. The same considerations are not applicable near the free ends of a continuous girder, where I may theoretically vanish. In the case before us, where the principal part of the stress arises not from the bending moments, but from the compression along the arch, the effect of the variation of I is very inconsiderable indeed.

* *Philosophical Magazine*, Vol. 40, 1870.

CHAPTER III.

ARCH RIB WITH FIXED ENDS AND HINGE JOINT AT THE CROWN.

Let the curve a of Fig. 3 represent the proportions of the arch we shall use to illustrate the method to be applied to arches of this character. The arch a is segmental in shape, and has a rise of one-fifth of the span. It is unnecessary to assume the particular dimensions in feet, as the above ratio is sufficient to determine the shape of the arch.

The arch is supposed to be fixed in the abutments, in such a manner that the position of a line drawn tangent to the curve a at either abutment is not changed in direction by any deflection which the arch may undergo. At the crown, however, is a joint, which is perfectly free to turn, and which will, then, not allow the propagation of any bending moment from one side to the other. In order that we may effect the construction more accurately, let us multiply the ordinates of the curve a by some convenient number, say 2, though a still larger multiplier would conduce to greater accuracy. We thus obtain the polygon d.

Having divided the span b into twelve equal parts $b_1 b_2$, etc, (a larger number of parts would be better for the discussion of an actual case), we lay off below the horizontal line b on the end verticals, lengths which express on some assumed scale the weights which may be supposed to be concentrated at the points of division of the arch. If $a\,l$ is the depth of the loading on the left and $al'=\frac{1}{2}al$ that on the right, then $b_6w_1+b_6'w_6'=$ the weight concentrated at a; $w_1w_2=$ the weight at a_1; $w_1'\,w_2'=$ the weight at a_1', etc. Using b as a pole, draw the equilibrium polygon c, whose extremities c_6 and c_6' bisect $w_3\,w_4$ and $w_3'\,w_4'$ respectively.

Now to find the closing line of this equilibrium polygon so that its ordinates shall be proportional to the bending moments of a straight girder of the same span, and of a uniform moment of inertia I, which is built in horizontally at the ends and has a hinge joint at its center; we notice in the first place that the bending moment at the hinge is zero, and hence the ordinate of the equilibrium polygon at this point vanishes. The closing line then passes through b the point in question. Furthermore it is evident that if we consider the parts of the girder at the right and left of the center as two separate girders whose ends are joined at the center, these ends have each the same deflection, by reason of this connection.

This is expressed by means of our equations by saying that $\Sigma(Mx)$ when the summation is extended from one end to the center is equal to $\Sigma(Mx)$ when the summation is extended from the other end to the center, for these are then proportional to the respective deflections of the center. We may then write it thus:

$$\Sigma_{b_6}^{b}(Mx)=\Sigma_{b_6'}^{b}(Mx)$$

The equation has this meaning, viz: that the center of gravity of the right and left moment areas taken together is in the center vertical: for, taking each moment M as a weight, x is its arm, and Mx its moment about the center.

In order to find in what direction to draw the closing line through b so that it shall cause the moment areas together to have their center of gravity in the center vertical through b, let us draw a second equilibrium polygon using the moment areas as a species of loading.

The area on the left included between any assumed closing line as bb_6 (or bh_6) and the polygon bc_6 may be considered to consist of a positive triangular area bc_6b_6 (or bc_6h_6) and a negative parabolic area $bc_3c_6c_0$; and similarly on the right a positive area $bc_6'b_6'$ (or $bc_6'h_6'$) and a negative area $bc_3'c_6'c_0'$.

At any convenient equal distances from the center as at p and p', lay off these loads to some convenient scale. It is, perhaps, most convenient to reduce the moment areas to equivalent triangles having each a base equal to half the span: then take the altitudes of the triangles as the loads. This we have done, so that $pp_1=\frac{4}{3}c_0c_3$, and $p'p_1'=\frac{4}{3}c_0'c_3'$. Now assume, for the instant, that closing line is b_6b_6', which of course is incorrect, and make $p_1p_2=b_6c_6$ and $p_1'p_2'=b_6'c_6'$, then these are the loads due to the positive triangular areas at the left and right respectively, while pp_1 and $p'p_1'$ are the negative parabolic loads.

Take o' as the pole of these loads, then pp' may be taken for the first side of the second equilibrium polygon. Draw $pq \parallel o'p_1$ and $p'q' \parallel o'p_1'$, and then from q and q' draw parallels to $o'p_2'$ respectively. These last sides intersect at q_2. The vertical through q_2 then contains the center of gravity of the moment areas when $b_6\,b_6'$ is assumed as the closing line.

A few trials will enable us to find the position of the closing line which causes the center of gravity to fall on the center vertical. We are able to conduct these trials so as to lead at once to the required closing line as follows. Since, evidently, $b_6c_6+b_6'c_6'=h_6c_6+h_6'c_6'$, it is seen that the sum of the positive loads is constant. Therefore make $p_2p_3=p_2'p_3'$ and use p_1p_3 and $p_1'p_3'$ as the positive loads, in the same manner as we used $p_1\,p_2$ and $p'_1p'_2$ previously.

