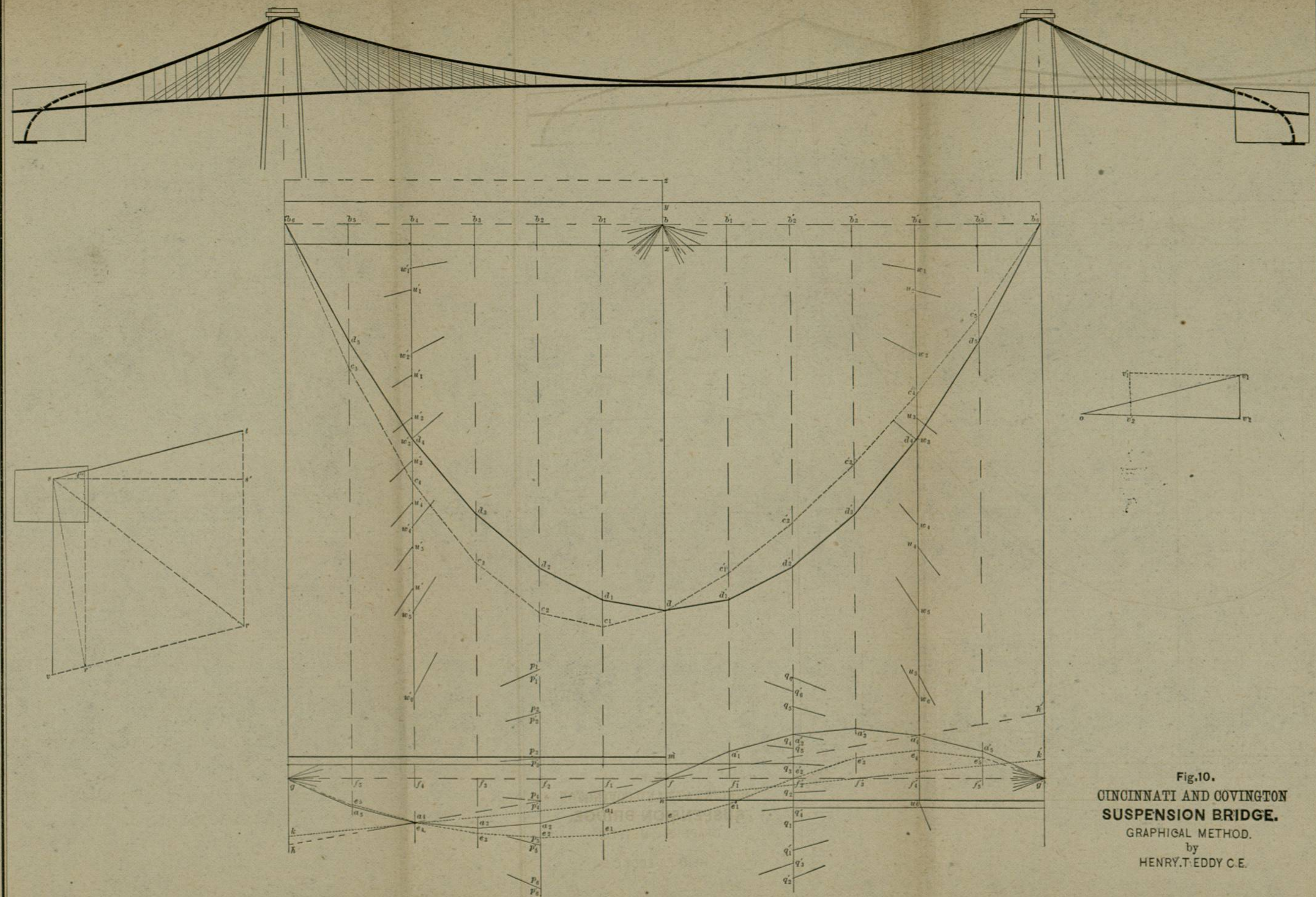

Fig. 10.

CINCINNATI AND COVINGTON

SUSPENSION BRIDGE.

GRAPHICAL METHOD.

by

HENRY T. EDDY C.E.

A NEW GENERAL METHOD

IN

GRAPHICAL STATICS.

A NEW GENERAL METHOD

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GRAPHICAL STATICS.

