The construction above given is applicable to any arch with three joints. The arch need not be symmetrical, and the three joints can be situated at any points of the arch as well as at the points chosen above.

CHAPTER VII.

THE ARCH RIB WITH ONE END JOINT.

Let the arch be represented by Fig. 8, in which the load, etc., is the same as in Fig. 6.

The closing line must pass through the joint, for at this joint the bending moment vanishes.

A second condition which must be fulfilled is, that the total deflection below the tangent at the fixed end of a straight girder having one end joint vanishes, for the position of the joint is fixed. This is expressed by the equation

$$\Sigma(Mx)=0,$$

in which the summation is extended from end to end.

This condition will enable us to draw the closing line of the polygon c, and also that of d. The problem may be thus stated:—In what direction shall a closing line such as c_6h' be drawn from c_6 so that the moment of the negative triangular area $c_6c_6'h'$ about c_6 shall be equal to the moment of the positive parabolic area c_6bc_6'

To solve this problem, first find the center of gravity of the parabolic area by taking it in parts. The parabolic area $c_6\ b\ c_2'$ is a segment of a single parabola whose area is $\frac{2}{3}b_6b_2'\times c_0c_2=\frac{1}{2}h_1\times b_6b_6'$, when h_1=the height of an equivalent triangle having the span for its base $\therefore\ h_1=\frac{8}{9}c_0c_2$.

Lay off $l_6b_6=c_0c_2$, and draw l_6b_3' $\therefore$ $b_5l_5=h_1$. Lay off $c_2'p_1=h_1$ as proportional to the weight of the parabolic area. Again, $c_2'p$ is proportional to the weight of the triangle $c_6c_2'c_6'$. The parabolic area $c_2'c_6'=\frac{2}{3}c_0'c_4'\times b_2'b_6'=\frac{1}{2}h_2\times b_6b_6'$, as before, $\therefore$ $h_2=\frac{4}{9}c_0'c_4'$, which may be found as h_1 was before.

Let $h_2=pp_2$, then on taking any pole, as c_2, of this weight line, we draw $qq_1 \parallel c_2c_2'$, since the left parabolic area has its center of gravity in the vertical through q_1, and the triangular area in that through q, we draw $qq_1' \parallel c_2p$, to the vertical through q_1', which contains the center of gravity of the right parabolic area. The position of q midway between the verticals containing b and b_2 is slightly to the right of its true position, as it should be at one-third of the distance from the vertical through b to that through b_2. This does not affect the nature of the process however.

Then $q_1q_2 \parallel c_2p_1$ and $q_1'q_2 \parallel c_2p_2$ give q_2 in the vertical through the center of gravity of the total positive area. The negative area, since it is triangular, has its center of gravity in the vertical through c_2'.

Now if the total positive bending moment be considered to be concentrated at its center of gravity and to act on a straight girder it will assume the shape rq_2r_1 of this second equilibrium polygon, and if a negative moment must be applied such that the deflection vanish, the remainder of the girder must be r_1r_2, a prolongation of rr_1. Now draw $c_2p_3 \parallel rr_1$, and we have $p_2p_3=c_6'h'$ the height of the triangle of negative area. Hence c_6h' is the closing line, fulfilling the required conditions.

Again, to draw the closing line b_6k' according to the same law, we know that the center of gravity of the polygonal area d is in the center vertical. To find the height p_1p', of an equivalent triangle having a base equal to the span, we may obtain an approximate result, as in Fig. 2, by taking one twelfth of the sum of the ordinates of the type bd, but it is much better to obtain an exact result by applying Simpson's rule which is simplified by the vanishing of the end ordinates. The rule is found to reduce in this case to the following:—The required height is one eighteenth of the sum of the ordinates with even subscripts plus one ninth of the sum of the rest.

Now this positive moment concentrated in the center vertical and a negative moment such as to cause no total deflection in a straight girder, will give as a second equilibrium polygon $rq_2'r_1'r_2'$; and if $c_2p_3' \parallel rr_1'$, then $p'p_3'=b_6'k'$ is the height of the triangular negative area, and the closing line is b_6k'.

Now the remaining condition is that the span is invariable, which is expressed by the equation

$$\Sigma(M_d-M_c)y=0,\ \text{or}\ \Sigma(M_dy)=\Sigma(M_cy).$$

Let us construct the deflection curve due to the moments M_d in a manner similar to that employed in Fig. 2. We lay off quantities dm_5, m_5m_4, etc., equal to one-fourth of the corresponding ordinates of the curve d, and dn_5, n_5n_4, etc., one-fourth of the ordinates of the curve c. We use one-fourth or any other fraction or multiple of both which may be convenient. By using b for a pole we obtain the deflection curves f and f' for the moments proportional to M_d, and the curves g and g' for those proportional to M_c.

Now, Prop. IV. requires that the ordinates of the polygon c should be increased so that gg' shall become equal to ff'. Make $di=gg'$ and $dj=ff'$ and draw as before the ratio lines di_0 and dj_0, then the vertical through t_1 is the new position of the load line.

Find the new length of bh which is ke, and with the new pole o, draw the polygon e starting at e. It must pass through b_6. The new pole o is found thus: draw $bv \parallel hh'$, then v divides the weight line into two parts, which are the vertical resistances of the abutments. From v_1 draw $v_1o \parallel kk'$, then the closing line of the polygon e has the direction kk'.

A single joint at any point of an unsymmetrical arch can be treated in a similar manner.

A thrust produced by temperature strains will be applied along the closing line kk', and the bending moments induced will be proportional to the ordinates of the polygon d from this closing line. The variation of span must be computed not for the horizontal span, but for the projections of it on the closing line kk'. The construction of this component of the total effect will be like that previously employed. Another effect will be caused in a line perpendicular to kk'. The variation of span for this construction, is the projection of the total horizontal variation on a line perpendicular to kk', and the bending moments induced by this force applied at b_6, and perpendicular to the closing line, will be proportional to the horizontal distances of the points of division from b_6. As these constructions are readily made, and the shearing and tangential stresses determined from them, it is not thought necessary to give them in detail.

