$M_0 i$, the effective bending moments, can be obtained by simple multiplication, since i is known at every point of the girder. Moreover, the vertical through the center of gravity of this positive effective moment area can be as readily found as that through the actual positive moment area. Call this vertical "the positive center vertical." Again, the negative moment areas proportional to $M_1 i$ and $M_2 i$ can be found from the triangular areas proportional to M_1 and M_2 by simple multiplication, and if we proceed to find the verticals through their centers of gravity we shall obtain the same verticals whatever be the magnitude of the negative triangular areas, since their vertical ordinates are all changed in the same ratio by assuming the negative areas differently. Let us call these verticals the "left" and "right" verticals of the span. In case $i=1$, as in Fig. 11, the left and right verticals divide the span at the one-third points. This matter will be treated more fully in connection with Fig. 13.

Again, let us call the line $t_1 t_1'$ "the third closing line." It is seen that, whatever may be the various positions of the tangent bt_1, the ordinate dn, between the third closing line and $t_1 q_1$ prolonged, is invariable; for the triangle $t_1 q_1 t_1'$ is invariable, being dependent on the positive load and pole distance alone. By similarity of triangles it then follows that the ordinate, such as lo', on any assumed vertical continues invariable; and when there is no negative load at t_1, then $bt_1 q_1$ becomes straight, o' coincides with b and n with p_1. Similar relations hold at the right of q_1. The quantity dp_1 is of the nature of a correction to be subtracted from the negative moment when the girder is fixed horizontally at the piers in order to find the negative moment when the tangent assumes a new position, for $np_1 = dn - dp_1$. The negative moments can consequently be found from the third closing line and the tangents at the piers; while the remaining lines $q_1 t_1$ and $q_1' t_1'$ will test the correctness of the work. Before applying these properties of the deflection polygon and its third closing line to a continuous girder, it is necessary to prove a geometrical theorem from Fig. 12.

Let the variable triangle xyz be such that the side xz always passes through

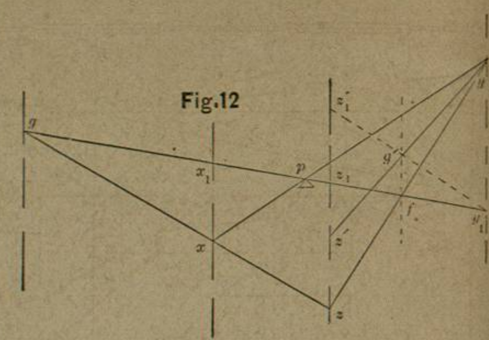

Fig.12

the fixed point g, the side xy always passes through the fixed point p, and the vertices xyz are always in the verticals through those points; then by the properties of homologous triangles the side yz also has a fixed point f in the straight line gp. Furthermore, if there is a point z' in the vertical through z, and in all positions of z it is at the same constant distance from z, then on the line yz' there is a fixed point g' where the vertical through f intersects yz'; for, if z' maintains its distance zz' invariable, then must any other point as g' remain constantly at the same vertical distance from f, as appears from similarity of triangles. But as f is fixed g' is also. When, for instance, the triangle xyz assumes the position $x_1 y_1 z_1$, then z' moves to z_1'.

Let us now apply the foregoing to the discussion of a continuous girder over three piers $p''pp'$ as shown in Fig. 13, in which the lengths of the spans have the ratio to each other of 2 to 3. Divide the total length of the girder into such a number of equal parts or panels, say 15, that one division shall fall at the intermediate pier, and let the number of lines in any panel of the type aa represent its relative moment of inertia. Assume the moment of inertia where there are three lines, as at a, a_4, etc., as the standard or I_0, then $i=1$ at a, $i=\frac{3}{2}$ at a_2, $i=\frac{3}{4}$ at a_6', etc.

Let the polygons c and c' be those due to the weights in the left and right spans respectively. Then the ordinates of the type bc are proportional to M_0 in the left span. The figure $bc_1 c_1'' c_2'' c_3'' c_3 c_4 c_5 c_5'' b_6$ is the positive effective moment area in the left span, and its ordinates are proportional to $M_0 i$. Its center of gravity has been found, by an equilibrium

