of the extrados, and from any convenient point on *bb*, as b_6' draw lines to *d* and *q*. These will enable us to find the ordinates *bd* of the ellipse of the extrados, from those of the circle, by decreasing the latter in the ratio of *bq* to *bd*. By this means, as many points as may be desired, can be found upon the intrados and extrados; and these curves may then be drawn with a curved ruler. We can use the arch ring so obtained for our construction, or multiply the ordinates by any convenient number, in case the arch is too flat for convenient work. Indeed we can use the semicircular ring itself if desirable. We shall in this construction employ the arch ring *ad* which has just been obtained.

We shall suppose that the material of the surcharge between the extrados and a horizontal line tangent at *d* causes by its weight a vertical pressure upon the arch. That this assumption is nearly correct in case this part of the masonry is made in the usual manner, cannot well be doubted. Rankine, however, in his Applied Mechanics assumes that the pressures are of an amount and in a direction due to the conjugate stresses of an homogeneous, elastic material, or of a material which like earth has an angle of slope due to internal friction. While this is a correct assumption, in case of the arch of a tunnel sustaining earth, it is incorrect for the case in hand, for the masonry of the surcharge needs only a vertical resistance to support it, and will of itself produce no active thrust, having a horizontal component.

This is further evident from Moseley's principle of least resistance, which is stated and proved by Rankine in the following terms:

"If the forces which balance each other in or upon a given body or structure, be distinguished into two systems, called respectively, *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

For the passive forces being caused by the application of the active forces to the body or structure, will not increase after the active forces have been balanced by them; and will, therefore, not increase beyond the least amount capable of balancing the active forces."

A surcharge of masonry can be sustained by vertical resistance alone, and therefore will exert of itself a pressure in no other direction upon the haunches of the arch. Nevertheless this surcharge will afford a resistance to horizontal pressure if produced by the arch itself. So that when we assume the pressures due to the surcharge to be vertical alone, we are assuming that the arch does not avail itself of one element of stability which may possibly be employed, but which the engineer will hesitate to rely upon, by reason of the inferior character of the masonry usually found in the surcharge. The difficulty is usually avoided, as in that beautiful structure, the London Bridge, by forming a reversed arch over the piers which can exert any needed horizontal pressure upon the haunches. This in effect increases by so much the thickness of the arch ring at and near the piers.

The pressure of earth will be treated in connection with the construction for the Retaining Wall. On combining the pressures there obtained with the weight, the load which a tunnel arch sustains, may be at once found, after which the equilibrium polygon may be drawn and a construction executed, similar in its general features to that about to be employed in the case before us.

Let us assume that the arch is loaded with a live load extending over the left half of the span, and having an intensity which when reduced to masonry of the same specific gravity as that of which the viaduct is built, would add a depth *df* to the surcharge. Now if the number of parts into which the span is divided be considerable, the weights which may be supposed to be concentrated at the points of division vary very approximately as the quantities of the type *af*. This approximation will be found to be sufficiently exact for ordinary cases; but should it be desired to make the construction exact, and also to take account of the effect of the obliquity of the joints in the arch ring, the reader will find the method for obtaining the centers of gravity, and constructing the weights, in Woodbury's Treatise on the Stability of the Arch pp. 405 *et seq.* in which is

given Poncelet's graphical solution of the arch.

With any convenient pole distance, as one half the span, lay off the weights. We have used b_6 as the pole and made $b_6w_1 = \frac{1}{2}$ the weight at the crown $= \frac{1}{4}(af+ad) = b_6'w_1'$, $w_1w_2 = a_1f_1$, $w_2w_3 = a_2f_2$, etc. Several of the weights near the ends of the span are omitted in the Figure; viz., w_4w_5, etc. From the force polygon so obtained, draw the equilibrium polygon *c* as previously explained.

The equilibrium polygon which expresses the real relations between the loading and the thrust along the arch, is evidently one whose ordinates are proportional to the ordinates of the polygon *c*.

It has been shown by Rankine, Woodbury and others, that for perfect stability,—*i.e*, in case no joint of the arch begins to open, and every joint bears over its entire surface,—that the point of application of the resultant pressure must everywhere fall within the middle third of the arch ring. For if at any joint the pressure reaches the limit zero, at the intrados or extrados, and uniformly increases to the edge farthest from that, the resultant pressure is applied at one third of the depth of the joint from the farther edge.

The locus of this point of application of the resultant pressure has been called the "curve of pressure," and is evidently the equilibrium curve due to the weights and to the actual thrust in the arch. If then it be possible to use such a pole distance, and such a position of the pole, that the equilibrium polygon can be inscribed within the inner third of the thickness of the arch ring, the arch is stable. It may readily occur that this is impossible, but in order to ensure sufficient stability, no distribution of live load should be possible, in which this condition is not fulfilled.

We can assume any three points at will, within this inner third, and cause a projection of the polygon *c* to pass through them, and then determine by inspection whether the entire projection lies within the prescribed limits. In order to so assume the points that a new trial may most likely be unnecessary, we take note of the well known fact, that in arches of this character, the curve of pressure is likely to fall without the prescribed limits near the crown and near the haunches. Let us assume *e* at the middle of the crown, e_5' at the middle of $a_5'd_5'$, and e_5 near the lower limit on a_5d_5. This last is taken near the lower limit, because the curvature of the left half of the polygon is more considerable than the other, and so at some point between it and the crown it may possibly rise to the upper limit. The same consideration would have induced us to raise e_5' to the upper limit, were it not likely that such a procedure would cause the polygon to rise above the upper limit on the right of e_5'.

Draw the closing line *kk* through e_5e_5', and the corresponding closing line *hh* through c_5c_5', and decrease all the ordinates of the type *hc* in the ratio of *hb* to *ke*, by help of the lines *bn* and *bl*, in a manner like that previously explained. For example $h_3c_3=n_3o_3$, and $l_3o_3=k_3e_3$. By this means we obtain the polygon *e* which is found to lie within the required limits. The arch is then stable: but is the polygon *e* the actual curve of pressures? Might not a different assumption respecting the three points through which it is to pass lead to a different polygon, which would also lie within the limits? It certainly might. Which of all the possible curves of pressure fulfilling the required condition, is to be chosen, is determined by Moseley's principle of least resistance, which applied to the case in hand, would oblige us to choose that curve of all those lying within the required limits, which has the least horizontal thrust, *i.e.* the smallest pole distance. It appears necessary to direct particular attention to this, as a recent publication on this subject asserts that the true pressure line is that which approaches nearest to the middle of the arch ring, so that the pressure on the most compressed joint edge is a minimum; a statement at variance with the theorem of least resistance as proved by Rankine.