CHAPTER VIII.

ARCH RIB WITH TWO JOINTS.

Let us take the two joints, one at the center and one at one end as represented in Fig. 9. Let the loading, etc., be as in Fig. 6.

The closing line evidently passes through the two joints, as at them the bending moment vanishes.

The remaining condition to be fulfilled is that the deflection of the right half of the arch in the direction of this line, shall be the same as that of the left half.

Let us then suppose that the straight girder $b_6'p'$ perpendicular to the closing line, is fixed at b_6' and bent first by the moments M_d giving us the deflection curve $b_6'f'$ when b_6' is taken as the pole, and the loads of the type mm are one-quarter of the corresponding ordinates of the polygon d; and secondly, by the moments M_c giving us the deflection curve $b_6'g'$ when drawn with the same pole, and the loads of the type nn also one-quarter of the corresponding ordinates of the polygon c. It should be noticed that the points at which these moments are supposed to be concentrated in the girder $b_6'p'$, are on the parallels to kk' through the points d_5, d_4, etc.

Similarly let ff_3 and f_3f_6 be the deflection curves of the straight girder d_6p (using d_3 as the pole distance), under the applied moments.

We have used now a pole distance differing from that used in the right half of the arch. These pole distances must have the same ratio that the quantity EI has for the two parts of arch. If EI is the same in both parts of the arch the same pole distance must be used to obtain the deflection curves in both sides of the middle. In the same manner the curves gg_3 and g_3g_6 are found. Now must the moments M_c causing the total deflection $p'g'-gg_6=\frac{1}{2}ai$ be elongated so that they shall cause a total deflection $pf'-ff_6=\frac{1}{2}aj$. The ratio lines ai_0, aj_0' will enable us to find the new position t_2 of the load line to effect this.

To find o the new pole, through v_2, which divides the load line into parts which are the vertical resistances of the piers, draw $v_2o \parallel b_6k$. Then draw the polygon e as in Fig. 7, starting from d. It must pass through b_6. We can find also whether ke_6' has the required ratio to hc_6' by the aid of the ratio lines, which will further test the accuracy of the work.

Any unsymmetrical arch with joints situated differently from the case considered can be treated by a like method.

The temperature strains should be treated like those in Fig. 8, which are caused by a thrust along the closing line. Those at right angles to this line vanish as the joints allow motion in this direction. The shearing and tangential stresses can be found as in Fig. 3.

Arches with more than three hinge joints are in unstable equilibrium, and can only be used in an inverted position as suspension bridges. These will be treated subsequently. If the joints, however, possess some stiffness so that they are no longer hinge joints, but are block-work joints, or analogous to such joints, we may still construct arches which are stable within certain limits although the number of joints is indefinitely increased. Such are stone or brick arches. These will also be treated subsequently.

The constructions in Figs. 6, 7, 8, 9, can be tested by a process like that employed in Figs. 2 and 3. In Fig. 2, for instance, we obtained the algebraic sum of the squares of the quantities of the type *ss*, and showed that such sum vanishes. We can obtain the same result in all cases.

CHAPTER IX.

THE CINCINNATI AND COVINGTON SUSPENSION BRIDGE. (Fig. 10.)

THE main span of this bridge has a length of 1057 feet from center to center of the towers, and the end spans are each 281 feet from the abutment to the center of the tower. The deflection of the cable is 89 feet at a mean temperature, or about 1—11.87th of the span. There is a single cable at each side of the bridge. Each of these cables is made up of 5200 No. 9 wires, each wire having a cross-section of 1–60th of a square inch and an estimated strength of 1620 lbs. Each of these cables has a diameter of 12½ inches, and an estimated strength of 4212 tons. Each cable rests at the tower upon a saddle of easy curvature, the saddle being supported by 32 rollers which run upon a cast iron bed-plate 8×11 feet, which forms part of the top of the tower. Since the bed-plate is horizontal this method of support ensures the exact perpendicularity of the force which the cables exert upon the towers, without its being necessary to make the inclination of the cable on both sides of the saddle the same. There is, therefore, no tendency by the cables to overturn the towers, and they need only be proportioned to bear the vertical stresses coming upon them.

As this bridge differs greatly in some respects from other suspension bridges, it seems necessary to describe its peculiarities somewhat minutely.

The roadway and sidewalks make a platform 36 feet wide, extending from abutment to abutment, 1619 feet. It is built of three thicknesses of plank solidly bolted together, in all 8 inches thick. This is strengthened by a double line of rolled I girders, 1630 feet long, running the entire length of the center of the platform. These I girders are arranged one line above the other, and across between them, at distances of 5 feet, run lateral I girders which are suspended from the cable. The upper line of girders is 9 inches deep, (and 30 lbs. per foot); the lower line is 12 inches deep (and 40 lbs. per foot). The lateral girders are 7 inches deep (and 20 lbs. per foot), and are firmly embraced between the double line of longitudinal girders. The girders of this center line are each 30 ft. long, and are spliced together by plates in the hollows of the I, but the holes through which the bolts pass are slots whose length is two or three times the diameter of the bolts. This makes a "slip joint" such as is often used in fastening the ends of the rails on a railroad. The slip joints permit the wooden planking of the roadway to expand and contract from variations of moisture and temperature without interference from the iron girders which are bolted to it.

There is also a line of wrought-iron truss-work about 10 feet deep extending from abutment to abutment on each side of the roadway, consisting of panels of 5 feet each, to each lower joint of which is fastened a lateral girder and a suspender from the cable. This trussing is a lattice, with vertical posts, and ties extending across two panels, and its chords are both made with slip joints every 30 feet.