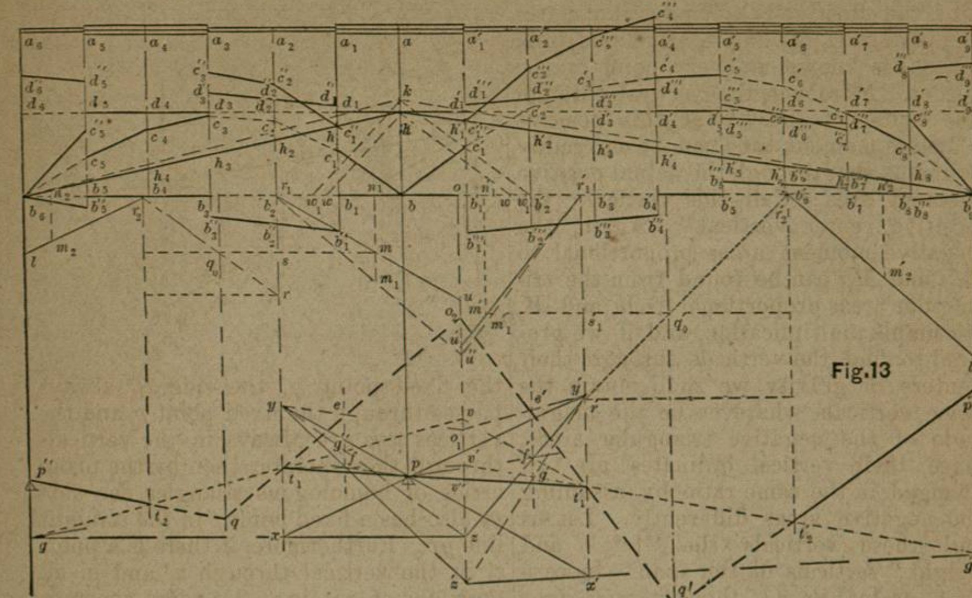

Fig.13

polygon not drawn, to lie in the positive center vertical qq_0. A similar positive effective moment area on the right has its center of gravity in the positive center vertical $q'q_0'$.

Now assume any negative area, as that included between the lines b and d, and draw the lines hb_6 and hb_9', dividing the negative area in each span into right and left triangular areas. Let the quantities of the type hb be proportional to M_1, hd to M_2, $h'b'$ to M_1', etc., then the ordinates of $bb_1 b_1'' b_2'' b_3'' b_3 b_5 b_5'' b_6 h$ are proportional to $M_1 i$, and the center of gravity of this area has been found to lie in the right negative vertical $t_1 r_1$. Similarly, the left negative vertical containing the center of gravity of the left negative effective moments, is $t_2 r_2$. In the right span $t_1' r_1'$ and $t_2' r_2'$ are the left and right verticals. As before stated, these verticals would not be changed in position by changing the position in any manner whatever of the line d by which the negative moments were assumed, for such change of position would change all the ordinates in the same ratio.

Let us find also the vertical containing the center of gravity of the effective moment area, corresponding to the actual moment area $b_6 hb_9'$. It is found by a polygon not drawn to be vo. Call vo "the negative center vertical." It is unchanged by moving the line d. If a polygon be drawn due to the effective moments as loads, two of its sides must intersect on vo, because it contains the center of gravity of contiguous loads. Now let rr_1 represent $\Sigma(M_0 i)$:—it is in fact one eighth of the sum of the ordinates $b_1 c_1 + b_1 c_1''$, etc., and hence is the height of a triangle having a base$=\frac{4}{3}bb_6$, and an area equal to the effective moment area in the left span. Also $r'r_1'$ is the height of a triangle having the same base, and an area equal to the effective moment area in the right span.

As previously explained, sr_1 is the amount of the right negative effective moment area in the left span, measured in the same manner, while sr is that on the left when the girder is fixed horizontally at the piers. We obtain $s'r_1'$ and $s'r'$ in the right span, in a similar manner. Now assume the arbitrary divisor $m=1$, and take the pole distance $r_1 n_1 = EI_0 \div n^2 n'$. Then as seen previously, if $mn_1 = sr_1$, ou is the constant intercept on the negative center vertical, between the third closing line in the left span, and a side of the type qt. Also ou' is a similar constant intercept on this vertical due to the right span. Make $r_2 n_2 = r_1 n_1$ and $n_2 m_2 = sr$, then lb_6 is a similar invariable intercept; as is $l'b_9'$, which is obtained in a similar manner.

Now the negative center vertical ov was obtained from the triangle $b_6 hb_9'$, *i.e.*

on the supposition that the actual moment over the pier is the same whether it be determined from the left or right of the pier. It is evident that while the girder is fixed horizontally at the intermediate pier, the moment at that pier is generally different on the two sides, at points infinitesimally near to it, but that when the constraint is removed an equalization takes place.

Since *ou* and *ou'* are derived from the positive effective moments, it appears that when the tangent at *p* is in such a position that the two third closing lines intercept a distance *uu'* on *ov* and the two lines of the type *qt* when prolonged intersect on *ov*, the moments over the pier will have become equalized.

We propose to determine the position of the tangent at *p* which will cause this to be true, by finding the proper position of the third closing lines in the two spans.

Move the invariable intercepts to a more convenient position, by making $o_1z=ou$, and $o_1z'=ou'$. Now by making the arbitrary divisor $m=1$, as we did, the ordinates of the deflection polygon became simply *D*, *i.e.*, they are of the same size in the drawing as in the girder, hence the difference of level of p'', p and p' must be made of the actual size. By changing *m* this can be increased or diminished at will.

Now we propose to determine two fixed points *g* and g'', through which the third closing line in the left span must pass, and similarly g''' and g' on the right.