Now to find the particular curve which has the least pole distance, it is evidently necessary that the curve should have its ordinates as large as possible. This may be accomplished very exactly, thus: above e_1 where the polygon approaches the upper limit more closely than at any other point near the crown, assume a new position of e_1 at the upper limit; and below e_4' where it approaches the lower limit most nearly on the right, assume a new position of e_4' at the lower limit. At the left e_3 may be retained. Now on passing the polygon through these points it will fulfill the second condition, which is imposed by the principle of least resistance.

A more direct method for making the polygon fulfill the required condition will be given in Fig. 18.

It is seen in the case before us, the changes are so minute that it is useless to find this new position of the polygon, and its horizontal thrust. The thrust obtained from the polygon *e* in its present position is sufficiently exact. The horizontal thrust in this case is found from the lines *bn* and *bl*. Since $2vv_2$ is the horizontal thrust, *i.e.* pole distance of the polygon *c*, $2vv_1$ is the horizontal thrust of the polygon *e*.

By using this pole distance and a pole properly placed, we might have drawn the polygon *e* with perhaps greater accuracy than by the process employed, but that being the process employed in Figs. 2, 3, etc., we have given this as an example of another process.

The joints in the arch ring should be approximately perpendicular to the direction of the pressure, *i.e.* normal to the curve of pressures.

With regard to what factor of safety is proper in structures of this kind, all engineers would agree that the material at the most exposed edge should never be subjected to a pressure greater than one fifth of its ultimate strength. Owing to the manner in which the pressure is assumed to be distributed in those joints where the point of application of the resultant is at one third the depth of the joint from the edge, its intensity at this edge is double the average intensity of the pressure over the entire joint. We are then led to the following conclusion, that the total horizontal thrust (or pressure on any joint) when divided by the area of the joint where this pressure is sustained ought to give a quotient at least ten times the ultimate strength of the material. The brick viaduct which we have treated is remarkable in using perhaps the smallest factor of safety in any known structure of this class, having at the most exposed edge a factor of only $3\frac{1}{2}$ instead of 5.

It may be desirable in a case like that under consideration, to discuss the changes occuring during the movement of the live load, and that this may be effected more readily, it is convenient to draw the equilibrium polygons due to the live and dead loads separately. The latter can be drawn once for all, while the former being due to a uniformly distributed load can be obtained with facility for different positions of the load. The polygon can be at once combined into a single polygon by adding the ordinates of the two together. Care must be taken, however, to add together only such as have the same pole distance. In case the construction which has been given should show that the arch is unstable, having no projection of the equilibrium polygon which can be inscribed within the middle third of the arch ring, it is possible either to change the shape of the arch slightly, or increase its thickness, or change the distribution of the loading. The last alternative is usually the best one, for the shape has been chosen from reasons of utility and taste, and the thickness from consideration of the factor of safety. If the center line of the arch ring (or any other line inscribed within the middle third) be considered to be an equilibrium polygon, and from a pole, lines be drawn parallel to the segments of this polygon, a weight line can be found which will represent the loading needed to make the arch stable. If this load line be compared with that previously obtained, it will be readily seen where a slight additional load must be placed, or else a hollow place made in the surcharge, such as will render the arch stable. In general, it may be remarked, that an additional load renders the curvature of the line of pressures sharper under it, while the removal of any load renders the curve straighter under it.

The foregoing construction is unrestricted, and applies to all unsymmetrical forms of arches or of loading, or both. As previously mentioned, a similar construction applies to the case of an arch sustaining the pressure of water or earth; in that case, however, the load is not applied vertically and the weight line becomes a polygon.

CHAPTER XIV.

RETAINING WALLS AND ABUTMENTS.

Let $aa'b'b$ in Fig. 15 represent the cross section of a wall of masonry which retains a bank of earth having a surface aa_6. Assume that the portion of the wall and earth under consideration is bounded by two planes parallel to the plane of the paper, and at a unit's distance from each other: then any plane containing the edge of the wall at *b*, as ba_0, ba_1, etc., cuts this solid in a longitudinal section, which is a rectangle having a width of one unit, and a length ba_0, ba_1, etc.

The resultant of the total pressure distributed over any one of these rectangles of the type *ba* is applied at one-third of that distance from *b*: *i.e.* the resultant pressure exerted by the earth against the rectangle at ba_0 is applied at a distance of $bk=\frac{1}{3}\,ba_0$ from *b*.

That the resultant is to be applied at this point, is due to the fact that the distributed pressure increases uniformly as we proceed from any point *a* of the surface toward *b*: the center of pressure is then at the point stated, as is well known.

Again, the direction of the pressures against any vertical plane, as that at ba_0, is parallel to the surface aa_6. This fact is usually overlooked by those who treat this subject, and some arbitrary assumption is made as to the direction of the pressure.

That the thrust of the earth against a vertical plane is parallel to the ground surface is proved analytically in Rankine's Applied Mechanics on page 127; which proof may be set forth in an elementary manner by considering the small parallelopiped *mn*, whose upper and lower surfaces are parallel to the ground surface. Since the pressure on any plane parallel to the surface of the ground is due to the weight of the earth above it, the pressure on such a plane is vertical and uniformly distributed. If *mn* were a rigid body, it would be held in equilibrium by these vertical pressures, which are, therefore, a system of forces