It is apparent that this whole arrangement of flooring with the girders and trusses attached to it possesses a very small amount of stiffness, in fact the stiffness is principally that of the flooring itself. It will permit a very large deflection, say 25 feet, up or down from its normal position without injury. Its office is something quite different from that of the ordinary stiffening truss of a suspension bridge. It certainly serves to distribute concentrated loads over short distances, but not to the extent required, if that were the sole means of preserving the cable in a fixed position under the action of moving loads. Its true function is to destroy all vibrations and undulations, and prevent their propagation from point to point by the enormous frictional resistance of these slip joints. When a wave does work against elastic forces, the reaction of those forces returns the wave with nearly its original intensity, but when it does work against friction it is itself destroyed.

The means relied on in this bridge to resist the effect of unbalanced loads is a system of stays extending from the top of the tower in straight lines to those parts of the roadway which would be most deflected by such loads. There are 76 such stays, 19 from the top of each tower. The longest stays extend so far as to leave only 350 feet., *i.e.*, a little over one-third of the span, in the center over which they do not extend. Each stay being a cable 2¼ inches in diameter has an estimated strength of 90 tons. They are attached every 15 feet to the roadway at the lower joints of the trussing, and are kept straight by being fastened to the suspenders where they cross them. This system is shown in Fig. 10 in which all the stays for one cable are drawn, together with every third suspender. The suspenders occur every 5 feet throughout the bridge but none are shown in the figure except those attached at the same points as the stays.

These stays must sustain the larger part of any unbalanced load, at the same time producing a thrust in the roadway against either the abutment or tower.

It is really an indeterminate question as to how the load is divided between the stays and trussing; and this the more, because of the manner in which the other extremities of the stays are attached. Of the nineteen stays carried to the top of one tower, the eight next the tower are fastened to the bed plate under the saddle, and so tend to pull the tower into the river; the remaining eleven are carried over the top of the tower, and rest on a small independent saddle, beside the main saddle, and are eight of them fastened to the middle portion of the side spans as shown in Fig. 10, while the other three are anchored to the abutment.

In view of the indeterminate nature of the problem, it has seemed best to suppose that the stays should be proportioned to bear the whole of any excess of loading of any portion of the bridge, over the uniformly distributed load (which latter is of course borne by the cable itself); and further that the truss really does bear some fraction of the unbalanced load, and that the bending moments have therefore the same relative amounts as if they sustained the entire unbalanced load. This fraction, however, is quite unknown owing to the impossibility of finding any approximate value of the moment of inertia I for the combined wood and iron work of the roadway.

This method of treatment has for our present purpose this advantage, that the construction made use of is the same as that which must be used when there are no stays at all, and the entire bending moments induced by the live loads are borne by the stiffness of the truss alone.

Now in order to determine the tension in any stay, as for instance that in the longest stay leading to the right hand tower, lay off v_1v_2 equal to the greatest unbalanced weight, which under any circumstances is concentrated at its lower extremity. This weight is sustained by the longitudinal resistance of the flooring, and the tension of the stay. The stresses induced in the stay and flooring by the weight, are found by drawing from v_1 and v_2 the lines v_1o and v_2o parallel respectively to the stay and the flooring. Then v_1o is the tension of the stay, and that of the other stays may be found in a similar manner.

It is impossible to determine with the same certainty how the stress ov_2 parallel to the flooring is sustained. It may be sustained entirely by the compression it produces in the part of the flooring between the weight and the tower or the abutment; or it may be sustained by the tension produced in the flooring at the left of the weight; or the stress ov_2 may be divided in any manner between these two parts of the flooring, so that v_1v_1' may represent the tension at the left, and ov_2' the compression at the right of the weight. It appears most probable that the induced stress is borne in the case before us by the compression of the flooring at the right, for the flooring is ill suited to bear tension both from the slip joints of the iron work and the want of other secure longitudinal fastenings; but on the contrary it is well designed to resist compression. The flooring must then be able at the tower to resist the sum of the compressions produced by all the unbalanced weights which can be at once concentrated at the extremities of the nineteen stays.

There is one considerable element of stiffness which has not been taken account of in this treatment of the stays, which serves very materially to diminish the maximum stresses to which they might otherwise be subjected. This is the intrinsic stiffness of the cable itself which is formed of seven equal subsidiary cables formed into a single cable, by placing six of them around the seventh central cable, and enclosing the whole by a substantial wrapping of wire, so that the entire cable having a diameter of $12\frac{1}{4}$ inches, affords a resistance to bending of from one sixth to one half that of a hollow cylinder of the same diameter and equal cross section of metal. Which of these fractions to adopt depends somewhat on the tightness and stiffness of the wrapping.

It is this intrinsic stiffness of the cable which is largely depended upon in the central part of the bridge, between the two longest stays, to resist the distortion caused by unbalanced weights.

As might be foreseen the distortions are actually much greater in the central part of the bridge than elsewhere, though they would have been by far the greater in those parts of the bridge where the stays are, had the stays not been used.

The center of a cable is comparatively stable while it is undergoing quite considerable oscillations, as may be readily seen by a simple experiment with a rope or chain.

Let us now determine the relative amount of the stresses in the stiffening truss, on the supposition that the actual stresses are some unknown fraction of the stresses which would be induced, if there were no stays, and the truss was the only means of stiffening the cable. We, therefore, have to determine only the total stresses, supposing there are no stays, and then divide each stress obtained by n (at present unknown) to obtain the results required. Let us draw the equilibrium polygon d which is due to a uniform load of depth xy, and which has a deflection bd six times the central deflection of the cable. The loading of the cable is so nearly uniform, that each of the ordinates of the type bd, may be considered with sufficient accuracy to be six times the corresponding ordinate of the cable. Any multiple other than six might have been used with the same facility. In order to cause the polygon to have the required deflection with any assumed pole distance it is necessary to assume the scale of weights in a particular manner, which may be determined easily in several ways. Let us find it thus:

Let W=one of the concentrated weights.
Let D=central deflection of cable.
Let S=span of the bridge
Let M=central bending moment due to the applied weights.