If the girder is free at p'' then as shown in connection with Fig. 11, the third closing line must pass through *g*, if $gp''=lb_6$. Draw *gz* as a tentative position of the third closing line, and complete the triangle $xy'z$ as in Fig. 12.

Then is xy' the tentative position of the tangent at *p*, and since the third closing line in the right span must pass through y', and make an intercept on the negative center vertical equal to uu', then $z'y'$ is its corresponding tentative position. But wherever *gz* may be drawn, every line making an intercept $=uu'$ and intersecting $t_1'r_1'$ in such a manner that the tangent passes through *p* must pass through the fixed point g', found as described in Fig. 12. Therefore the third closing line in the right span passes through g'. Similarly, if there were more spans still at the right of these, we should use g' for the determination of another fixed point, as we have used *g* to determine it.

Now find g''' and g'' precisely as *g* and g' have been found, and draw the third closing lines t_1t_2 and $t_1't_2'$. If t_1t_1' passes through *p* the construction is accurate. Make $uu''=vv''$, then is n_1m_1 the negative effective moment at the left, and $n_1'm_1'$ that at the right of the pier.

Let *bw* be the effective moment area corresponding to the triangle hbb_6, and measured in the same manner as the positive area was, by taking one eighth of its ordinates, and let $bw_1=n_1m_1$; then as the effective moment *bw* is to the actual moment *bh* corresponding to it, so is the effective moment bw_1 or n_1m_1 to the actual moment *bk* corresponding to it. The same moment *bk* is also found from $n_1'm_1'$, by an analogous construction at the right of *b*, which tests the accuracy of the work.

Several other tests remain which we will briefly mention.

Prolong $p''t_2$ to *q*, and $p't_2'$ to q', then qt_1 and $q't_1'$ must intersect on the negative center vertical at o_2 so that $o_2v''=ou''$. Also vv' must be equal to uu'. Again t_1v' passes through *f*, and $t_1'v$ through f'. Also yo_1 intersects qo_2 on the fixed vertical fg'' at *e*, and $y'o_1$ intersects $q'o_2$ on the fixed vertical $f'g'$ at e'. That these must be so is evident from a consideration of what occurs during a supposed revolution of the tangent t_1t_1', to the position xy'.

Now having determined the moment *bk* over the pier, kb_6 and kb_6' are the true closing lines of the moment polygons *c* and c'. Call these closing lines *k*, then the ordinates of the type *kc* will represent the bending moments at different points of the girder. The points of the contra flexure are at the points where the closing lines intersect the polygons *c* and c'. The directions of the closing lines will permit at once the determination of the resistances at the piers and the shearing stresses at any point.

The particular difference between the construction in case of constant and of variable moment of inertia, is seen to be in the positions of the center verticals positive and negative, and the right and left verticals.

The small change in their position due to the variation in the moment of inertia, is the justification of the remarks previously made respecting the close approximation of the two cases.

It is seen that the process here developed can be applied with equal facility to a girder with any number of spans. Also if the moment of inertia varies continuously instead of suddenly, as assumed in Fig. 13, the panels can be taken short enough to approximate with any required degree of accuracy to this case.

CHAPTER XI.

THE THEOREM OF THREE MOMENTS.

The preceding construction has been in reality founded on the theorem of three moments, but when the equation expressing that theorem is written in the usual manner, the relationship is difficult to see. Indeed the equation as given by Weyrauch* for the girder having a variable moment of inertia, is of so complicated a nature that it may be thought hopeless to attempt to associate mechanical ideas with the terms of the equation, in any clearly defined relationship. We propose to derive and express the equation in a novel manner, which will at once be easy to understand, and not difficult of interpretation in connection with the preceding construction.

Let us assume the general equation of deflections in the form.

$$D=\Sigma(Mx\div EI),\ \text{or}\ D.EI_0=\Sigma(Mix) \qquad (7)$$

in which *I* is the variable moment of inertia, I_0 some particular value of *I* assumed as the standard of comparison, $i=I_0\div I$, and *x* is measured horizontally from the point as origin, where the deflection *D* is taken to the point of application of the actual bending moment *M*. The quantity *Mi* is called the effective bending moment, and the deflection *D* is the length of the perpendicular from the origin to the line tangent to the deflection curve at point to which the summation is extended.

Now consider two contiguous spans of a continuous girder of several spans, and let *acb* denote the piers, *c* being the intermediate pier. Let the span $ac=l$ and $bc=l'$. Take the origin at *a* and extend the summation to *c*, calling the deflection at *a*, D_a. When the origin is at *b* and the summation extends to *c*, let the deflection be D_b. Let also y_a, y_b and y_c be the heights of *a*, *b* and *c* respectively above some datum level. Then, as may be readily seen,

$$D_a = y_a - y_c - lt_c\ ,$$
$$D_b = y_b - y_c - l't_c',$$

if t_c is the tangent of the acute angle at *c* on the side towards *a* between the tangent line of the deflection curve at *c* and the horizontal, and t_c' is the tangent of the corresponding acute angle on the side of *c* towards *b*.