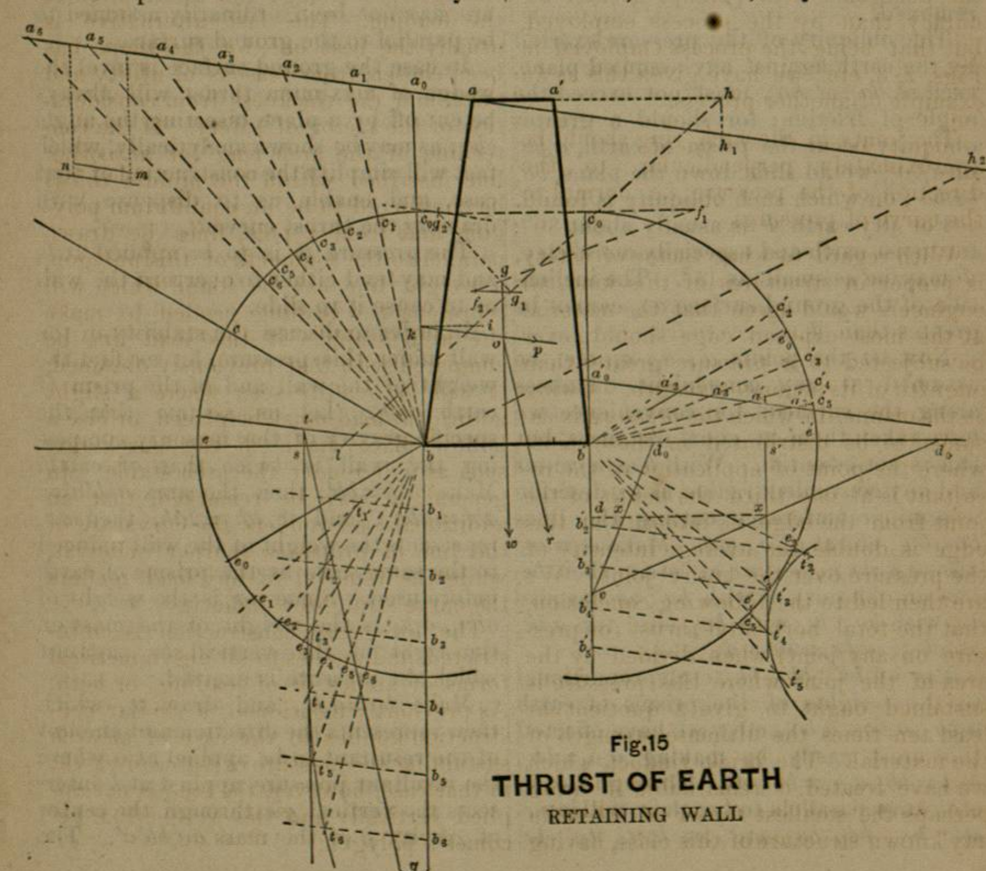


Fig.15
THRUST OF EARTH
RETAINING WALL

in equilibrium; but as mn is not rigid it must be confined by pressures distributed over each end surface, which last are distributed in the same manner on each end, because each is at the same depth below the surface. Now the vertical pressures and end pressures hold mn in equilibrium: they therefore form a system in equilibrium. But the vertical pressures are independently in equilibrium, therefore the end pressures alone form a system which is independently in equilibrium. That this may occur, and no couple be introduced, these must directly oppose each other; *i.e.* be parallel to the ground line aa_5.

Draw $kp \parallel aa_4$, it then represents the position and direction of the resultant pressure upon the vertical ba_0. Draw the horizontal ki, then is the angle ikp called the *obliquity* of the pressure, it being the angle between the direction of the pressure and the normal to the plane upon which the pressure acts.

Let $ebc=\Phi$ be the *angle of friction*, *i.e.* the inclination which the surface of ground would assume if the wall were removed.

The obliquity of the pressure exerted by the earth against any assumed plane, such as ba_3 or ba_4, must not exceed the angle of friction; for should a greater obliquity occur the prism of earth, a_0ba_3 or a_0ba_4, would slide down the plane, ba_3 or ba_4, on which such obliquity is found.

For dry earth Φ is usually about 30°; for moist earth and especially moist clay, Φ may be as small as 15°. The inclination of the ground surface aa_4 cannot be greater than Φ.

Now let the points a_1, a_2, a_3, etc., be assumed at any convenient distances along the surface: for convenience we have taken them at equal distances, but this is not essential. With b as a center and any convenient radius, as bc, describe a semi-circumference cutting the lines ba_1, ba_2, etc. at c_1, c_2, etc. Make $ee_0=ec$; also $e_0e_1=c_0c_1$, $e_0e_2=c_0c_2$, etc.: then be_0 has an obliquity Φ with ba_0, as has also be_1 with ba_1, be_2 with ba_2, etc.; for $a_0be_0 =a_1be_1=a_2be_2=90°+\Phi$.

Lay off bb_1, bb_2, bb_3, etc., proportional to the weights of the prisms of earth a_0ba_1, a_0ba_2, a_0ba_3, etc.: we have effected this most easily by making $a_0a_1=bb_1$, $a_0a_2=bb_2$, $a_0a_3=bb_3$, etc. Through b, b_1 b_2, etc., draw parallels to kp; these will intersect be_0, be_1, be_2, etc., at b, t_1, t_2, etc. Then is bb_1t_1 the triangle of forces holding the prism a_0ba_1 in equilibrium, just as it is about to slide down the plane ba_1, for bb_1 represents the weight of the prism, b_1t_1 is the known direction of the thrust against ba_0, and bt_1 is the direction of the thrust against ba_1 when it is just on the point of sliding: then is t_1b_1 the greatest pressure which the prism can exert against ba_0. Similarly t_2b_2 is the greatest pressure which the prism a_0ba_2 can exert. Now draw the curve $t_1t_2t_3$, etc., and a vertical tangent intersecting the parallel to the surface through b at t; then is tb the greatest pressure which the earth can exert against ba_0. This greatest pressure is exerted approximately by the prism or wedge of earth cut off by the plane ba_4, for the pressure which it exerts against the vertical plane through b is almost exactly $b_4t_4=bt$. This is Coulomb's "wedge of maximum thrust" correctly obtained: previous determinations of it have been erroneous when the ground surface was not level, for in that case the direction of the pressure has not been ordinarily assumed to be parallel to the ground surface.

In case the ground surface is level the wedge of maximum thrust will always be cut off by a plane bisecting the angle cbc_0, as may be shown analytically, which fact will simplify the construction of that case, and enable us to dispense with drawing the thrust curve tt.

The pressure tb is to be applied at k, and may tend either to overturn the wall or to cause it to slide.

In order to discuss the stability of the wall under this pressure, let us find the weight of the wall and of the prism of earth aba_0. Let us assume that the specific gravity of the masonry composing the wall is twice that of earth. Make $a'h=bb'$, then the area $abb'a'= abh=abh_1$; and if $ah_2=2ah$, then ah_2 represents the weight of the wall reduced to the same scale as the prisms of earth before used. Since aa_0 is the weight of aba_0, a_0h_2 is the weight of the mass on the right of the vertical ba_0 against which the pressure is exerted.

Make $bq=a_0h_2$, and draw tq, which then represents the direction and amount of the resultant to be applied at o where the resultant pressure applied at k intersects the vertical gw through the center of gravity g of the mass $aa_0bb'a'$. The

center of gravity g is constructed in the following manner. Lay off $a'h=bb'$, and $bl=aa'$; and join hl. Join also the middle points of ab and $a'b'$: the line so drawn intersects hl at g_1 the center of gravity of $aa'b'b$. Find also the center of gravity g_2, of aba_0, which lies at the intersection of a line parallel to aa_0, and cutting ba_0 at a distance of $\frac{1}{3}\,ba_0$ from a_0 and of a line from b bisecting aa_0. Through g_2 and g_1 draw parallels, and lay off g_2f_1 and g_1f_2 on them proportional to the weights applied at g_1 and g_2 respectively. We have found it convenient to make $g_2f_1=\frac{1}{2}ah_2$, and $g_1f_2=\frac{1}{2}$ aa_0. Then f_1f_2 divides g_1g_2 inversely as the applied weights; and g, the point of intersection, is the required center of gravity.