Then, if the pole distance$=\frac{1}{3}S$, $M=\frac{1}{3}S\times 6D=2SD$, for the moment is the product of the pole distance by the ordinate of the equilibrium polygon. Again, computing the central moment from the applied forces,

$$M=\tfrac{11}{2}W\times\tfrac{1}{2}S-5W\times\tfrac{1}{4}S=\tfrac{3}{2}WS,$$

in which the first term of the right hand member is moment of the resistance of the piers, and the second term is the moment of the concentrated weights applied at their center of gravity.

$$\therefore\ \tfrac{3}{2}WS=2SD\ \therefore\ W=\tfrac{4}{3}D,$$

Hence, if one-third of the span is to represent the pole distance or true horizontal tension of an equilibrium curve having six times the deflection of the cable, each concentrated weight when the span is divided into twelve equal parts, is represented by a length equal to $\frac{4}{3}$ of the deflection of the cable. The true horizontal tension of the cable will be six times that of the equilibrium polygon, or it will be represented, in the scale used, by a line twice the length of the span. Now taking b as the pole, at distances $bb_4=bb_4'=\frac{1}{3}S$, lay off $b_4'w_1=b_4w_1'=\frac{1}{2}W=\frac{2}{3}D$, so that they together represent the weight concentrated at b; and let $w_1w_2=W$, represent the weight concentrated at b_2, etc. Then can the equilibrium polygon d be constructed by making $dd_1 \parallel bw_1$, $d_1d_2 \parallel bw_2$, etc. If $bd=6D$ the polygon must pass through b_6 and b_6', which tests the accuracy of the work.

Now to investigate the effect of an unbalanced load covering one-half the span, let us take one half the load on the right half of the span and place it upon its left, so that xz and xb represent the relative intensity of the loading upon the left and right half of the span respectively, the total load being the same as before. If it is desirable to consider that the total load has been increased by the unbalanced load we have simply to change the scale so that the same length of load line as before, (viz, $b_4'w_6+b_4w_6'$) shall represent the total loading. This will give a new value to the horizontal tension also.

Now let a new equilibrium polygon c be drawn, which is due to the new distribution of the concentrated weights. It is necessary to have the closing line of this polygon c horizontal, and this may be accomplished either, by drawing the polygon in any position and laying off the ordinates of the type bc equal to those in the polygon so drawn, or better as is done in this Figure by laying off in each weight line that part of the total load which is borne by each pier, which is readily computed, as follows. The distance of the center of gravity of the loading divides the span in the ratio of 17 to 27. Hence $\frac{27}{44}$ and $\frac{17}{44}$ of the total load are the resistances of the piers, or since the total load$=11W$, we have $b_4'u_6=\frac{27}{4}W$ and $b_4u_6'=\frac{17}{4}W$. Now make u_6u_5=the weight concentrated at b_5, etc., and $b_4'u_2+b_4u_1$= that at b_1. Then draw the polygon c.

The polygon c has the same central deflection as the polygon d; for compute as before,

$$\therefore\ M=\tfrac{17}{4}W\times\tfrac{1}{2}S-\tfrac{5}{2}W\times\tfrac{1}{4}S=\tfrac{3}{2}WS$$

in which the first term of the second member is the moment of the resistance of the right pier, and the second term is the moment of the concentrated weights applied at their center of gravity.

By similar computations we may prove the following equalities;

$$d_5c_5=d_1c_1=-d_1'c_1'=-d_5'c_5'\ ;$$
$$d_4c_4=d_2c_2=-d_2'c_2'=-d_4'c_4'\ ;$$
$$d_3c_3=-d_3'c_3'.$$

The quantities of the type dc are proportional to the bending moments which the stiffening truss must sustain if it preserves the cable in its original shape, when acted on by an unbalanced load of depth bx, on the supposition that the truss has hinge joints at its ends, and is by them fastened to the piers. For in that case the cable is in the condition of an arch with hinge joints at its ends. The condition which then holds is this:

$$\Sigma(M_dy)=\Sigma(M_cy)$$

or,

$$\Sigma(M_d-M_c)\,y=0\ \therefore\ \Sigma(cd)\,y=0.$$

This last is fulfilled as is seen by the above equations, for to every product such as $+b_1d_1\times d_1c_1$ corresponds another $-b_1'd_1'\times d_1'c_1'$ of the same magnitude but opposite sign.

The polygon c could have been obtained by a second equilibrium polygon in a manner precisely like that used before, but as it appears useful to show the connection between the methods of treating the arch rib which is itself stiff, and the flexible arch or cable, which is stiffened by a separate truss, we have departed from our previously employed method for determining the polygon c, as it is easy to do when both c and d are parabolic.

Now let us compute the bending moment

$$=d_5c_5\times\tfrac{1}{3}S=M^c-M_d$$
$$M_c=\tfrac{27}{4}W\times\tfrac{1}{12}S=\tfrac{9}{16}WS$$
$$M_d=\tfrac{11}{2}N\times\tfrac{1}{12}S=\tfrac{11}{24}WS$$
$$\therefore\ M_c-M_d=\tfrac{5}{48}WS.$$

Compute also the bending moment at the vertical through b_4,

$$M_c=\tfrac{27}{4}W\times\tfrac{1}{6}S-\tfrac{3}{2}W\times\tfrac{1}{12}S=WS$$
$$M_d=\tfrac{11}{2}W\times\tfrac{1}{6}S-W\times\tfrac{1}{12}S=\tfrac{5}{6}WS$$
$$\therefore\ M_c-M_d=\tfrac{1}{6}WS$$

Similar computations may be made for the remaining points, and this noteworthy result will be found true, that the bending moments induced in the stiffening truss by the assumed loading, are the same as would have been induced by a positive loading on the left of a depth yz, and a negative loading on the right of an equal depth yb. For compute the moments due to such loading at the points b_5 and b_4.