This will be equivalent to assuming a new position of the closing line. The only change in the second equilibrium polygon will be in the position of the last two sides. These must now be drawn parallel to $o'p_3$ and $o'p_3'$ respectively; and they intersect at q_3. The vertical through q_3 contains the center of gravity for this assumed closing line. Another trial gives us q_4.

Now if the direction of the closing line had changed gradually, then the intersection of the last sides of the second equilibrium polygon would have described a curve through q_2, q_3 and q_4. If one of these points, as q_3, is near the center vertical, then the arc of a circle $q_2q_3\,q_4$, will intersect it at q_5 indefinitely near to the point where the true locus of the points of intersection would intersect the center vertical.

Let us assume that q_5 is then determined with sufficient exactness by the circular arc $q_2q_3q_4$, and draw qq_5 and $q'q_5$ as the last two sides of the second equilibrium polygon. Now draw $o'p_5 \parallel qq_5$ and $o'p_5' \parallel q'q_5$, then $p_1p_5=c_6h_6$ and $p_1'p_5'=c_6'h_6'$ are the required positive loads, and h_6bh_6' is the position of the closing line such that the center of gravity of the moment areas is in the center vertical.

It is evident that the closing line of the polygon d considered as itself an equilibrium polygon is the horizontal line through d, for that will cause the center of gravity of the moment areas on the left and right, between it and the polygon d, to fall on the center vertical.

The next step in the construction is to apply Prop. IV, for the determination of the bending moments.

That Prop. IV is true for an arch of this kind is evident; for, the loading causes bending moments proportional to the ordinates h_2c_2, h_3c_3, etc., while the arch itself is fitted to neutralize, in virtue of its shape, moments which are proportional to k_2d_2, k_3d_3, etc. The differences of the moments represented by these ordinates are what actually produce bending in the arch.

Now the ordinates of the type hc are not drawn to the same scale as those of the type kd, for each was assumed regardless of the other. In order that we may find the ratio in which the ordinates hc must be changed to lay them off on the same scale as kd it is necessary to use another equation of condition imposed by the nature of the joint and supports, viz:

$$\Sigma_{b_6}^{a}(M_a-M_c)y=\Sigma_{b_6'}^{a}(M_a-M_c)y$$

or

$$\Sigma_{b_6}^{d}(M_d-M_c)y=\Sigma_{b_6'}^{d}(M_d-M_c)y$$

The left hand side of the equation is the horizontal displacement (*i.e.*, the total deflection) of the extremity a of the left half of the arch, due to the actual bending moments (M_d-M_c) acting upon it: and the right hand side is the horizontal displacement of a the extremity of the right half of the arch due to the moments actually bending it. These are equal because connected by the joint.

The construction of the deflection curves due to these moments will enable us to find the desired ratio.

The ordinates kd and hc are rather longer than can be used conveniently, to represent the intensity of the moments concentrated at d_1, d_2, etc, and c_1, c_2, etc.: so we will use the halves of these quantities instead. Therefore lay off $dm_6=\frac{1}{4}k_6b_6$, $m_6m_5=\frac{1}{2}k_5\,d_5$, $m_5m_4=\frac{1}{2}k_4d_4$, etc., and also $dn_6=\frac{1}{4}h_6c_6$, $n_6n_5=\frac{1}{2}h_5c_5$, etc.

We use only one-quarter of each end ordinate because the moment area supposed to be concentrated at each end has only one half the width of the moment areas concentrated at the remaining points of division.

Using b as a pole we find the deflection curve fb due to the moment M_a or M_d and the deflection curve gb due to the moments M_c on the left. On the right we should find a deflection $df'=df$ not drawn, and similarly a deflection dg' not equal to dg.

Now the equation we are using requires that the ordinates hc shall be elongated so that when used as weights the deflections shall be identical: *i.e.*, we must have $df=\frac{1}{2}gg'$. To effect the elongation, lay off $aj=df$ and $ai=\frac{1}{2}gg'$; and at any convenient distance on the horizontals ii_0 and jj_0 draw the vertical i_0j_0; then the lines ai_0 and aj_0 will effect the required elongation. For example, lay off $ai_6=h_6c_6$, from which we obtain $aj_6=k_6e_6$ for the left end ordinate, and similarly $aj_6'=k_6'e_6'$.

The pole distance tt_1, of the original polygon c must be shortened in the same ratio in which the ordinates are elongated. Hence the new pole distance of the polygon e is tt_2.

Since k_6k_6' is the closing line of the polygon e, and is horizontal, the pole of e is o, on the horizontal through h_6; for, h_6w_6 is the part of the applied weight sustained by the left support.

Now if the weight line be moved up to t_2 so that the applied weights are u_1u_1' at the center, etc., and o is the pole, the polygon e may be described starting from d, and it will finally cut off the end ordinates k_6e_6 and $k_6'e_6'$ before obtained. Then will the ordinates of the type de be proportional to the moments actually bending the arch, and the moments will be equal to the products of de by tt_2, in which de is measured on the scale of distance, and tt_2 on the scale adopted for the weights w_1w_2, etc.