Let or be parallel to tq; since it intersects bb' so far within the base, the wall has sufficient stability against overturning. The base of the wall is so much greater than is necessary for the support of the weight resting upon it, that engineers have not found it necessary that the resultant pressure should intersect the base within the middle third of the joint. The practice of English engineers, as stated by Rankine, is to permit this intersection to approach as near b' as $\frac{1}{8}bb'$, while French engineers permit it to approach as near as $\frac{1}{5}bb'$ only. In all cases of buttresses, piers, chimneys, or other structures which call into play some fraction of the ultimate strength of the material, or ultimate resistance of the foundation as great as one tenth, or one fifteenth, the point should not approach b' nearer than $\frac{1}{3}$ bb'.

Again, let the angle of friction between the wall and the earth under it be Φ': then in order that the thrust at k may not cause the wall to slide, the angle wor must be less than Φ'.

When, however, the angle Φ' is less than wor it becomes necessary to gain additional stability by some means, as for example by continuing the wall below the surface of the ground lying in front of it. Let $a_0'a_5'$ be the surface of the ground which is to afford a *passive* resistance to the thrust of the wall: then in a manner precisely analogous to that just employed for finding the greatest active pressure which earth can exert against a vertical plane, we now find the least passive pressure which the earth in front of the wall will sustain without sliding up some plane such as $b'a_3'$ or $b'a_4'$, etc. The difference in the two cases is that in the former case friction hindered the earth from sliding down, while it now hinders it from sliding up the plane on which it rests.

Lay off $e'e_0'=ee_0$; then taking any points $a_2'a_3'$, etc. on the ground surface, make $e_0'e_2'=c_0'c_2'$, $e_0'e_3'=c_0'c_3'$, etc.

Lay off $b'b_2'=a_0'a_2'$, etc., and drawing parallels through b_2', b_3', etc., we obtain the thrust curve $t_2't_3'$, etc.

The small prism of earth between $b'a_0'$ and the wall adds to the stability of the wall, and can be made to enter the construction if desired, in the same manner as did aba_0.

The vertical tangent through s' shows us that the earth in front of the wall can withstand a thrust having a horizontal component $b's'$ measured on a scale such that $b'b_2'=a_0'a_2'$ is the weight of the prism of earth $a_0'b'a_2'$.

This scale is different from that used on the left. To reduce them to the same scale lay off from b', the distances $b'd_0$ and $b'd_0'$ proportional to the perpendiculars from b on aa_4 and b' on $a_1'a_4'$ respectively. In the case before us, as the ground surfaces are parallel, we have made $b'd_0=ba_0$ and $b'd_0'=b'a_0'$.

Then from any convenient point on $b'b_4'$, as v, draw vd_0 and vd_0': these lines will reduce from one scale to the other. We find then that $x'd$ is the thrust on the scale at the left corresponding to $xd=b's'$ on the right: *i.e.*, the earth under the surface assumed at the right can withstand something over one fourth of the thrust sb at the left.

It will be found that a certain small portion of the earth near a_0' has a thrust curve on the left of b', but as it is not needed in our solution it is omitted.

If any pressure is required in pounds, as for example sb, it is founds as follows: —the length of ah_2 is to that of sb as the weight of $bb'aa'$ in lbs. is to the pressure sb in lbs.

Frequently the ground surface is not a plane, and when this is the case it often consists of two planes as ad, da_7 Fig. 16. In that case, draw some convenient line as ad_1, and lay off ad_1, d_1d_2, etc. at will, which for convenience we have made equal. Draw d_1a_1, d_2a_2, etc. parallel to bd, and join ba_1, ba_2, etc.: then are the

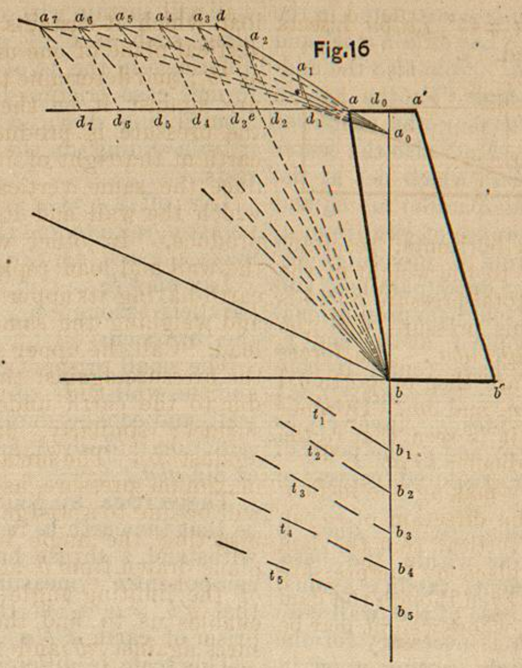


Fig.16

triangles bda, bda_1, bda_2, bda_3, etc. proportional in area to the lines ea, ea_1, etc. Hence the weights of the prisms of earth baa_1, baa_2, etc., are proportional to ad_1, ad_2, etc.

In case ab slopes backward the part of the wall at the left of the vertical ba_0 rests upon the earth below it sufficiently to produce the same pressure which would be produced if baa_0 were a prism of earth. The weights of the wedges which produce pressures, and which are to be laid off below b, are then proportional to $d_0d_1=bb_1$, $d_0d_2=bb_2$, etc. The direction of the pressures of the prisms at the right of bd are parallel to ad; but upon taking a larger prism the direction may be assumed to be parallel to a_0a_3, a_0a_4, etc., which is very approximately correct. Now draw $b_1t_1 \parallel a_0a_1$, $b_4t_4 \parallel a_0a_4$, etc.; and complete the construction for pressure precisely as in Fig. 15, using for resultant pressure the direction and amount of that due to the wedge of maximum pressure thus obtained.

In finding the stability of the wall, it will be necessary to find the weight and center of gravity of the wall itself, minus a prism of earth baa_0, instead of plus this prism as in Fig. 15; for it is now sustained by the earth back of the wall.

When the back of the wall has any other form than that above treated, the vertical plane against which the pressure is determined should still pass through the lower back edge of the wall.

In case the wall is found to be likely to slide upon its foundations when these are level, a sloping foundation is frequently employed, such that it shall be nearly perpendicular to the resultant pressure upon the base of the wall. The construction employed in Fig. 15 applies equally to this case.

The investigation of the stability of any abutment, buttress, or pier, against overturning and against sliding, is the same as that of the retaining wall in Fig. 15. As soon as the amount, direction, and point of application, of the pressure exerted against such a structure is determined, it is to be treated precisely as was the resultant pressure kp in Fig. 15.