The resistance of the pier due to such loading $= \frac{5}{4} W$

$$\therefore M_5 = \tfrac{5}{4} W \times \tfrac{1}{12} S = \tfrac{5}{48} WS$$

and

$$M_4 = \tfrac{5}{4} W \times \tfrac{1}{6} S - \tfrac{1}{2} W \times \tfrac{1}{12} S = \tfrac{1}{6} WS, \text{ etc.}$$

We arrive then at this conception of the stresses to which the stiffening truss is subjected, viz:—the truss is loaded with the applied weights acting downward, and is drawn upward by a uniformly distributed negative loading, whose total amount is equal to the positive loading, so that the load actually applied at any point may be considered to be the algebraic sum of the two loads of different signs which are there applied. This conception might have been derived at once from a consideration of the fact that the cable can sustain only a uniform load, if it is to retain its shape; but it appears useful in several regards to show the numerical agreement of this statement with Prop. IV of which in fact it is a particular case. It is unnecessary to make a general proof of this agreement, but instead we will now state a proposition respecting stiffening trusses, the truth of which is sufficiently evident from considerations previously adduced.

Prop. VI. The stresses induced in the stiffening truss of a flexible cable or arch, by any loading, is the same as that which would be induced in it by the application to it of a combined positive and negative loading distributed in the following manner, viz: the positive loading is the actual loading, and the negative loading is equal numerically to the positive loading, but is so distributed as to cause no bending moments in the cable or arch, *i.e.*, the cable or arch is the equilibrium polygon for this negative loading.

By flexible cable or arch is meant one which has hinge joints at the points where it supports the stiffening truss. It need not actually have hinge joints at these points: the condition is sufficiently fulfilled if it is considerably more flexible than the truss which it supports.

The truth of Prop. VI has been recognized by previous writers upon this subject in the particular case of the parabolic suspension cable, and it has been erroneously applied to the determination of the bending moments in the arch rib in general. It is inaccurate for this purpose in two particulars, inasmuch as in the first place the arch to which it is applied is not parabolic, though the negative loading due to it is assumed to be uniform, and in the second place the horizontal thrust is not the same for the different kinds of arch rib, while this assumes the same thrust for all, viz: that arising from a flexible arch or one with three or more joints.

A similar proposition has been introduced into a recent publication on this subject*, but in that work the truss stiffens a simple parabolic cable, and the truss is not supposed to be fastened to the piers, so that it may rise from either pier whenever its resistance becomes negative. As this should not be permitted in a practical construction the case will not be discussed. In accordance with Prop. VI let us determine anew the bending moments due to an unbalanced load on the left of an intensity denoted by bz. As before seen this produces the same effect as a positive loading of an intensity $yz = fm = \frac{1}{2} bz$ on the left, and a negative loading of an intensity $yb = fn = \frac{1}{2} bz$. Now using g as a pole with a pole distance of gf_2 = one third of the span lay off the concentrated weight $p_1 p_2$ = that applied at b_1, etc., on the same scale as the weights were laid off in the previous construction, and in such a position that g is opposite the middle of the total load, which will cause the closing line to be horizontal. Then draw the equilibrium polygon a due to these weights. The ordinates of the type af are by Prop. VI proportional to the bending moments induced in the stiffening truss by the unbalanced load when the truss is simply fastened to the

* Graphical Statics, A. J. Du Bois, p. 329, published by John Wiley & Son, New York.

piers at the ends, and, as we have seen, each of the quantities af is identical with the corresponding quantity cd.

If the stiffening truss is fixed horizontally at its ends a closing line hh' must be drawn in such a position that $\Sigma(M) = 0$, and as it is evident that it must divide the equilibrium polygon symmetrically it passes through f its central point.

As stated in a previous article, the maximum bending moments at certain points of the span are caused when the unbalanced load covers somewhat more than half of the span. In the case of a parabolic cable or arch the maximum maximorum bending moment is caused when this load extends over two-thirds of the span, as is proved by Rankine in his Applied Mechanics by an analytic process. Let the load extend then over all except the right hand third of the span with an intensity represented by $bz = q_3 q_4'$. Then if $f_2' q_3 = \frac{1}{2} f_2' q_4'$, the truss may by Prop. VI be considered to sustain a positive load of the intensity $f_2' q_3$ on the left of b_2', and a negative load of the intensity $f_2' q_4'$ on the right of b_2'. Using g' as the pole and the same pole distance as before, lay off the weight $q_6 q_5$ concentrated at b_5, etc., so that g' is opposite the middle of the weight line. We thus obtain the equilibrium polygon e, in which the ordinates of the type ef are proportional to the bending moments of the truss under the assumed loading, when its ends are simply fastened to the piers.

Now bd was the ordinate of an equilibrium polygon having the same horizontal tension, and under a load of the same intensity covering the entire span. It will be found that $bd = \frac{27}{4} f_2 e_2$, which may be stated thus:—the greatest bending moment induced in the stiffening truss, by an unbalanced load of uniform intensity is four twenty-sevenths of that produced in a simple truss under a load of the same intensity covering the entire span. This result was obtained by Rankine analytically. If the truss is fixed horizontally at its ends, we must draw a closing line kk', which fulfills the conditions before used for the straight girder fixed at the ends, as discussed previously in connection with the St. Louis Arch. By the construction of a second equilibrium polygon, as there given, we find the position of kk'; then the ordinates ke will be proportional to the bending moments of the stiffening truss.