The accuracy of the construction is finally tested by taking $\Sigma(ds)^2=0$, an equation deduced from $\Sigma''(M_d-M_c)y=0$, as explained in the previous article upon the St. Louis Arch. It is unnecessary to explain the details of this construction since as appears from Fig. 3 it is in all respects like that in Fig. 2.

Now let us find the intensity of the tangential compression along the arch and of the shearing normal to the arch. Since the pole distance tt_2 refers to the difference of ordinates between the polygons d and e, whose ordinates are double the actual ordinates, if we wish now to return to the actual arch a whose ordinates are halves of the ordinates of d, we must take a pole distance $tt_3=2tt_2$ and move the weight line so that it is the vertical through t_3. Then tt_3 is the actual horizontal thrust of this arch due to the weights; and ov_6 is the resultant stress in the segment a_5b_6 of the arch, which may be resolved into two components or_6 and v_6r_6 respectively parallel and perpendicular to a_5b_6.

Then are or_6 and v_6r_6 respectively, the thrust directly along, and the shear directly across the segment a_5b_6 of the arch. Similarly or_5 and v_5r_5 represent the thrust along, and the shear across the segment a_5a_4, and so on for other segments. These quantities are all measured in the same scale as that of the applied weights.

The shear changes sign twice, as will be seen from inspection of the directions in which the quantities of the type vr are drawn. The shear is zero wherever the curves d and e are parallel to each other. Thus the shear is nearly zero at b_6, at a_2 and at some point between a_2' and a_3'.

The maxima and minima shearing stresses are to be found where the inclination between the tangents to the curves d and e are greatest.

The statements made in the previous article, respecting the position of the moving load which causes maximum bending moments, are applicable to this kind of arch also.

The maximum normal shearing stress will occur for the parts of the arch near the center, when the moving load is near its present position, covering one half of the arch. But the maximum normal shearing stress near the ends, may occur when the arch is entirely covered by the moving load, or when it may occur when the moving load is near its present position, it being dependent upon the rise of the arch, and the ratio between the moving and permanent load.

The maximum tangential compressions occur when the moving load covers the entire arch. The stresses obtained by the foregoing constructions, go upon the supposition that the arch has a constant cross-section, so that its moment of inertia does not vary, and no account is taken of the stresses caused by any changes of the length of the arch rib, due to variations of temperature or other causes. These latter stresses we shall now investigate for both of the kinds of arches which have been treated.

CHAPTER IV.

TEMPERATURE STRAINS.

It is convenient to classify all strains and stresses arising from a variation in the length of the arch, under the head of temperature, as such stresses could evidently have been brought about by suitable variations of temperature.

The stresses of this kind which are of sufficient magnitude to be worthy of consideration, besides temperature stresses are of two kinds, viz. the elastic shortening of the arch under the compression to which it is subjected, and the yielding of the abutments, under the horizontal thrust applied to them by the arch. This latter may be elastic or otherwise. It was, I believe, neglected in the computation of the St. Louis Arch, and no doubt with sufficient reason, as the other stresses of this kind were estimated with a sufficient margin to cover this also. Anything which makes the true span of the arch differ from its actual span causes strains of this character. By true span is meant the span which the arch would have if laid flat on its side on a plane surface in such a position that there are no bending moments at any point of it, while the actual span is the distance between the piers when the arch is in position. If the arch be built in position, but joined at the wrong temperature the true and actual spans do not agree and excessive temperature strains are caused.

Taking the coefficient of expansion of steel as ordinarily given, a change of ±80°F. from the mean temperature would cause the St. Louis Arch to be fitted to a span of about $3\frac{1}{4}$ inches, greater or less than at the mean.

The problem we wish to solve then is very approximately this: What horizontal thrust must be applied to increase or decrease the span of this arch by $3\frac{1}{4}$ inches, and what other stresses are induced by this thrust. In Fig. 4 the half span is represented on the same scale as in Fig. 2. The only forces applied to the half arch are an unknown horizontal thrust H at b_8 and an equal opposite thrust H at a. The arch is in the same condition as it would be if Fig. 4 represented half of a gothic arch of a span $=2ab$, of which a was one abutment, and b_8 was the new crown at which a weight of $2H$ was applied. The gothic arch would be continuous at the crown, but the abutment a would be mounted on rollers, so that although the direction of a tangent at a could not be changed, nevertheless the abutment could afford no resistance to keep the ends from moving apart, *i.e.* there is no thrust in the direction of ab, any more than there is along an ordinary straight girder.

In order to facilitate the accurate construction, let us multiply the ordinates of a by 3 and use the polygon d instead. Now the real equilibrium polygon of the applied forces H, is the straight line kk_8. By real equilibrium polygon is meant, that one which has for its pole distance, the actual thrust of the arch. As we are now considering this arch, H is the applied force, and the thrust spoken of is at right angles to H. We have just shown this thrust to be zero. We have then to construct an equilibrium polygon for the applied force H with a pole distance of zero. The polygon is infinitely deep in the direction of H, and hence is a line parallel to H. This fixes its direction.