Now if we consider equation (7) to refer to the span *l*, the moment *M* may be taken to be made up of three parts, viz:—M_0 caused by the weights on the girder, M_1 dependent on the moment M_c at *c*, and M_2 dependent on the moment M_a at *a*. The moments in the span l' may be resolved in a similar manner. We may then write the equations of deflections in the two spans when the summation extends over each entire span as follows:

$$EI_0(y_a-y_c-lt_c)=\Sigma_c^a(M_0ix)-\Sigma_c^a(M_1ix)$$
$$-\Sigma_c^a(M_2ix)\ .\ .\ .\ .\ . \qquad (8)$$

$$EI_0(y_b-y_c-l't_c')=\Sigma_c^b(M_0'i'x')$$
$$-\Sigma_c^b(M_1'i'x')-\Sigma_c^b(M_2'i'x') \qquad (9)$$

in which *x* is measured from *a*, and x' from *b* towards *c*. Now if the girder is originally straight, $t_c = -t_c'$, hence we can combine these two equations so as to eliminate t_c and t_c', and the resulting equation will express a relationship between the heights of the piers, the bending moments (positive and negative), their points of application and the moments of inertia; of which quantities the negative bending moments are alone unknown. The equation we should thus obtain would be the general equation of which the ordinary expression of the theorem of three moments is a particular case. Before we write this general equation it is desirable to introduce certain modifications of form which do not diminish its generality. Suppose that

$$\bar{x}_1\Sigma_c^a(M_1i)=\Sigma_c^a(M_1ix)$$

then is $\bar{x}_1$ the distance from *a* to the center of gravity of the negative effective

* Allgemeine Theorie und Berechnung der Continuirlichen und Einfachen Trager. Jakob I. Weyrauch. Leipzig 1873.

moment area next to c. As was shown in connection with Fig. 13, the position of this center of gravity is independent of the magnitude of M_1 or M_c and may be found from the equation,

$$\bar{x}_1=\frac{\int_c^a ix^2dx}{\int_c^a ixdx} \quad . \; . \; . \; . \quad (10)$$

for M_1 is proportional to x. Similarly it may be shown that

$$\bar{x}_2=\frac{\int_c^a i(l-x)xdx}{\int_c^a i(l-x)dx} \quad . \; . \quad (11)$$

is the distance of the center of gravity of the negative effective moment area next to a.

Again, suppose that

$$i_1\Sigma_c^a(M_1)=\Sigma_c^a(Mi)$$

then is i_1 an average value of i for the negative effective moment area next to c, which is likewise independent of the magnitude of M_1, as appears from reasoning like that just adduced respecting $\bar{x}_1$. Hence i_1 may be found from the equation

$$i_1=\frac{\int_c^a ixdx}{\int_c^a xdx} \quad . \; . \; . \quad (12)$$

Similarly it may be shown that

$$i_2=\frac{\int_c^a i(l-x)dx}{\int_c^a (l-x)dx} \quad . \; . \quad (13)$$

in which i_2 is the average value of i for the negative effective moment area next to a.

The integrals in equations (10), (11), (12), (13), and in others like them referring to the span l', which contain i must be integrated differently, in case i is discontinuous, as it usually is in a truss, from the case where i varies continuously. When i is discontinuous the integral extending from c to a must be separated into the sum of several integrals, each of which must extend over that portion of the span l in which i varies continuously.

Furthermore we have

$$\Sigma_c^a(M_1)=\tfrac{1}{2}M_c l \quad . \; . \; . \quad (14)$$

since each member of this equation represents the negative actual moment area next to c in the span l.

Similarly, we have the equations

$$\Sigma_c^a(M_2)=\tfrac{1}{2}M_a l\;,\;\; \Sigma_c^b(M_1')=\tfrac{1}{2}M_c'l',$$
$$\Sigma_c^b(M_2'l')=\tfrac{1}{2}M_b l'.$$

If there is no constraint at the pier then must $M_c = M_c'$.

Now making the substitutions in equations (8) and (9), which have been indicated in the developments just completed, and then eliminating t_c and t_c',

$$EI_0\left\{\frac{y_a-y_c}{l}+\frac{y_b-y_c}{l'}\right\}-\frac{\bar{x}_0 i_0}{l}\Sigma_c^a(M_0)-$$
$$\frac{\bar{x}_0'i_0'}{l'}\Sigma_c^b(M_0')=\tfrac{1}{2}[M_a\bar{x}_2 i_2+M_c(\bar{x}_1 i_1+\bar{x}_1'i_1')$$
$$+M_b\bar{x}_2'i_2'] \quad . \; . \; . \quad (15)$$

in which $\bar{x}_0$ is the distance from a of the center of gravity of the positive effective moment area due to the weights in the span l, and $\bar{x}_0'$ is a similar distance from b in the span l', while i_0 and i_0' are average values of i for these areas derived from the equations in each span,

$$i_0=\Sigma(M_0 i)\div\Sigma(M_0).$$

It may frequently be best to leave the expressions containing the positive moments in their original form as expressed in equations (8) and (9).