The shearing stress in the truss is obtained from the loading which causes the bending moment, in the same manner as that in any simple truss. The horizontal tension in the cable, is the same whenever the total load on the span is the same, and is not changed by any alteration in the distribution of the loading, which fact is evident from Prop. VI. The maximum tension of the cable is found when the live load extends over the entire span, and is to be obtained from a force polygon which gives for its equilibrium polygon the curve of the cable itself, as would be done by using the weights $w_1 w_2$, etc., and a pole distance of six times bb_4 = twice the span.

The temperature strains of a stiffening truss of a suspension bridge are more severe than those of the truss stiffening an arch, because the total elongation of the cable in the side spans as well in the main span, is transmitted to the main span and produces a deflection at its center. This is one reason why stays furnish a method of bracing, particularly applicable to suspension bridges. But supposing that the truss bears part of the bending moment due to the elongation of the cable, it is evident that when the truss is simply fastened to the piers, the bending moments so induced are proportional to the ordinates of the type bd, for by the elongation of the cable, it transfers part of its uniformly distributed weight to the truss.

That load which the cable still sustains, is uniformly distributed, if the cable still remains parabolic, therefore that transferred to the truss is uniformly distributed.

When the truss is fixed horizontally at the piers, the closing line of the curve d must be changed so that $\Sigma(M) = 0$, and the bending moments induced by variations of temperature, will be proportional to the ordinates between the curve d and this new closing line.

It remains only to discuss the stability of the towers and anchorage abutments. The horizontal force tending to overturn the piers comes from a few stays only, as was previously stated, and is of such small amount that it need not be considered.

The weight of the abutment in the case before us is almost exactly the same as the ultimate strength of the cable. Suppose that $st = sv$ are the lines representing these quantities in their position relatively to the abutment. Since their resultant sv intersects the base beyond the face of the abutment, the abutment would tip over before the cable could be torn asunder. And since the angle vsr is greater than the angle of friction between the abutment and the ground it stands on, the abutment if standing on the surface of the ground, would slide before the cable could be torn asunder.

The smallest value which the factor of safety for the cable assumes under a maximum loading is computed to be six. Take $st' = \frac{1}{6} st$ as the greatest tension ever induced in the cable, then sr' the resultant of sv and st' cuts the base so far within the face that it is apparent that the abutment has sufficient stability against overturning, and the angle vsr' is so much smaller than the least value of the angle of friction between the abutment and the earth under it, that the abutment would not be near the point of sliding even if it stood on the surface of the ground. It should be noticed tha all the suspenders in the side span assist in reducing the tension of the cable as we approach the abutment, and conduce by so much to its stability. Also the thrust of the roadway may assist the stability of the abutment, both with respect to overturning and sliding.

CHAPTER X.

THE CONTINUOUS GIRDER WITH VARIABLE CROSS-SECTION.

In the foregoing chapters the discussion of arches of various kinds has been shown to be dependent upon that of the straight girder; but as no graphical discussion has, up to the present time, been published which treats the girder having a variable cross-section and moment of inertia, our discussion has been limited to the case of arches with a constant moment of inertia.

Certain remarks were made, however, in the first chapter tending to show the close approximation of the results in case of a constant moment of inertia to those obtained when the moment of inertia is variable. We, in this chapter, propose a new solution of the continuous girder in the most general case of variable moment of inertia, the girder resting on piers having any different heights consistent with the limits of elasticity of the girder. This solution will verify the remarks made, and enable us easily to see the manner in which the variation of the moment of inertia affects the distribution of the bending moments, and by means of it the arch rib with variable moment of inertia can be treated directly.

Besides the importance of the continuous girder in case it constitutes the entire bridge by itself, we may remark that the continuous girder is peculiarly suited to serve as the stiffening truss of any arched bridge of several spans in which the arches are flexible. Indeed, it is the conviction of the writer that the stiff arch rib adopted in the construction of the St. Louis Bridge was a costly mistake, and that, if a metal arch was desirable, a flexible arch rib with stiffening truss was far cheaper and in every way preferable.

Let us write the equation of deflections in the form

$$mD \,.\, \frac{EI_0}{mn^2n'} = \Sigma\left(\frac{Mi}{nn'} \cdot \frac{x}{n}\right)$$

in which n is the number by which any horizontal dimension of the girder must be divided to obtain the corresponding dimension in the drawing, n' is the divisor by which force must be divided to obtain the length by which it is to be represented in the drawing, m is an arbitrary divisor which enables us to use such a pole distance for the second equilibrium polygon as may be most convenient, I_0 is the moment of inertia of the girder at any particular cross section assumed as a standard with which the values of I at other cross sections are compared, and $i = I_0 \div I$ is the ratio of I_0 (the standard moment of inertia), to I (that at any other cross-section). For the purpose of demonstrating the general properties of girders, the equation need not be encumbered with the coefficients mnn', but for purposes of explaining the graphical construction they are very useful, and can be at once introduced into the equation when needed.

In the equation

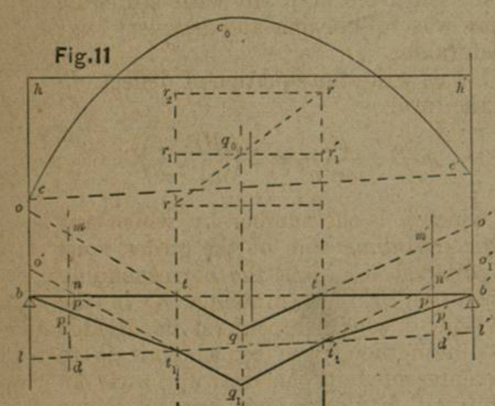
$$D \,.\, EI_0 = \Sigma_a^o (Mix)$$

the quantity D is the deflection of any point O of the girder below the tangent at the point a where the summation begins, and M is the actual bending moment at any point between O and a. These moments M at any point consist in general of three quantities, represented in the construction by the positive ordinate of the equilibrium polygon due to the weights, and by the two negative ordinates of the triangles into which we have divided the negative moment area. If we distinguish these components of M by letting M_0 represent that due to the weights, while M_1 and M_2 represent the components due to the left and right negative areas respectively, the equation of deflections becomes

$$D \,.\, EI_0 = \Sigma_a^o(M_0 ix) - \Sigma_a^o(M_1 ix) - \Sigma_a^o(M_2 ix)$$

Now let us take O at a pier at one end of a span and extend the summation over the entire span.