Its position is fixed from the consideration that the total bending is zero, (because the direction of the tangents at the extremities a and b_8 are unchanged), which is expressed by the equation

$$\Sigma(M_d)=0.$$

This gives us the same closing line through k which we found in Fig. 2, and the ordinates of the type kd, are proportional to the moments caused by the horizontal thrust H.

Now lay off $dm_8=\frac{1}{2}k_8b_8$, $m_8m_7=k_7d_7$, etc., as in Fig. 2.

The problem of finally determining H, will be solved in two steps:

1°. We shall find the actual values of the moments to which the ordinates kd are proportional;

2°. We shall find H by dividing either of these moments by its arm.

By considering the equation

$$D_yEI=\Sigma(My)$$

given in Chapter I, in which D_y is the horizontal displacement, it is seen that if the actual moments are used for weights, and EI for the pole distance, we shall obtain, as the second equilibrium polygon, a deflection curve whose ordinates are the actual deflections due to the moments. By actual moments, actual deflections, etc, is meant, that all of the quantities in the equation are laid off to the scale of distance, say *one* n^{th} of the actual size.

Now let the equation be written

$$nD_y \cdot \frac{1}{n} EI = \Sigma(My).$$

From which it is seen that if the ordinates be multiplied by n, so that on the paper they are of the same size as in the arch, we must use *one* n^{th} of the former pole distance, all else remaining unchanged.

Now for the St. Louis Arch, $EI=$ 39680000 foot tons. Let us take 100 tons to the inch, as the scale of force: and since $bd=3$ inches, the scale of distance n is found from the proportion

$$3 \text{ in.} :: 51.8 \text{ ft.} :: 1 : n = 210 \text{ nearly,}$$

$$\text{and } EI \div 100\, n^2 = 9 \text{ in. nearly,}$$

which is the pole distance necessary to use with the actual deflection $\frac{1}{2}$ of $3\frac{1}{4}$ in.$=1\frac{5}{8}$in., in order that the moments may be measured to scale. As it is inconvenient to use so large a distance as 9 in. on our paper, let us take $\frac{3}{8}$ of 9 in. $=3\frac{1}{3}$ in. nearly $=dz$ for the pole distance, and $\frac{8}{3}$ of $1\frac{5}{8}$ in. $=4\frac{1}{3}$ in. $=dy$, for the deflection.

Now with z as a pole and the weights dm_8, m_8m_7, etc, draw the deflection curve bf, having the deflection $=df$. The moments M_d must be increased in such a ratio that the deflection will be increased from df to dy. Therefore draw the straight lines bf and by, which will enable us to effect the increase in the required ratio. For example, the moment $dm_1=bi$ is increased to bj, and $dm_8=bj$ is increased to bj_8. Now measuring bj in inches and multiplying by 210 and by 100, we have found that 21000 $bj=1809$ foot tons$=$the moment at d or a.

And again, 21000 $bj_8=3747$ foot tons $=$the moment at b_8.

By measurement 210 $dk=17$ ft. and 210 $bk=34.8$ ft.

$$\therefore H = 1809 \div 17 = 106 \text{ tons}, +$$

$$\text{or } H=3747 \div 34.8 = 108 \text{ tons } -.$$

These results should be identical, and the difference between them of less than 2 per cent. is due to the error occasioned by using the polygon d instead of the curve of the ellipse, and to small errors in measurement. With a larger figure and the subdivision of the span into a greater number of parts this error could be reduced. The value of H found for the St. Louis Arch by computation was 104 tons, but that was not on the supposition of a uniform moment of inertia I, and should be less than the value we have obtained.

Now this horizontal thrust H due to temperature and to any other thrusts of like nature as compression, etc, is of the nature of a correction to the thrust due to the applied weights. Thus in Fig. 2 we found $3ov'$ to be the thrust due to the applied weights, and on applying the correction we must use the two thrusts $3ov'+H$ and $3ov'-H$ as pole distances to obtain equilibrium polygons whose ordinates reckoned from the arch a will, when multiplied by its pole distance, give the true bending moments. The tangential and normal stresses can then be determined by resolution, precisely as in Fig. 3.

If it, however, appears desirable to compute separately the strains due to H, this may be more readily done than in combination with the stresses already obtained. We have already seen sufficiently how the bending moments due to H are found. In fact the moments are such as would be produced by applying H at the point where the horizontal through k cuts the polygon d, for this is the point of no moment, and may be considered for the instant as a free end of each segment, to each of which H is applied causing the moments due to its arm and intensity.

To find the tangential stress and shear, lay off in Fig. 4 $av=H$ and on it as a diameter describe a semicircle, and draw $ar_8 \parallel a_7a_8$, $ar_7 \parallel a_6a_7$ etc.; then will ar_8 be the component of H along a_7a_8, and vr_8 be the component of H directly across the same segment. In a similar manner the quantities of which ar on the type are the tangential stresses and the quantities vr are the shearing stresses caused by H.

The scale used for this last construction is about fifty tons to the inch.