Equation (15) expresses the theorem of three moments in its most general form.

Let us now derive from equation (15), the ordinary equation expressing the theorem of three moments, for a girder having a constant cross section. In this case $i=1$, and we wish to find the value of the term $\Sigma(M_0 x)$ in each span. Let M_0 be caused by several weights P applied at distances z from a, then the moment due to a single weight P at its point of application is

$$M_z = Pz(l-z)\div l,$$

which may be taken as the height of the triangular moment area whose base is l which is caused by P. This triangle whose area is $\tfrac{1}{2}M_z l$ is the component of $\Sigma(M_0)$ due to P and can be applied as a concentrated bending moment at its center of gravity at a distance x from a.

Now $x=\tfrac{1}{3}(l+z)$, and taking all the weights P at once

$$\Sigma^a(M_0 x)=\tfrac{1}{6}\Sigma_c^a[P(l^2-z^2)z].$$

Also in equation (15) we have in this case

$$\bar{x}_1=\tfrac{1}{3}l,\;\; \bar{x}_2=\tfrac{2}{3}l,\;\; \bar{x}_1'=\tfrac{1}{3}l',\;\; \bar{x}_2'=\tfrac{2}{3}l'$$

$$\therefore\; 6EI\left\{\frac{y_a-y_c}{l}+\frac{y_b-y_c}{l'}\right\}$$
$$-\frac{1}{l}\Sigma_c^a[P(l^2-z^2)z]-\frac{1}{l'}\Sigma_c^b[P'(l'^2-z'^2)z']$$
$$=M_a l+2M_c(l+l')+M_b l' \quad . \quad (16)$$

Equation (16) then expresses the theorem of three moments for a girder having a constant moment of inertia I, and deflected by weights applied in the span l at distances z from a, and also by weights in the span l' at distances z' from b.

Let us also take the particular case of equation (15) when the moment of inertia is invariable and the piers on a level; then $i=1$, and if we let A_0 and A_0' be the positive moment areas due to the weights we have

$$6\left\{\frac{1}{l}A_0\bar{x}_0+\frac{1}{l'}A_0'\bar{x}'_0\right\}=$$
$$M_a l+2M_c(l+l')+M_b l' \quad . \; . \quad (17)$$

This form of the equation of three moments was first given by Greene.*

The advantage to be derived in discussing this theorem in terms of the bending moments, instead of the applied weights is evident both in the analytical and the graphical treatment. The extreme complexity of the ordinary formulae arises from their being obtained in terms of the weights.

In order to complete the analytic solution of the continuous girder in the general case of equation (15), it is only necessary to use the well known equations,

$$M=M_c+S_c z_0-\Sigma_c^o(Pz_0) \quad . \; . \quad (18)$$

$$S_c=\frac{1}{l}[M_a-M_c+\Sigma_c^a(Pz)] \quad . \; . \quad (19)$$

$$S_c'=\frac{1}{l'}[M_b-M_c+\Sigma_c^b(Pz')] \quad . \quad (20)$$

$$R_c=S_c+S_c' \quad . \; . \; . \; . \; . \quad (21)$$

$$S=S_c-\Sigma_c^o(P) \quad . \; . \; . \; . \; . \quad (22)$$

In (18) M is the bending moment at any point O in the span l, S_c is the shear at c due to the weights in the span l, and z_0 is the distance from O towards c of the applied forces P and S_c in the segment Oc.

* Graphical Method for the Analysis of Bridge Trusses. Chas. E. Greene. Published by D. Van Nostrand. New York, 1875.

Equation (19) is derived from (18) by taking O at a, and (20) is obtained similarly in the span l'. R_c is the reaction of the pier at c. S is the shear at O in the span l. These equations also complete the solution of the cases treated in (16) and (17).

CHAPTER XII.

THE FLEXIBLE ARCH RIB AND STIFFENING TRUSS.

Whenever the moment of inertia of an arch rib is so small, that it cannot afford a sufficient resistance to hold in equilibrium the bending moments due to the weights, it may be termed a flexible rib.

It must have a sufficient cross section to resist the compression directly along the rib, but needs to be stiffened by a truss, which will most conveniently be made straight and horizontal. The rib may have a large number of hinge joints which must be rigidly connected with the truss, usually by vertical parts. It is then perfectly flexible.

If, however, the rib be continuous without joints, or have blockwork joints, it may nevertheless be treated as if perfectly flexible, as this supposition will be approximately correct and on the side of safety, for the bending moments induced in the truss will be very nearly as great as if the rib were perfectly flexible, in case the same weight would cause a much greater deflection in the rib than in the truss. It will be sufficient to describe the construction for the flexible rib without a figure, as the construction can afford no difficulties after the constructions already given have been mastered.