Fig. 11



If the piers are b and b' as in Fig. 11, let us suppose that O coincides with b and a with b'; also suppose for the instant that I is constant, so that $i = 1$ at all points of the girder. Then we have

$$D_b \,.\, EI = \bar{x}_0 \Sigma_{b'}^b (M_0) - \bar{x}_1 \Sigma_{b'}^b (M_1) - \bar{x}_2 \Sigma_{b'}^b (M_2)$$

in which D_b is the deflection of b below the tangent at b', $\bar{x}_0$ is the distance of the center of gravity of the moment area due to the applied weights from b, while $\bar{x}_1$ and $\bar{x}_2$ are the distances of the centers of gravity of the negative areas from b. In Fig. 11 let cc_0c' be the positive area due to the weights and representing $\Sigma_{b'}^b (M_0)$, while $\Sigma_{b'}^b (M_1)$ and $\Sigma_{b'}^b (M_2)$ are represented by hcc' and $hh'c'$ respectively. Let the center of gravity of cc_0c' be in qq_0, while the centers of the two negative areas are in tr and $t'r'$. Let the height of a triangle on some assumed base, and equivalent in area to cc_0c', be rr_2, then by a process like that in Fig. 2 it is evident that rr_1 and r_1r_2 are the heights of the right and left negative triangles, having the assumed base, on the supposition that the girder is fixed horizontally over the piers.

Now introducing the constants mnn' into the last equation and into the equation before that, the relation of the quantities is such that if the moments be applied as weights at their centers of gravity with the pole distance $pt = EI \div mn^2n'$, the equilibrium polygon so obtained will be tangent at the piers to the exaggerated deflection curve obtained when the distributed moments are used as weights; and the deflection at the pier b from the tangent at b' will be the same as that of this exaggerated deflection curve, and vice versa.

Let $pm = r_1r_2$, $p'm' = rr_1$ and $pt = p't$, then t and t' constitute the pole, pm and $p'm'$ the negative loads, and $pm + p'm'$ the positive load. Then is $btqt'b'$ the equilibrium polygon for these loads. The deflection of b below $b't'$ vanishes as it should in case the girder is fixed horizontally over the pier.

Now let the direction of the tangents at the piers be changed so that the tangents to the exaggerated deflection curve assume the directions bt_1 and $b't_1'$. Then the load line and force polygon assume a new position, such that t_1 and t_1' form the pole, and $dn = pm$ and $d'n' = p'm'$ comprise the positive load while np_1 and $n'p_1'$ are the new negative loads which will cause the equilibrium polygon $bt_1q_1t_1'b'$, which is due to them, to have its sides bt_1 and $b't_1'$ in the directions assumed.

There are several relations of quantities in this figure to which we wish to direct attention. It is evident, in case I is not constant, that from the area cc_0c' whose ordinates are proportional to M_0, the actual bending moments due to the weights, another area whose ordinates are proportional to

M_0i, the effective bending moments, can be obtained by simple multiplication, since i is known at every point of the girder. Moreover, the vertical through the center of gravity of this positive effective moment area can be as readily found as that through the actual positive moment area. Call this vertical "the positive center vertical." Again, the negative moment areas proportional to M_1i and M_2i can be found from the triangular areas proportional to M_1 and M_2 by simple multiplication, and if we proceed to find the verticals through their centers of gravity we shall obtain the same verticals whatever be the magnitude of the negative triangular areas, since their vertical ordinates are all changed in the same ratio by assuming the negative areas differently. Let us call these verticals the "left" and "right" verticals of the span. In case $i=1$, as in Fig. 11, the left and right verticals divide the span at the one-third points. This matter will be treated more fully in connection with Fig. 13.

Again, let us call the line t_1t_1' "the third closing line." It is seen that, whatever may be the various positions of the tangent bt_1, the ordinate dn, between the third closing line and t_1q_1 prolonged, is invariable; for the triangle $t_1q_1t_1'$ is invariable, being dependent on the positive load and pole distance alone. By similarity of triangles it then follows that the ordinate, such as lo', on any assumed vertical continues invariable; and when there is no negative load at t_1, then bt_1q_1 becomes straight, o' coincides with b and n with p_1. Similar relations hold at the right of q_1. The quantity dp_1 is of the nature of a correction to be subtracted from the negative moment when the girder is fixed horizontally at the piers in order to find the negative moment when the tangent assumes a new position, for $np_1=dn-dp_1$. The negative moments can consequently be found from the third closing line and the tangents at the piers; while the remaining lines q_1t_1 and $q_1't_1'$ will test the correctness of the work. Before applying these properties of the deflection polygon and its third closing line to a continuous girder, it is necessary to prove a geometrical theorem from Fig. 12.

Let the variable triangle xyz be such that the side xz always passes through

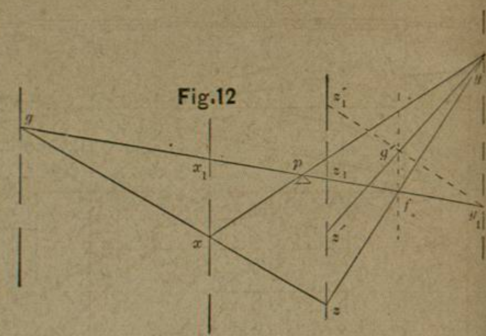


Fig.12

the fixed point g, the side xy always passes through the fixed point p, and the vertices xyz are always in the verticals through those points; then by the properties of homologous triangles the side yz also has a fixed point f in the straight line gp. Furthermore, if there is a point z' in the vertical through z, and in all positions of z it is at the same constant distance from z, then on the line yz' there is a fixed point g' where the vertical through f intersects yz'; for, if z' maintains its distance zz' invariable, then must any other point as g' remain constantly at the same vertical distance from f, as appears from similarity of triangles. But as f is fixed g' is also. When, for instance, the triangle xyz assumes the position $x_1y_1z_1$, then z' moves to z_1'.