Now H is positive or negative according as the temperature is increased above or diminished below the mean, and these tangential and normal components, of course, change sign with H.

It should also be noticed in this connection that thrusts and bending moments, which are numerically equal but of opposite sign, are induced by equal contractions and expansions.

The stresses due to variation of temperature in the arch of Fig. 3, having a center joint, are constructed in Fig. 5.

It is evident from reasoning similar to that employed for the case just discussed, that the closing line dk_8 of the polygon d is the equilibrium polygon of the thrust H induced by variation of temperature. Suppose we have changed the equation of deflections to the form,

$$mD_y \cdot \frac{EI}{mn^2} = \Sigma\left(\frac{M}{n} \cdot \frac{y}{n}\right),$$

in which, if $mD_y=dy$ and $EI \div mn^2=dz$, then the moments M and the ordinates y will be laid off on the scale of 1 to n. This is equivalent to doing what was done in the previous case, where m was equal to $\frac{8}{3}$. The remainder of the process is that previously employed.

It should be noticed that we have in Figs. 4 and 5, incidentally discussed two new forms of arches, viz: in Fig. 4 that of an arch having its ends fixed in direction, but not in position; *i.e.*, its ends may slide but not turn, and in Fig. 5, that of an arch sliding freely and turning freely at the ends. The first of these arches has the same bending moments as a straight girder, fixed in direction at the ends, and the second of them has the same bending moments as a simple girder supported at its ends.

Errata.—The measurements of Fig. 4 given on page 24 do not agree with the scale on which the drawing is engraved. The following equations and quantities agree with the dimensions of Fig. 4, and are to be substituted instead of those given on page 24.

Let the scale of force be 100 tons to the inch, and since $bd=4\frac{1}{2}$ inches, $4\frac{1}{2}$ in. : 51.8 ft. :: 1 : $n=140$ nearly, and $EI \div 100n^2=20$ in. nearly, which is the pole distance to use with the actual deflection of the half span$=1\frac{5}{8}$ in.

Now take one fourth of this pole distance $=5$ in. $=dz$, and four times the deflection$=6\frac{1}{2}$ in. $=dy$, as being more convenient to use; the moments, which are the products of the deflections by the pole distance, will be unchanged by this process.

Now increase the ordinates in such a ratio that the deflection will be increased from df to dy. For example, the moment $dm_1=bi$ is increased to bj, and $dm_8=bi_8$ is increased to bj_8. Now by measuring bj in inches and multiplying by 140 and by 100 we have found 14000 $bj=$ 1809 foot tons$=$the moment at a or d. And again, 14000 $bj_8=3747$ foot tons $=$ the moment at b_8.

$$\text{By measurement, } 140\ dk=17 \text{ ft.}$$

$$\text{and } 140\ bk=34.8 \text{ ft.}$$

$$\therefore H=1809 \div 17=106 \text{ tons } +,$$

$$\text{or } H=3747 \div 34.8=108 \text{ tons } -.$$

Near the bottom of the second column of page 24, instead of ar_8, ar_7, vr_8, ar, vr, read av_8. av_7, vv_8, av, vv.

The scale used in the last construction in Fig. 4, is about $33\frac{1}{3}$ tons to the inch.

UNSYMMETRICAL ARCHES.

The constructions which have been given have been simplified somewhat by the symmetry of the right and left hand halves of the arch, but the methods which have been used are equally applicable if such symmetry does not exist, as it does not, if, for example, the abutments are of different heights.

In particular, for the unsymmetrical arch, its closing line is not in general horizontal, and must be found precisely as that for the equilibrium polygon due to the applied weights.

If, in Fig. 3, the hinge joint is not situated at the center, the arch is unsymmetrical, and the determination of the closing line due to the applied weights, is not quite so simple as in Fig. 3. It will be necessary to draw the trial lines through the joint by which the curve of errors q is found.

CHAPTER V.

ARCH RIB WITH END JOINTS.

Let the curve a of the arch to be treated have a span of six times the rise, as represented in Fig. 6, and having divided the span into twelve equal parts, make the ordinates of the type bd twice the ordinates ab.

Let a uniform load having a depth xy cover the two-thirds of the span at the left, and a uniform load having a depth

$xy' = \frac{1}{2}xy$ cover the one-third of the span at the right. Assume any pole distance, as of one-third of the span, and lay off $b_4w_1 = xy$ = one-half of the load supposed to be concentrated at the center; $w_1w_2 = 2xy$ = the load concentrated above b_1, etc. Similarly at the left make $b_4'w_1' = xy$ = one-half the load above b; $w_1'w_2' = 2xy$ = the load above b_1'; $w_2'w_3' = xy + xy' = \frac{3}{2}xy$ = the load above b_2'; $w_3'w_4' = xy$ = the load above b_3', etc.

From this force polygon draw the equilibrium polygon c, just as in Figs. 2 and 3.