Lay off on some assumed scale the applied weights as a load line, and let us call this vertical load line ww'. Divide the span into some convenient number of equal parts by verticals, which will divide the curve a of the rib into segments. From some point b as a pole draw a pencil of rays parallel to the segments of a, and across this pencil draw a vertical line uu', at such a distance from b that the distance uu' between the extreme rays of the pencil is equal to ww'. Then the segments of uu' made by the rays of the pencil are the loads which the arch rib would sustain in virtue of its being an equilibrium polygon, and they would induce no bending moments if applied to the arch. The actual loads in general are differently distributed. By Prop. VI the bending moments induced in the truss are those due to the difference between the weight actually resting on the arch at each point, and the weight of the same total amount distributed as shown by the segments of the line *uu'*.

Now lay off a load line *vv'* made up of weights which are these differences of the segments of *uu'* and *ww'*, taking care to observe the signs of these differences. The algebraic sum of all the weights *vv'* vanishes when the weights which rest on the piers are included, as appears from inspection of the construction in the lower part of Fig. 10. The construction above described will differ from that in Fig. 10 in one particular. The rib will not in general be parabolic, and the loads which it will sustain in virtue of its being an equilibrium polygon will not be uniformly distributed, hence the differences which are found as the loading of the stiffening truss do not generally constitute a uniformly distributed load.

The horizontal thrust of the arch is the distance of *uu'* from *b* measured on the scale on which the loads are laid off, and the thrust along the arch at any point is length of the corresponding ray of the pencil between *b* and *uu'*. These thrusts depend only on the total weight sustained, while the bending moments of the stiffening truss depend on the manner in which it is distributed, and on the shape of the arch.

Having determined thus the weights applied to the stiffening truss, it is to be treated as a straight girder by methods previously explained according to the way in which it is supported at the piers.

The effect of variations of temperature is to make the crown of the arch rise and fall by an amount which can be readily determined with sufficient exactness, (see Rankine's Applied Mechanics Art. 169). This rise or fall of the arch produces bending moments in the stiffening truss, which is fastened to the tops of the piers, which are the same as would be produced by a positive or negative loading, causing the same deflection at the center and distributed in the same manner as the segments of *uu'*: for it is such a distribution of loads or pressures which the rib can sustain and produce. A similar set of moments can be induced in the stiffening truss by lengthening the posts between the rib and truss.

When this deflection and the value of *EI* in the truss are known, these moments can be at once constructed by methods like those already employed. A judicious amount of cambering of this kind is of great use in giving the structure what may be called "initial stiffness." The St. Louis Arch is wanting in initial stiffness to such an extent that the weight of a single person is sufficient to cause a considerable tremor over an entire span. This would not have been possible had the bridge consisted of an arch stiffened by a truss which was anchored to the piers in such a state of bending tension as to exert considerable pressure upon the arch. This tension of the truss would be relieved to some extent during the passage of a live load.

The arch rib with stiffening truss, is a form of which many wooden bridges were erected in Pennsylvania in the earlier days of American railroad building, but its theory does not seem to have been well understood by all who erected them, as the stiffening truss was itself usually made strong enough to bear the applied weights, and the arch was added for additional security and stiffness, while instead of anchoring the truss to the piers and causing it to exert a pressure on the arch, a far different distribution of pressures was adopted. Quite a number of bridges of this pattern are figured by Haupt* from the designs of the builders, but most of them show by the manner of bracing near the piers that the engineers who designed them did not know how to take advantage of the peculiarities of this combination. This further appears from the fact, that the trussing is not usually continuous.

A good example, however, of this combination constructed on correct principles is very fully described by Haupt on pages 169 *et seq.* of his treatise. It is a wooden bridge over the Susquehanna River, 5½ miles from Harrisburg on the

* Theory of Bridge Construction. Herman Haupt, A.M. New York. 1853.

Pennsylvania Railroad, and was designed by Haupt. It consists of twenty-three spans of 160 feet each from center to center of piers. The arches have each a span of 149½ feet and a rise of 20 ft. 10 in., and are stiffened by a Howe Truss which is continuous over the piers and fastened to them. It was erected in 1849. Those parts which were protected from the weather have remained intact, while other parts have been replaced, as often as they have decayed, by pieces of the original dimensions. This bridge, though not designed for the heavy traffic of these days, still stands after twenty-eight years of use, a proof of the real value of this kind of combination in bridge building.

CHAPTER XIII.

THE ARCH OF MASONRY.

Arches of stone and brick have joints which are stiff up to a certain limit beyond which they are unstable. The loading and shape of the arch must be so adjusted to each other that this limit shall not be exceeded. This will appear in the course of the ensuing discussion.

Let us take for discussion the brick arch erected by Brunel near Maidenhead England, to serve as a railway viaduct.

It is in the form of an elliptic ring, as represented in Fig. 14, having a span of 128 ft. with a rise of 24¼ feet. The thickness of the ring at the crown is 5¼ ft., while at the pier the horizontal thickness is 7 ft. 2 inches.