Let us now apply the foregoing to the discussion of a continuous girder over three piers $p''pp'$ as shown in Fig. 13, in which the lengths of the spans have the ratio to each other of 2 to 3. Divide the total length of the girder into such a number of equal parts or panels, say 15, that one division shall fall at the intermediate pier, and let the number of lines in any panel of the type aa represent its relative moment of inertia. Assume the moment of inertia where there are three lines, as at a, a_4, etc., as the standard or I_0, then $i=1$ at a, $i=\frac{3}{2}$ at a_2, $i=\frac{3}{4}$ at a_6', etc.

Let the polygons c and c' be those due to the weights in the left and right spans respectively. Then the ordinates of the type bc are proportional to M_0 in the left span. The figure $bc_1c_1''c_2''c_2''c_3c_4c_5c_6''b_6$ is the positive effective moment area in the left span, and its ordinates are proportional to M_0i. Its center of gravity has been found, by an equilibrium

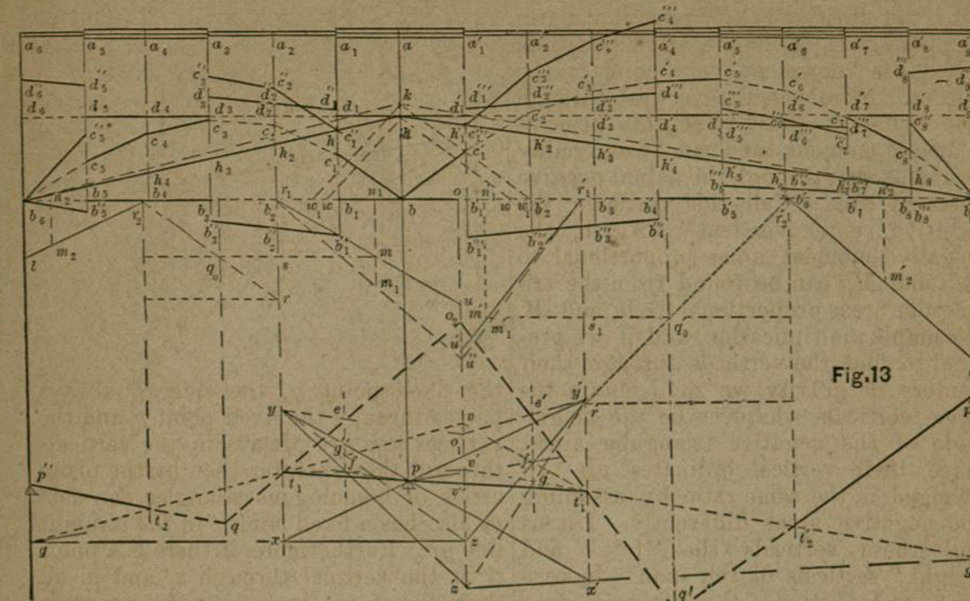


Fig.13

polygon not drawn, to lie in the positive center vertical qq_0. A similar positive effective moment area on the right has its center of gravity in the positive center vertical $q'q_0'$.

Now assume any negative area, as that included between the lines b and d, and draw the lines hb_6 and hb_6', dividing the negative area in each span into right and left triangular areas. Let the quantities of the type hb be proportional to M_1, hd to M_2, $h'b'$ to M_1', etc., then the ordinates of $bb_1b_1''b_2''b_3''b_4b_5b_6''b_6h$ are proportional to M_1i, and the center of gravity of this area has been found to lie in the right negative vertical t_1r_1. Similarly, the left negative vertical containing the center of gravity of the left negative effective moments, is t_2r_2. In the right span $t_1'r_1'$ and $t_2'r_2'$ are the left and right verticals. As before stated, these verticals would not be changed in position by changing the position in any manner whatever of the line d by which the negative moments were assumed, for such change of position would change all the ordinates in the same ratio.

Let us find also the vertical containing the center of gravity of the effective moment area, corresponding to the actual moment area b_6hb_6'. It is found by a polygon not drawn to be vo. Call vo "the negative center vertical." It is unchanged by moving the line d. If a polygon be drawn due to the effective moments as loads, two of its sides must intersect on vo, because it contains the center of gravity of contiguous loads. Now let rr_1 represent $\Sigma(M_0i)$:—it is in fact one eighth of the sum of the ordinates $b_1c_1+b_1c_1''$, etc., and hence is the height of a triangle having a base$=\frac{4}{3}bb_6$, and an area equal to the effective moment area in the left span. Also $r'r_1'$ is the height of a triangle having the same base, and an area equal to the effective moment area in the right span.

As previously explained, sr_1 is the amount of the right negative effective moment area in the left span, measured in the same manner, while sr is that on the left when the girder is fixed horizontally at the piers. We obtain $s'r_1'$ and $s'r'$ in the right span, in a similar manner. Now assume the arbitrary divisor $m=1$, and take the pole distance $r_1n_1=EI_0\div n''n'$. Then as seen previously, if $mn_1=sr_1$, ou is the constant intercept on the negative center vertical, between the third closing line in the left span, and a side of the type qt. Also ou' is a similar constant intercept on this vertical due to the right span. Make $r_2n_2=r_1n_1$ and $n_2m_2=sr$, then lb_6 is a similar invariable intercept; as is $l'b_6'$, which is obtained in a similar manner.

Now the negative center vertical ov was obtained from the triangle b_6hb_6', i.e.