In the case of a reservoir wall or dam, the construction is simplified from the fact that, since the surface of water is level and the angle of friction vanishes, the resultant pressure is perpendicular to the surface upon which the water presses. It is useful to examine this as a case of our previous construction. In Fig. 17, let abb' be the cross-section of the dam; then the wedge of maximum pressure against ba_0 is cut off by the

plane ba_1 when $cba_1=45°$, *i.e.* ba_1 bisects cba_0 as before stated.

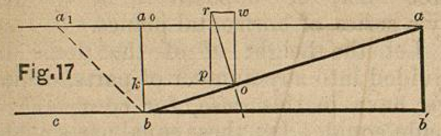


Fig.17

This produces a horizontal resultant pressure at k equal to the weight of the wedge. Now the total pressure on ab is the resultant of this pressure, and the weight of the wedge aba_0. The forces to be compounded are then proportional to the lines $a_1a_0=bv_0$ and aa_0. By similarity of triangles it is seen that ro the resultant is perpendicular to ab.

It is seen that by making the inclination of ab small, the direction of ro can be made so nearly vertical that the dam will be retained in place by the pressure of the water alone, even though the dam be a wooden frame, whose weight may be disregarded.

We can now construct the actual pressures to which the arch of a tunnel surcharged with water or earth is subjected. Suppose, for example, we wish to find the pressure of such a surcharge on the voussoir $a_4d_4d_5a_5$ Fig. 14. Find the resultant pressure against a vertical plane extending from d_5 to the upper surface of the surface and call it p_5. Draw a horizontal through d_4 and let its intersection with the vertical just mentioned be called d_5''. Find the resultant pressure against the vertical plane extending from d_5'' to the surface, and call it p_5'. Now let $p_5''=p_5-p_5'$ and let it be applied at such a point of d_5d_5'' that p_5 shall be the resultant of p_5' and p_5''. Then will the resultant pressure against the voussoir be the resultant of p_5'' and the weight of that part of the surcharge directly above it.

FOUNDATIONS IN EARTH.

A method similar to that employed in the determination of the pressure of earth against a retaining wall, or a tunnel arch, enables us to investigate the stability of the foundations of a wall standing in earth.

Suppose in Fig. 15 that the wall $abb'a'$ is a foundation wall, and that the pressure which it exerts upon the plane bb' is vertical, being due to its own weight and the weight of the building or other load which it sustains. Now consider a vertical plane of one unit in height, say, as bb_1; and determine the resultant pressure against it on the supposition that the pressure is produced by a depth of earth at the right of it, sufficient to produce the same vertical pressure on bb' which the wall and its load do actually produce. In other words we suppose the wall and load replaced by a bank of earth having its upper surface horizontal and weighing the same as the wall and load. Call the upper surface z, and find the pressure against the vertical plane zb due to the earth under the given level surface; similarly, find the pressure against zb_1. The surface being level, the maximum pressure, as previously stated will be due to a wedge cut off by a plane bisecting the angle between bz and a plane drawn from b at the inclination Φ, of the limiting angle of friction. This enables us to find the horizontal pressures against zb and zb_1 directly: their difference is the resultant active pressure against bb_1.

Next, it must be determined what passive pressure the earth at the left of bb_1 can support. The passive resistance of the earth under the surface a against the plane ab as well as that against the plane ab_1 can be found exactly as that was previously found under the surface a'. The difference of these resistances is the resistance which it is possible for bb_1 to support. Indeed bb_1 could support this pressure and afford this resistance even if the active pressure against ab were, at the limit of its resistance, which it is not. The limiting resistance which is thus obtained, is then so far within the limits of stability, that ordinarily, no further factor of safety is needed, and the stability of the foundation is secured, if the active pressure against bb_1 does not exceed the passive resistance. This construction should be made on the basis of the smallest angle of friction Φ which the earth assumes when wet; that being smaller than for dry earth, and hence giving a greater active pressure at the right, and a less resistance at the left.

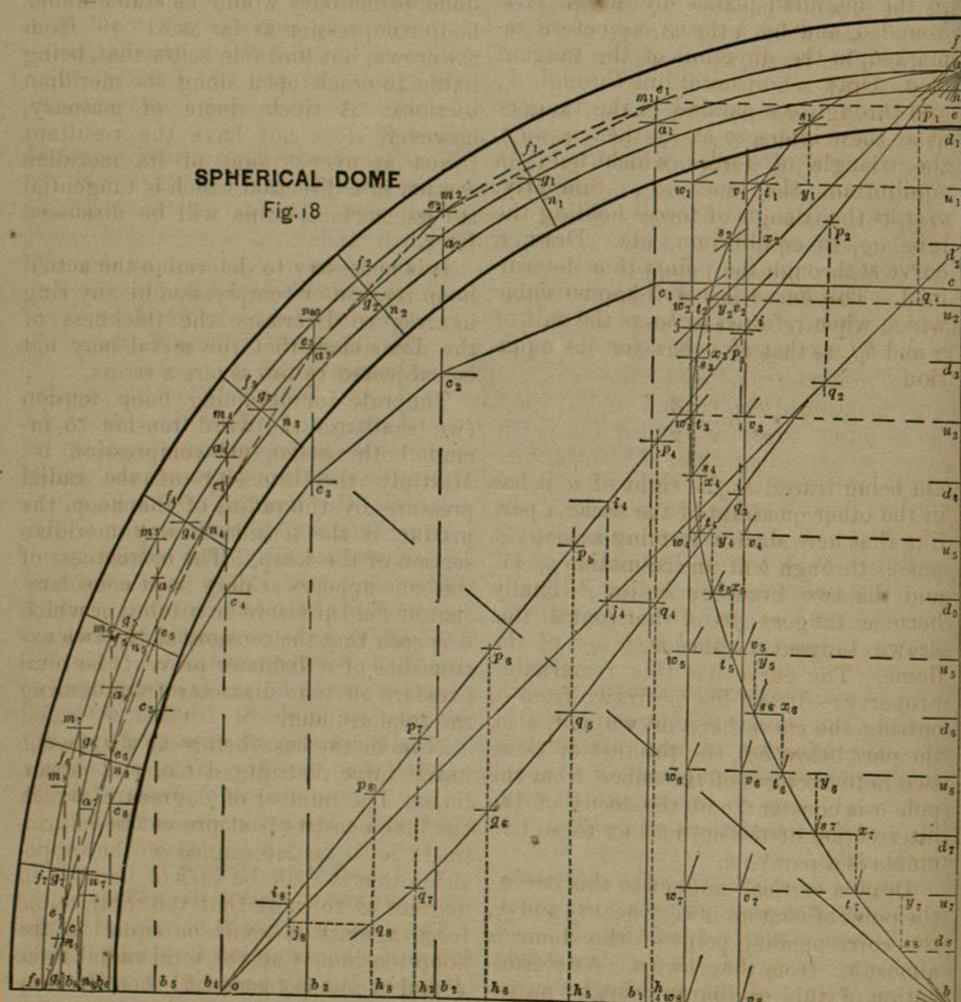
CHAPTER XV.