All general processes used in the graphical computation of statical problems consist, in their last analysis, in a systematized application of the proposition known as the "parallelogram of forces," which states that if two forces be applied to a material point, and if they be represented in magnitude and direction by two determinate straight lines, then their resultant is represented in magnitude and direction by the diagonal of a parallelogram, two of whose sides are the just mentioned determinate lines. This is the basis of all grapho-statical construction, but the methods by which it is systematized, and the auxiliary ideas incorporated in the processes, have so enlarged its possibilities of usefulness, that Graphical Statics may perhaps claim to be a science of itself;—the science of the geometrical treatment of force.

In order to introduce to the public a new set of auxiliary ideas, which shall constitute a new method, of a character equally general with that now in use and known as the "equilibrium polygon method," it has seemed best to give, in the first place, a brief review of the principal ideas already employed by the cultivators of this science.

RECIPROCAL FIGURES.

When a framed structure, such as a roof or bridge truss, is subjected to the action of certain weights or forces, these applied forces form a system which is in equilibrium. Now any system of forces in equilibrium may be represented in magnitude and direction by the sides of a closed polygon, a fact which follows at once from the doctrine of the parallelogram of forces. Such a polygon is called the polygon of the applied forces.

Again, the forces which act at any joint of a frame are in equilibrium, and hence there is a closed polygon of the forces acting at each joint. The forces which meet at a joint of a frame are the longitudinal tensions or compressions of the pieces meeting at that joint, together with any of the applied forces whose point of application may be the joint in question. Draw a diagram of the frame and the applied forces all of which we will suppose lie in a single plane. Call this the "frame diagram:" it represents the position and direction of all the forces acting in and upon the frame. The frame diagram necessarily has at least three lines meeting at each joint. A piece which constitutes part of the frame does not necessarily have both its extremities attached at joints of the frame; one extremity may be firmly attached to any immovable object. The frame diagram is, therefore, not necessarily made up of closed figures.

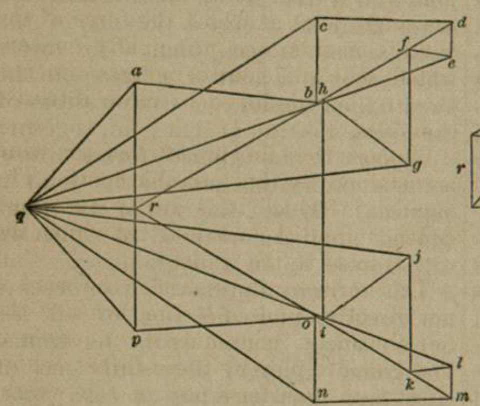
Now draw the closed polygon of the forces applied to the frame, and at each of the joints where forces are applied draw the closed polygon of the forces which meet at that joint, using so far as possible the lines already drawn as sides of the new polygons, and at the same time draw polygons for the forces acting at each of the remaining joints. If this process be effected with care as to the order of procedure, as well as to the order in which the forces follow each other in the polygon of the applied forces, then the resulting "diagram of forces," which is formed of the combination of the polygon of the applied forces with the polygons for each joint, will contain in it a single line and no more parallel to each line of the frame diagram. In that case the force diagram is said to be a reciprocal figure to the frame diagram. If sufficient care is not exercised in the particulars mentioned some of the lines in the force diagram will have to be repeated, and the figure drawn will not be the reciprocal of the frame diagram, nevertheless it will give a correct construction of the quantities sought.

If the frame diagram and the force diagram are both closed figures then they are mutually reciprocal. The properties of reciprocal figures were clearly set forth by Professor James Clerk Maxwell, in the *Philosophical Magazine*, vol. 27, 1864; in which is stated, what is also evident from considerations already adduced above, that mutually "reciprocal figures are mechanically reciprocal; that is, either may be taken as representing a system of points (*i.e.* joints) and the other as representing the magnitudes of the forces acting between them."

The subject has also been treated by Professor B. Cremona in a memoir entitled "Le figure reciproche nelle statica grafica." Milan, 1872.

We shall now give examples of this method of computing the forces acting between the joints of a frame, together with certain extensions by which we are enabled to treat moving loads, etc. The method is correctly called "Clerk Maxwell's Method." The notation employed, which is particularly suitable for the treatment of reciprocal diagrams, is due to R. H. Bow, C.E.; and is used by him in his work entiled "Economics of Construction." London, 1873. In this work will be found a very large number of frame and force diagrams drawn by this method.