Now the closing line of the equilibrium polygon for a straight girder with ends free to turn, must evidently pass so that the end moments vanish. Hence c_6c_6' is the closing line of the polygon c, and b_6b_6' is the closing line of the polygon d, drawn according to the same law. The remaining condition by which to determine the bending moments is:

$$\Sigma(M_d - M_c)y = 0 \quad \therefore \quad \Sigma(M_d y) = \Sigma(M_c y)$$

which is the equation expressing the condition that the span is invariable, the summation being extended from end to end of the arch.

This summation is effected first as in Figs. 2 and 3, by laying off as loads quantities proportional to the applied moments concentrated at the points of division of the arch, and thus finding the second equilibrium polygon, or deflection polygon of two upright girders, bent by these moments.

Let us take one-fourth of each of the ordinates bd for these loads, *i.e.* $bm = \frac{1}{4}$ of $\frac{1}{2}bd$; $mm_1 = \frac{1}{4}b_1d_1$, etc.: also bn, nn_1, etc., equal to similar fractions of the ordinates of the curve c. Using d as the pole for this load, we obtain the total deflection bf_6 on the left, and the same on the right (not drawn) due to the bending moments M_d.

Similarly g_6g_6' is the total deflection right and left due to the moments M_c.

Now the equation of condition requires that $\frac{1}{2}g_6g_6' = bf_6$. That this may occur, the ordinates of the polygon c must be elongated in the ratio of these deflections. To effect this, make $ai = \frac{1}{2}g_6g_6'$ and $aj = bf_6$, and on the horizontals through i and j at a convenient distance draw the vertical i_0j_0; then the lines ai_0 and aj_0 will effect the required elongation, as previously explained. To obtain the center ordinate be, for example, make $ai' = bh$ $\therefore$ $aj' = be$. To find the new pole o, draw bv parallel to c_6c_6' and vo horizontal, as before explained.

If ai_0 cuts the load line at t_1 and the horizontal through t_1 cuts aj_0 at t_2, then the vertical through t_2 is the new position of the load line and tt_2 is the new horizontal thrust.

Now using o as the pole of the load line u_1u_1' etc., through t_2 draw the equilibrium polygon starting from e. It must pass through b_6 and b_6', which tests the accuracy of the construction.

The construction may now be completed just as in Fig. 3, by doubling the pole distance, and finding the tangential thrust along the arch and the normal shear directly across the arch in the segments into which it is divided. The maximum thrust and tangential stress is obtained when the line load covers the entire span.

To compute the effect of changes of temperature and other causes of like nature in producing thrust, shear, bending moment etc., let us put the equation of deflections in the following form:

$$mD_y \cdot \frac{EI}{mn^2n'} = \Sigma\left(\frac{M}{nn'} \cdot \frac{y}{n}\right) \quad . \quad \text{(D)}$$

This equation may perhaps put in more intelligible form the processes used in Figs. 4 and 5, and is the equation which should be used as the basis for the discussion of temperature strains in the arch. In equation (D) n is the number by which the rise of the arch must be divided to reduce it to bd, *i.e.*, it is the scale of the vertical ordinates of the type bd, in Fig. 6, so that if bd was on the same scale as the arch itself, n would be unity. Again, n' is the scale of force, *i.e.*, the number of tons to the inch; and m is a number introduced for convenience so that any assumed pole distance p may be used for the pole distance of the second equilibrium polygon. In Fig. 6, $p = bd$.

We find m from the equation,

$$p = \frac{EI}{mn^2n'}, \quad \therefore \quad m = \frac{EI}{pn^2n'},$$

from which m may be computed, for EI is a certain known number of foot tons when the cross-section of the rib is given, p is a number of inches assumed in the drawing, n and n' are also assumed. Now D_y is the number of inches by which the span is increased or decreased by the change of temperature, and mD_y is at once laid off on the drawing.

The quantities in equation (D) are so related to each other, that the left-hand member is the product of the pole distance and ordinate of the second equilibrium polygon, while the right-hand member is the bending moment produced by the loading $M \div nn'$, which loading is proportional to M. The curve f was constructed with this loading, and only needs to have its loads and ordinates elongated in the ratio of bf_6 to $\frac{1}{2}mD_y$ to determine the values of $M \div nn'$ at the various points of division of the arch. One-half of each quantity is used, because we need to use but one-half the arch in this computation. Two lines drawn, as in Figs. 4 and 5, effect the required elongation.

The foregoing discussion is on the implied assumption that the horizontal thrust caused by variation of temperature is applied in the closing line bb_6 of the arch, which is so evident from previous discussions as to require no proof here.

The quantity determined by the foregoing process is $M \div nn' = q$ say, a certain number of inches. Then $M = nn'q$, and $H = M \div y = n'q \div \frac{y}{n}$, in which $\frac{y}{n}$ is the length of the ordinate in inches on the drawing at the point at which M is applied.

The determination of the shearing and tangential stress induced by H is found by using H as the diameter of a circle, in which are inscribed triangles, whose sides are respectively parallel and perpendicular to the segments of the arch, precisely as was done in Figs. 4 and 5.

The whole discussion of the arch with end joints may be applied to an unsymmetrical arch with end joints. In that case, it would be necessary to draw a curve f' at the right as well as f at the left, and the two would be unlike, as g and g' are. This, however, would afford no difficulty either in determining the stresses due to the loads, or to the variations of temperature.