SPHERICAL DOME OF METAL.

The dome which will be treated in the following construction is hemispherical in shape; but the proposed construction applies equally to domes of any different form generated by the revolution of the arc of some curve about a vertical axis: such forms are elliptic, parabolic or hyperbolic domes, as well as pointed or gothic domes, etc. Let the quadrant *aa* in Fig. 18, represent the part of the meridian section of a thin metallic dome between the crown and the springing circle. The metallic dome is supposed to be so thin that its thickness need not be represented in the Figure: the thickness of a dome of masonry, however, is a matter of prime importance and will be treated subsequently.

In a thin metallic dome the only thrust along a meridian section is necessarily in a direction tangent to that section at each point of it. This consideration will enable us to determine this thrust as well as the hoop tension or compression along any of the conical rings into which the dome may be supposed to be divided by a series of horizontal planes.

Let the height *ab* of the dome be divided into any number of parts, which we have in this case, for convenience, made equal. Let these equal parts of the type *du* be the distances between horizontal planes such that the planes through the points d_1, d_2, etc., cut small circles from the hemisphere which pass through the point a_1, a_2, etc., and similarly the planes through u_1, u_2, etc., cut small circles which pass through g_1, g_2, etc. Now suppose the thickness of this dome to be uniform, and if *ab* be taken to represent the weight of a quadrantal lune of the dome included between two meridian planes making some small angle with each other; then



SPHERICAL DOME
Fig. 18

from the well-known expression for the area of the zone of a sphere it appears that ad_1 will represent the weight of that part of the lune above a_1d_1. Similarly au_1 is the weight of the lune ag_1; ad_2 the weight of aa_2, etc.

This method of obtaining the weight applies of course in case the dome is any segment of a sphere less than a hemisphere and of uniform thickness. If the thickness increases from the crown, the weights of the zones cut by equi-distant horizontal planes increase directly as the thickness. In case the dome is not spherical the weights must be determined by some process suited to the form of the dome and its variation in thickness.

Now the weight of the lune aa_1 is sustained by a horizontal thrust which is the resultant of the horizontal pressures in the meridian planes by which it is bounded, and by a thrust, as before remarked, in the direction of the tangent at *a*. Draw a horizontal line through d_1, and through *a* a parallel to the tangent at *a*: these intersect at s_1, then is ad_1s_1 the triangle of forces which hold in equilibrium the lune aa_1. Similarly, au_1t_1 is the triangle of forces holding the lune ag_1 in equilibrium, etc. Draw a curve *st* through the points thus determined. This curve is a well-known cubic which when referred to *ba* as the axis of *x* and bg_5 as that of *y* has for its equation

$$\frac{y^2}{x^2}=\frac{r-x}{r+x}$$

On being traced at the right of *a* it has in the other quadrant of the dome a part like that here drawn forming a loop; it passes through *b* at an inclination of 45° and the two branches below *b* finally become tangent to a horizontal line drawn tangent to the circle *aa* of the dome. The curve has this remarkable property:—If any line be drawn from *a*, cutting the curve here drawn and, also, the part below bg_5, the product of these two radii vectores of the curve from the pole *a* is constant, and the locus of the intersection of the normals at these two points is a parabola.

Draw a vertical tangent to this curve: the point of contact is very near t_3, and g_3, the corresponding point of the dome is almost 52° from the crown *a*. A determination of this maximum point by means of the equation gives the height of it above *b* as $\frac{1}{2}(\sqrt{5}-1)\,r$, corresponding to about 51°49′. Now consider any zone, as, for example, that whose meridian section is g_1a_2: the upper edge is subjected to a thrust whose radial horizontal component is proportional to u_1t_1, while the horizontal thrust against its lower edge is proportional to d_2s_2, and the difference s_2x_2 between these radial forces produces a hoop compression around the zone proportional to s_2x_2. It will be seen that these differences which are of the type *sx* or *ty*, change sign at t_3. Hence all parts of the dome above 51° 49′ from the crown, are subjected to a hoop compression which vanishes at that distance from *a*, while all parts of the dome below this are subjected to hoop tension. This may be stated by saying that a thin dome of masonry would be stable under hoop compression as far as 51° 49′ from the crown, but unstable below that, being liable to crack open along its meridian sections. A thick dome of masonry, however, does not have the resultant thrust at every point of its meridian section in a direction which is tangential to its surface,—this will be discussed later.

It is necessary to determine the actual hoop tension or compression in any ring in order to determine the thickness of the dome such that the metal may not be subjected to too severe a stress.

The rule for obtaining hoop tension (we shall use the word tension to include both tension and compression) is: Multiply the intensity of the radial pressure by the radius of the hoop, the product is the tension at any meridian section of the hoop. The correctness of this rule appears at once from consideration of fluid pressure in a tube, in which it is seen that the tensions at the two extremities of a diameter prevent the total pressure on that diameter from tearing the tube asunder.

Now in the case before us t_1y_1 is the radial force distributed along a certain lune. The number of degrees of which the lune consists is at present undetermined: let it be determined on the supposition that it shall be such a number of degrees as to cause that the total radial force against it shall be equal to the hoop tension. Call the total radial force *P* and the hoop tension *T*, then the lune is to be such that $P=T$. Also let θ be the number of degrees in the lune, then $90°\div\theta$ is the number of lunes in a quarter of the dome, and $90\ P\div\theta$ is the radial force against a quarter of the dome, which last must be divided by $\frac{1}{2}\pi$ to obtain the hoop tension; because if p is the intensity of radial pressure, $\frac{1}{2}\pi rp$ is the total pressure against a quadrant and rp, as previously stated, is the hoop tension. The ratio of these is $\frac{1}{2}\pi$, and by this we must divide the total radial pressure in every case to obtain hoop tension

$$\therefore \frac{180\ P}{\theta\pi}=T, \quad \therefore\ \theta=\frac{180°}{\pi}$$

$$\text{for } P=T \quad \therefore\ \theta=57°.3-$$

This is the number of degrees of which the lune must consist in order that when ab represents its weight, t_1y_1 shall represent the hoop tension in the meridian section a_1g_1. The expression we have found is independent of the radius of the ring, and hence holds for any other ring as g_1a_2, in which s_2x_2 is the hoop tension, etc. To find what fraction this lune is of the whole dome, divide θ by 360°

$$\therefore \frac{\theta}{360}=\frac{180}{360\pi}=\frac{1}{2\pi}=\frac{4}{25}\text{nearly,}$$

from which the scale of weight is easily found, thus; let W be the total weight of the dome and r its radius, then

$2\pi r : W :: 1 : n$, the weight per unit, or the hoop tension per unit of the distances ty or sx.