Divide the span into an even number of equal parts of the type *bb*, and with a radius of half the span describe the semicircle *gg*. Let ba=24¼ ft. be the rise of the intrados, and from any convenient point on the line *bb* as b_5 draw lines to *a* and *g*. These lines will enable us to find the ordinates *ba* of the ellipse of the intrados from the ordinates *bg* of the circle, by decreasing the latter in the ratio of *bg* to *ba*. For example, draw a horizontal through g_3 cutting b_5g at i_3, then a vertical through i_3, cutting b_5a at j_3, then will a horizontal through j_3 cut off a_3b_3 the ordinate of the ellipse corresponding to b_3g_3 in the circle, as appears from known properties of the ellipse.

Similarly let bq=64 ft. + 7 ft. 2 in., and with *bq* as radius describe a semicircle. Let bd=24¼ ft. + 5¼ ft. be the rise

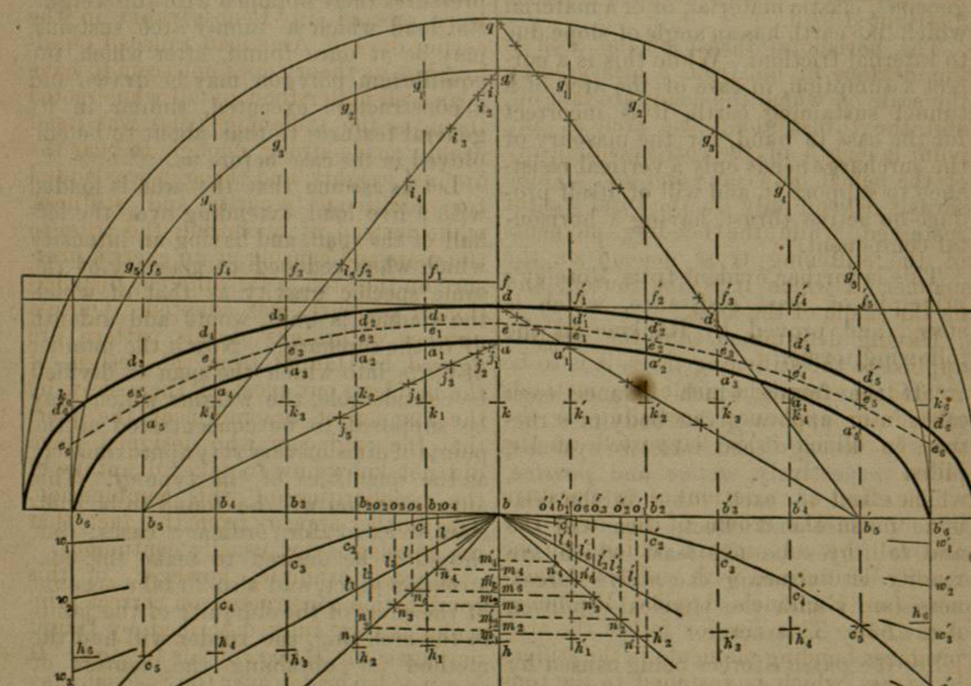

Fig.14
ARCH OF MASONRY
MAIDENHEAD RAILWAY VIADUCT

of the extrados, and from any convenient point on bb, as b_5' draw lines to d and q. These will enable us to find the ordinates bd of the ellipse of the extrados, from those of the circle, by decreasing the latter in the ratio of bq to bd. By this means, as many points as may be desired, can be found upon the intrados and extrados; and these curves may then be drawn with a curved ruler. We can use the arch ring so obtained for our construction, or multiply the ordinates by any convenient number, in case the arch is too flat for convenient work. Indeed we can use the semicircular ring itself if desirable. We shall in this construction employ the arch ring ad which has just been obtained.

We shall suppose that the material of the surcharge between the extrados and a horizontal line tangent at d causes by its weight a vertical pressure upon the arch. That this assumption is nearly correct in case this part of the masonry is made in the usual manner, cannot well be doubted. Rankine, however, in his Applied Mechanics assumes that the pressures are of an amount and in a direction due to the conjugate stresses of an homogeneous, elastic material, or of a material which like earth has an angle of slope due to internal friction. While this is a correct assumption, in case of the arch of a tunnel sustaining earth, it is incorrect for the case in hand, for the masonry of the surcharge needs only a vertical resistance to support it, and will of itself produce no active thrust, having a horizontal component.

This is further evident from Moseley's principle of least resistance, which is stated and proved by Rankine in the following terms:

"If the forces which balance each other in or upon a given body or structure, be distinguished into two systems, called respectively, *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

For the passive forces being caused by the application of the active forces to the body or structure, will not increase after the active forces have been balanced by them; and will, therefore, not increase beyond the least amount capable of balancing the active forces."