When the live load extends over two-thirds of the span, as in the Fig., the maximum bending moment is nearly in the middle of that live load, and is very approximately the largest which can be induced by a live load of this intensity, while the greatest moment of opposite sign is found near the middle of the unloaded third of the span.

If the curve of the arch were a parabola instead of the segment of a circle, these statements would be exact and not approximate, as may be proved analytically. This matter will be further treated hereafter.

CHAPTER VI.

ARCH RIB WITH THREE JOINTS.

Let the joints be at the center and ends of the arch, as seen in Fig. 7. Let the loading and shape of the arch be the same as that used in Fig. 6. Now since the bending moment must vanish at each of the joints, the true equilibrium curve must pass through each of the joints; *i. e.*, every ordinate of the polygon c must be elongated in the ratio of db to bh. To effect this, make $di = bh$, and at a convenient distance on the horizontals through b and i draw the vertical i_0b_4. Then the ratio lines di_0 and db_4 will enable us to elongate as required, or to find the new pole distance ti, diminished in the same ratio, by drawing the horizontal ti through i_0. The new pole o is found in the same manner as in Fig. 6.

Now with the new pole o and the new load line through t, we can draw the polygon e starting at d. It must then pass through b_6 and b_6' which tests the accuracy of the construction.

The maximum thrust, and tangential stress is attained when the live load covers the entire span.

Variations in length due to changes of temperature induce no bending moments in this arch, but there may be slight alteration in the thrust, etc., produced by the slight rising or falling of the crown due to the elongation or shortening of the arch. This is so small a displacement that it is of no importance to compute the stresses due to it. We have for the same reason, in the previous and subsequent constructions, omitted to compute the stresses arising from the displacement which the arch undergoes at various points by reason of its being bent. It would be quite possible to give a complete investigation of these stresses by analogous methods.

The construction above given is applicable to any arch with three joints. The arch need not be symmetrical, and the three joints can be situated at any points of the arch as well as at the points chosen above.

CHAPTER VII.

THE ARCH RIB WITH ONE END JOINT.

Let the arch be represented by Fig. 8, in which the load, etc., is the same as in Fig. 6.

The closing line must pass through the joint, for at this joint the bending moment vanishes.

A second condition which must be fulfilled is, that the total deflection below the tangent at the fixed end of a straight girder having one end joint vanishes, for the position of the joint is fixed. This is expressed by the equation

$$\Sigma(Mx)=0,$$

in which the summation is extended from end to end.

This condition will enable us to draw the closing line of the polygon c, and also that of d. The problem may be thus stated:—In what direction shall a closing line such as c_6h' be drawn from c_6 so that the moment of the negative triangular area $c_6c_6'h'$ about c_6 shall be equal to the moment of the positive parabolic area c_6bc_6'

To solve this problem, first find the center of gravity of the parabolic area by taking it in parts. The parabolic area $c_6\ b\ c_2'$ is a segment of a single parabola whose area is $\frac{2}{3}b_6b_2'\times c_0c_2=\frac{1}{2}h_1\times b_6b_6'$, when h_1=the height of an equivalent triangle having the span for its base $\therefore\ h_1=\frac{8}{9}c_0c_2$.

Lay off $l_6b_6=c_0c_2$, and draw $l_6b_3'\ \therefore\ b_5l_5=h_1$. Lay off $c_2'p_1=h_1$ as proportional to the weight of the parabolic area. Again, $c_2'p$ is proportional to the weight of the triangle $c_6c_2'c_6'$. The parabolic area $c_2'c_6'=\frac{2}{3}c_0'c_4'\times b_2'b_6'=\frac{1}{2}h_2\times b_6b_6'$, as before, $\therefore\ h_2=\frac{4}{9}c_0'c_4'$, which may be found as h_1 was before.

Let $h_2=pp_2$, then on taking any pole, as c_2, of this weight line, we draw $qq_1 \parallel c_2c_2'$, since the left parabolic area has its center of gravity in the vertical through q_1, and the triangular area in that through q, we draw $qq_1' \parallel c_2p$, to the vertical through q_1', which contains the center of gravity of the right parabolic area. The position of q midway between the verticals containing b and b_2 is slightly to the right of its true position, as it should be at one-third of the distance from the vertical through b to that through b_2. This does not affect the nature of the process however.

Then $q_1q_2 \parallel c_2p_1$ and $q_1'q_2 \parallel c_2p_2$ give q_2 in the vertical through the center of gravity of the total positive area. The negative area, since it is triangular, has its center of gravity in the vertical through c_2'.

Now if the total positive bending moment be considered to be concentrated at its center of gravity and to act on a straight girder it will assume the shape rq_2r_1 of this second equilibrium polygon, and if a negative moment must be applied such that the deflection vanish, the remainder of the girder must be r_1r_2, a prolongation of rr_1. Now draw $c_2p_3 \parallel rr_1$, and we have $p_2p_3=c_6'h'$ the height of the triangle of negative area. Hence c_6h' is the closing line, fulfilling the required conditions.