Distances at or as, on the same scale, represent the thrust tangential to the dome in the direction of the meridian sections, and uniformly distributed over an arc of $57°.3-$: *e.g.* if we divide at_2 measured as a force by $\theta\times u_2g_2$ measured as a distance we shall obtain the intensity of the meridian compression at the joint cut from the dome by the horizontal plane through a_2.

Analogous constructions hold for domes not spherical and not of uniform thickness. Approximate results may be obtained by assuming a spherical dome, or a series of spherical zones approximating in shape to the form which it is desired to treat.

CHAPTER XVI.

SPHERICAL DOME OF MASONRY.

Let the dome treated be that in Fig. 18 in which the uniform thickness of the masonry is one-sixteenth of the internal diameter or one-eighth of the radius of the intrados. Divide ab the radius of the center line into any convenient number of equal parts, say eight, at u_1, u_2, etc.: a much larger number would be preferable in actual construction. At the points a_1, a_2, etc., on the same levels with u_1, u_2, etc. pass conical joints normal to the dome, so that b is the vertex of each of the cones.

If we consider a lune between meridian planes making a small angle with each other, the center of gravity of the parts of the lune between the conical joints lie at g_1, g_2, etc. on the horizontal midway between the previous horizontals. These points are not exactly upon the central line aa, but if the number of horizontals is large, the difference is inappreciable. We assume them upon aa. That they fall upon the horizontals through d_1, d_2, etc., midway between those through u_1, u_2, etc., is a consequence of the equality in area between spherical zones of the same height.

In finding the volume of a sphere it may be considered that we take the sum of a series of elementary cones whose bases form the surface of the sphere, and whose height is the radius. Hence, if any equal portions of the surface of a sphere be taken and sectorial solids be formed on them as bases and having their vertices at the center, then the sectorial solids have equal volumes. The lunes of which we treat are equal fractions of such equal solids.

Draw the verticals of the type bg through the centers of gravity g_1, g_2, etc. The weights applied at these points are equal and may be represented by au_1, $u_1u_2=w_1w_2$, etc. Use a as the pole and w_1w_8 as the weight line; and, beginning at the point f_8, draw the equilibrium polygon c due to the weights.

We have used for pole distance the greatest horizontal thrust which it is possible for any segment of the dome to exert upon the part below it, when the hoop compression extends to 51° 49′ from the crown.

Below the point where the compression vanishes we shall not assume that the bond of the masonry is such that it can resist the hoop tension which is developed. The upper part of the dome will be then carried by the parts of the lunes below this point by their united action as a series of masonry arches standing side by side.

Now it is seen that the curve of equilibrium c, drawn with this assumed horizontal thrust falls within the curve of the lune, which signifies that the dome will not exert so great a thrust as that assumed. By the principle of least resistance, no greater horizontal thrust will be called into action than is necessary to cause the dome to stand, if stability is possible. If a less thrust than that just employed be all that is developed in the dome, then the point where the hoop compression vanishes is not so far as 51° 49′ from the crown, and a longer portion of the lune acts as an arch, than has been supposed by previous writers on this subject,* none of whom, so far as known, have given a correct process for the solution of the problem, although the results arrived at have been somewhat approximately correct.

To ensure stability, the equilibrium curve must be inscribed within the inner third of that part of the meridian section of the lune which is to act as an arch; as appears from the same reasons which were stated in connection with arches of masonry.

And, further, the hoop compression will vanish at that level of the dome where the equilibrium curve, in departing from the crown, first becomes more nearly vertical than the tangent of the meridian section; for above that point the greatest thrust that the dome can exert, cannot be so great as at this point where the thrust of the arch-lune is equal to that of the dome.

Now to determine in what ratio the ordinates of the curve c must be elongated to give those of the curve e which fulfills the required conditions, we draw the line fo, and cut it at p_1, p_2, etc. by the horizontals m_1p_1, m_2p_2, etc., the quantities mb being the ordinates of exterior of the inner third. Again draw verticals through p_1, p_2, etc., and cut them at q_1, q_2, q_3, etc. by horizontals through c_1, c_2, c_3, etc. Through these points draw the curve qq, whose ordinates are of the type qh. Some one of these ordinates is to be elongated to its corresponding ph, and in such a manner that no qh shall then become longer than its corresponding ph. To effect this, draw oq_2 tangent to the curve qq; then will oq_2 enable us to effect the required elongation: *e.g.* let the horizontal through c_4 cut oq_2 at j_4, and then the vertical through j_4 cuts fo at i_4, then is e_4 (which is on the same level with i_4) the new position of c_4. Similarly, we may find the remaining points of the curve e; but it is better to determine the new pole distance, and use this method as a test only.

The curve qq made use of in this construction for finding the ratio lines for so elongating the ordinates of the curve c, that the new ordinates shall be those of a curve e tangent to the exterior line of the inner third, may be applied with equal facility to the construction for the arch of masonry. This furnishes us with a direct method in place of the tentative one employed in connection with Fig. 14.

To find the new pole distance, draw $fj \parallel oq_2$ cutting ww at j, then will i the intersection of the horizontal through j, be the new position of the weight line vv, having its pole distance from a diminished in the required ratio.

The equilibrium curve e will be parallel to the curve of the dome at the points where the new weight line vv cuts the curve st. It should be noticed that the pole distance which we have now determined is still a little too large because the polygon e is circumscribed about the true equilibrium curve; and as the polygon has an angle in the limiting curve mm the equilibrium curve is not yet high enough to be tangent to the limiting curve. If the number of divisions had originally been larger (which the size of our Figure did not permit) this matter would be rectified.

The polygon e is seen at e_5 to fall just without the required limits, this would be partly rectified by slightly decreasing the pole distance as just suggested; the point, however, would still remain just without the limit after the pole distance is decreased, and by so much is the dome unstable. A dome of which the thick-

* See a paper read before the Royal Inst. of British Architects, "on the Mathematical Theory of Domes," Feb. 6th, 1871. By Edmund Beckett Denison, L.L.D., Q.C., F.R.A.S.

ness is one fifteenth of the internal diameter, is almost exactly stable.

It is a remarkable fact that a semi-cylindrical arch of uniform thickness and without surcharge must be almost exactly three times as thick, viz., the thickness must be about one fifth the span in order that it may be possible to inscribe the equilibrium curve within the inner third.