A surcharge of masonry can be sustained by vertical resistance alone, and therefore will exert of itself a pressure in no other direction upon the haunches of the arch. Nevertheless this surcharge will afford a resistance to horizontal pressure if produced by the arch itself. So that when we assume the pressures due to the surcharge to be vertical alone, we are assuming that the arch does not avail itself of one element of stability which may possibly be employed, but which the engineer will hesitate to rely upon, by reason of the inferior character of the masonry usually found in the surcharge. The difficulty is usually avoided, as in that beautiful structure, the London Bridge, by forming a reversed arch over the piers which can exert any needed horizontal pressure upon the haunches. This in effect increases by so much the thickness of the arch ring at and near the piers.

The pressure of earth will be treated in connection with the construction for the Retaining Wall. On combining the pressures there obtained with the weight, the load which a tunnel arch sustains, may be at once found, after which the equilibrium polygon may be drawn and a construction executed, similar in its general features to that about to be employed in the case before us.

Let us assume that the arch is loaded with a live load extending over the left half of the span, and having an intensity which when reduced to masonry of the same specific gravity as that of which the viaduct is built, would add a depth df to the surcharge. Now if the number of parts into which the span is divided be considerable, the weights which may be supposed to be concentrated at the points of division vary very approximately as the quantities of the type af. This approximation will be found to be sufficiently exact for ordinary cases; but should it be desired to make the construction exact, and also to take account of the effect of the obliquity of the joints in the arch ring, the reader will find the method for obtaining the centers of gravity, and constructing the weights, in Woodbury's Treatise on the Stability of the Arch pp. 405 *et seq.* in which is

given Poncelet's graphical solution of the arch.

With any convenient pole distance, as one half the span, lay off the weights. We have used b_6 as the pole and made $b_6w_1 = \frac{1}{2}$ the weight at the crown $= \frac{1}{4}(af+ad) = b_6'w_1'$, $w_1w_2 = a_1f_1$, $w_2w_3 = a_2f_2$, etc. Several of the weights near the ends of the span are omitted in the Figure; viz., w_4w_5, etc. From the force polygon so obtained, draw the equilibrium polygon c as previously explained.

The equilibrium polygon which expresses the real relations between the loading and the thrust along the arch, is evidently one whose ordinates are proportional to the ordinates of the polygon c.

It has been shown by Rankine, Woodbury and others, that for perfect stability,—*i.e*, in case no joint of the arch begins to open, and every joint bears over its entire surface,—that the point of application of the resultant pressure must everywhere fall within the middle third of the arch ring. For if at any joint the pressure reaches the limit zero, at the intrados or extrados, and uniformly increases to the edge farthest from that, the resultant pressure is applied at one third of the depth of the joint from the farther edge.

The locus of this point of application of the resultant pressure has been called the "curve of pressure," and is evidently the equilibrium curve due to the weights and to the actual thrust in the arch. If then it be possible to use such a pole distance, and such a position of the pole, that the equilibrium polygon can be inscribed within the inner third of the thickness of the arch ring, the arch is stable. It may readily occur that this is impossible, but in order to ensure sufficient stability, no distribution of live load should be possible, in which this condition is not fulfilled.

We can assume any three points at will, within this inner third, and cause a projection of the polygon c to pass through them, and then determine by inspection whether the entire projection lies within the prescribed limits. In order to so assume the points that a new trial may most likely be unnecessary, we take note of the well known fact, that in arches of this character, the curve of pressure is likely to fall without the prescribed limits near the crown and near the haunches. Let us assume e at the middle of the crown, e_5' at the middle of $a_5'd_5'$, and e_5 near the lower limit on a_5d_5. This last is taken near the lower limit, because the curvature of the left half of the polygon is more considerable than the other, and so at some point between it and the crown it may possibly rise to the upper limit. The same consideration would have induced us to raise e_5' to the upper limit, were it not likely that such a procedure would cause the polygon to rise above the upper limit on the right of e_5'.

Draw the closing line kk through e_5e_5', and the corresponding closing line hh through c_5c_5', and decrease all the ordinates of the type hc in the ratio of hb to ke, by help of the lines bn and bl, in a manner like that previously explained. For example $h_3c_3 = n_3o_3$, and $l_3o_3 = k_3e_3$. By this means we obtain the polygon e which is found to lie within the required limits. The arch is then stable: but is the polygon e the actual curve of pressures? Might not a different assumption respecting the three points through which it is to pass lead to a different polygon, which would also lie within the limits? It certainly might. Which of all the possible curves of pressure fulfilling the required condition, is to be chosen, is determined by Moseley's principle of least resistance, which applied to the case in hand, would oblige us to choose that curve of all those lying within the required limits, which has the least horizontal thrust, *i.e.* the smallest pole distance. It appears necessary to direct particular attention to this, as a recent publication on this subject asserts that the true pressure line is that which approaches nearest to the middle of the arch ring, so that the pressure on the most compressed joint edge is a minimum; a statement at variance with the theorem of least resistance as proved by Rankine.

Now to find the particular curve which has the least pole distance, it is evidently necessary that the curve should have its ordinates as large as possible. This may be accomplished very exactly, thus: above e_1 where the polygon approaches the upper limit more closely than at any other point near the crown, assume a new position of e_1 at the upper limit; and be-