Again, to draw the closing line b_6k' according to the same law, we know that the center of gravity of the polygonal area d is in the center vertical. To find the height p_2p', of an equivalent triangle having a base equal to the span, we may obtain an approximate result, as in Fig. 2, by taking one twelfth of the sum of the ordinates of the type bd, but it is much better to obtain an exact result by applying Simpson's rule which is simplified by the vanishing of the end ordinates. The rule is found to reduce in this case to the following:—The required height is one eighteenth of the sum of the ordinates with even subscripts plus one ninth of the sum of the rest.

Now this positive moment concentrated in the center vertical and a negative moment such as to cause no total deflection in a straight girder, will give as a second equilibrium polygon $rq_2'r_1'r_2'$; and if $c_2p_3' \parallel rr_1'$, then $p'p_3'=b_6'k'$ is the height of the triangular negative area, and the closing line is b_6k'.

Now the remaining condition is that the span is invariable, which is expressed by the equation

$$\Sigma(M_d-M_e)y=0,\ \text{or}\ \Sigma(M_dy)=\Sigma(M_cy).$$

Let us construct the deflection curve due to the moments M_d in a manner similar to that employed in Fig. 2. We lay off quantities dm_5, m_5m_4, etc., equal to one-fourth of the corresponding ordinates of the curve d, and dn_5, n_5n_4, etc., one-fourth of the ordinates of the curve c. We use one-fourth or any other fraction or multiple of both which may be convenient. By using b for a pole we obtain the deflection curves f and f' for the moments proportional to M_d, and the curves g and g' for those proportional to M_c.

Now, Prop. IV. requires that the ordinates of the polygon c should be increased so that gg' shall become equal to ff'. Make $di=gg'$ and $dj=ff'$ and draw as before the ratio lines di_0 and dj_0, then the vertical through t_1 is the new position of the load line.

Find the new length of bh which is ke, and with the new pole o, draw the polygon e starting at e. It must pass through b_6. The new pole o is found thus: draw $bv \parallel hh'$, then v divides the weight line into two parts, which are the vertical resistances of the abutments. From v_1 draw $v_1o \parallel kk'$, then the closing line of the polygon e has the direction kk'.

A single joint at any point of an unsymmetrical arch can be treated in a similar manner.

A thrust produced by temperature strains will be applied along the closing line kk', and the bending moments induced will be proportional to the ordinates of the polygon d from this closing line. The variation of span must be computed not for the horizontal span, but for the projections of it on the closing line kk'. The construction of this component of the total effect will be like that previously employed. Another effect will be caused in a line perpendicular to kk'. The variation of span for this construction, is the projection of the total horizontal variation on a line perpendicular to kk', and the bending moments induced by this force applied at b_6, and perpendicular to the closing line, will be proportional to the horizontal distances of the points of division from b_6. As these constructions are readily made, and the shearing and tangential stresses determined from them, it is not thought necessary to give them in detail.

CHAPTER VIII.

ARCH RIB WITH TWO JOINTS.

Let us take the two joints, one at the center and one at one end as represented in Fig. 9. Let the loading, etc., be as in Fig. 6.

The closing line evidently passes through the two joints, as at them the bending moment vanishes.

The remaining condition to be fulfilled is that the deflection of the right half of the arch in the direction of this line, shall be the same as that of the left half.

Let us then suppose that the straight girder $b_6'p'$ perpendicular to the closing line, is fixed at b_6' and bent first by the moments M_d giving us the deflection curve $b_6'f'$ when b_6' is taken as the pole, and the loads of the type mm are one-quarter of the corresponding ordinates of the polygon d; and secondly, by the moments M_c giving us the deflection curve $b_6'g'$ when drawn with the same pole, and the loads of the type nn also one-quarter of the corresponding ordinates of the polygon c. It should be noticed that the points at which these moments are supposed to be concentrated in the girder $b_6'p'$, are on the parallels to kk' through the points d_5, d_6, etc.

Similarly let ff_3 and f_3f_6 be the deflection curves of the straight girder d_6p (using d_3 as the pole distance), under the applied moments.

We have used now a pole distance differing from that used in the right half of the arch. These pole distances must have the same ratio that the quantity EI has for the two parts of arch. If EI is the same in both parts of the arch the same pole distance must be used to obtain the deflection curves in both sides of the middle. In the same manner the curves gg_3 and g_3g_6 are found. Now must the moments M_c causing the total deflection $p'g'-gg_6=\frac{1}{2}ai$ be elongated so that they shall cause a total deflection $pf'-ff_6=\frac{1}{2}aj$. The ratio lines ai_0, aj_0' will enable us to find the new position t_2 of the load line to effect this.

To find o the new pole, through v_2, which divides the load line into parts which are the vertical resistances of the piers, draw $v_2o \parallel b_6k$. Then draw the polygon e as in Fig. 7, starting from d. It must pass through b_6. We can find also whether ke_6' has the required ratio to hc_6' by the aid of the ratio lines, which will further test the accuracy of the work.