The only large hemispherical dome, of which I have the dimensions, which is thick enough to be perfectly stable without extraneous aid such as hoops or ties, is the Gol Goomuz at Beejapore, India. It has an internal diameter of 137½ feet, and a thickness of 10 feet, it being slightly thicker than necessary, but it probably carries a load upon the crown which requires the additional thickness.

The hemispherical dome of uniform thickness is a very faulty arrangement of material. It is only necessary to make the dome so light and thin for 51° 49′ from the crown that it cannot exert so great a horizontal thrust as do the thicker lunes below, to take complete advantage of the real strength of this form of structure. A dome whose thickness gradually decreases toward the crown takes a partial advantage of this, but nothing short of a quite sudden change near this point appears to be completely effective.

The necessary thickness to withstand the hoop compression and the meridian thrust can be found as previously shown in the dome of metal.

Domes are usually crowned with a lantern or pinnacle, whose weight must be first laid off below the pole a after having been reduced to the same unit as that of the zones of the dome.

Likewise when there is an eye, at the crown or below, the weight of the material necessary to fill the eye must be subtracted, so that a is then to be placed below its present position. The construction is then to be completed in the same manner as in Fig, 18.

It is at once seen that the effect of an additional weight, as of a lantern, at the crown, since it moves the point a upward a certain distance, will be to cause the curve st to have all its points except b to the left of their present position, and especially the points in the upper part of the curve, thus making the point of no hoop tension much nearer the crown than in the metallic dome. It will be noticed that the addition of very small weight at the crown will cause the point m_2 of no hoop tension in the dome of masonry to approach almost to the crown, so that then the lunes will act entirely as stone arches with the exception of a very small segment at the crown.

On the contrary, the removal of a segment at the crown, or the decrease of the thickness, or any device for making the upper part of the dome lighter will remove the point of no hoop tension further from the crown, both for the dome of metal and of masonry. In any dome of masonry the thickness above the point of no hoop tension, as determined by the curve st, need be only such as to withstand the two compressions to which it is subjected, viz; hoop compression and meridian compression: while below that the lunes acting as arches must be thick enough to cause a horizontal thrust equal to the maximum radial thrust of the dome above the point of no hoop tension.

Several large domes are constructed of more than one shell, to give increased security to the tall lanterns surmounting them: St. Peter's, at Rome, is double, and the Pantheon, at Paris, is triple. The different shells should all spring from the same thick zone below the point of no hoop tension; and the lunes of this thick zone should be able to afford a horizontal thrust equal to the sum of the radial thrusts of all the shells standing upon it.

Attention to this will secure the stability in itself of any dome of masonry spherical or otherwise; and, though I here offer no proof of the assertion, I am led to believe that this is the solution of the problem of constructing the dome of a minimum weight of material, on the supposition that the meridian joints can afford no resistance to hoop tension.

Now, in fact, it is a common device to ensure the stability of large domes by encircling them with iron hoops or chains, or by embedding ties in the masonry; and this case appears to be of sufficient importance to demand our attention.

If the hoop encircles the dome at 51° 49′ or any other less distance from the crown the dome will be a true dome at all points above the hoop. Suppose the hoop to be at 51° 49′, then the curve e should, below that point, be made to pass through the points f_3 and f_8, from which it is seen that the dome may be made thinner than at present, and the horizontal thrust caused will be less. The tension of the hoop would be that due to a radial thrust which is the difference between that given by the curve st for this point and the horizontal thrust (pole distance) of the polygon e when it passes through f_3 and f_8. That the curve e passes through these last mentioned points is a consequence of the principle of least resistance.

Again, suppose another hoop encircles the dome at f_5; the curve e must pass through f_8 and f_5, and in this part of the lune will have a corresponding horizontal thrust. The curve e must also pass through f_5 and f_3, but in this part of the lune will have a horizontal thrust corresponding to it, differing from that in the part between f_8 and f_5: indeed the horizontal thrust in the segment of a dome above any hoop depends exclusively upon that segment and and is unaffected by the zone below the hoop. The tension sustained by the hoop is, however, due to the radial force, which is the difference of the horizontal thrusts of the zones above and below the hoop.

It is seen that the introduction of a second hoop will still further diminish the thickness of lune necessary to sustain the dome, unless indeed the thickness is required to sustain the meridian compression.

Had a single hoop been introduced at f_5 with none above that point, the dome above f_5 should then be investigated, just as if the springing circle was situated at that point. The curve e must then start from f_5, as it before did from f_8, and be made to become tangent to the limiting curve at some point between f_5 and the crown.

By the method here employed for finding the tension of a hoop it is possible to discuss at once the stresses induced in the important modern domes constructed with rings and ribs of metal and having the intermediate panels closed with glass.

On introducing a large number of rings at small distances from each other, it will be seen that the discussion just given leads to the method previously given for the dome of metal.

The dome of St. Paul's, London, is one which has excited much adverse criticism by reason of the novel means employed to overcome the difficulties inherent in so large a dome at so great a height above the foundations of the building. The exterior dome consists of a framework of oak sustained by conical dome of brick which forms the core. There is also a parabolic brick dome under the cone which forms no essential part of the system. Since the conical dome in general presents some peculiarities worthy of notice we will give an investigation of that form of structure as our concluding construction.

CHAPTER XVII.

CONICAL DOME OF METAL.

In Fig. 19, let bd be the axis of the frustum of a metallic cone cut by a vertical plane in the meridian section a. The cone is supposed to have a uniform thickness too small to be regarded in comparison with its other dimensions. Suppose the frustum to be cut by a series of equi-distant horizontal planes as at g_1, g_2, etc., into a series of frustra or rings: then the weight of each ring is proportional to its convex surface. The convex surface of any ring $=2\pi r\times$ slant height; when r is half the sum of the radii of the two bases, *i.e.*, r is the mean radius. Consequently, the weights of these rings, or any given fraction of them included between two meridian planes, is proportional to their mean radii. Let us draw these mean radii d_1a_1, d_2a_2, etc., between the horizontals through g_1,g_2, etc., and use some convenient fraction, say $\frac{1}{3}$, of these quantities of the type da as the weights. The line ii cuts off $\frac{1}{3}$ of each of these: then lay off $du_1=d_1i_1$ as the weight of the ring ag_1, lay off $u_1u_2=d_2i_2$, $u_2u_3=d_3i_3$, etc., as the weights of the rings g_1g_2, g_2g_3, etc.

Draw the line $dt \parallel aa$, it corresponds to the curve st of Fig. 18; then the quantities of the type tu represent the horizontal radial thrust which the cone exerts upon the part below it, while the radial thrust borne by any ring is the difference between two successive quantities of the type tu, *i.e.*, the radial thrust in the ring g_7g_8 is represented by t_8y_8,