Let the right hand part of Fig. 1



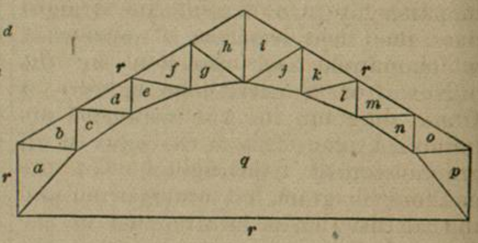


Fig.1.
ROOF TRUSS
TEMPERATURE STRESSES

represent a roof truss having an inclination of 30° to the horizon, of which the lower chord is a polygon inscribed in an arc of 60° of a circle. If the lower extremities of the truss abut against immovable walls a change of temperature causes an horizontal force between these lower joints, the effect of which upon the different pieces of the truss is to be constructed. No other weights or forces are now considered except those due to this horizontal force. This force is considered thus apart from all others because it is a force between two joints, and must enable us to obtain a pair of mutually reciprocal figures, such as weights and other applied forces seldom give.

It is seen that the force between these joints might be supposed to be caused by a tie joining these points; and in general it may be stated that the diagram of forces due to any cambering or stress induced in a frame by "keying" pieces, is mutually reciprocal to the frame diagram.

Let any piece of the frame be denoted by the letters in the spaces on each side of it; thus the pieces of the lower chord are *qa*, *qc*, *qe*, etc.; and those of the upper chord are *rb*, *rd*, etc., while *ab*, *bc*, etc., are pieces of the bracing, and *qr* is the tie whose tension produces the stress under consideration.

In the force diagram upon the left, let *qr* represent, on some assumed scale of tons to the inch, the tension in the piece *qr*; and complete the triangle *aqr* with its sides parallel to the pieces which converge to the joint *aqr*; then must this triangle represent the forces which are in equilibrium at that joint. Next, with *ar* as one side, complete the triangle *abr*, by making its sides parallel to the pieces meeting at the joint of the same name:—its sides will represent the forces in equilibrium at that joint. In a similar manner we proceed from joint to joint, using the stresses already obtained in determining those at the successive joints.

It is not possible to determine in general more than two unknown stresses in passing to a new joint, unless aided by some considerations of symmetry which may exist at such a joint as *ghijq*.

Now from the left hand figure as a frame diagram, in which stresses are induced by causing tension in the tie *qr*, we can construct the right hand figure as a force diagram, but it must be noticed in that case that *rb*, *rh*, *rf*, *rd* are separate and distinct pieces meeting at the joint *r*, although they all lie in the same right line, and that the same is true along the line *o i k m*.

One or two considerations of a general nature should be recalled in this connection.

A polygon encloses the space *q*; in the reciprocal figure the lines parallel to its sides must all diverge from the point *q*: and if the upper chord had been a polygon, instead of being of uniform slope, the lines parallel to its sides would diverge from the point *r*. As it is, *ra*, *rb*, *rd*, *rm* etc., form the rays of such a pencil, in which several rays are superposed one upon another.

The determination of the question as to whether the stress in a given piece is tension or compression is effected by following the polygon for any joint completely around and noting whether the forces act toward or from the joint: *e.g.* at the point *fghrf*, from following the diagrams of preceding joints in the manner stated, it will be found that *fg* is under tension, and acts from the joint; consequently, *gh* which acts toward the joint is under compression, as are also the two remaining pieces. Hence if the tension in the tie *qr* be replaced by an equal compression in a part, tending to move the lower extremities of the roof from each other, the sign of every stress in the roof will be changed, but the numerical amount will remain unchanged, and no change will be made in the force diagram.

ROOF TRUSS.

As another example let us take a roof truss represented in Fig. 2, acted upon by the equal weights *fe*, *ed*, *dd'*, etc. Suppose that the effect of the wind against the right hand side of the truss is such as to cause a deviation of the force applied at the joint *a'b'e'f'* of the amount indicated in the figure. Such a deviation may of course occur at several joints of a roof, but the treatment of the single joint at which the force of the wind is, in this case, principally concentrated, will sufficiently indicate the method to be employed in more intricate examples.

Suppose that this pressure of the wind is sustained by the left abutment. The manner in which it is really sustained depends upon the method by which the roof is fixed to the walls.

This horizontal pressure of the wind is not directly opposed to the thrust of the left abutment, consequently a couple is brought into play by these forces, whose effect is to transfer a part of the weight from the right to the left abutment. To compute the amount of this effect, draw an horizontal line through this joint (or in case the wind acts at several joints the horizontal line has to be drawn through the center of action of the wind pressure) and prolong it until it intersects the vertical at the right abutment at 3. Let 14 be equal to the pressure of the wind. Join 13 and prolong 13 until it intersects the vertical through 4 at 5, then is 45 the amount by which the weight upon the left abutment is increased, and that

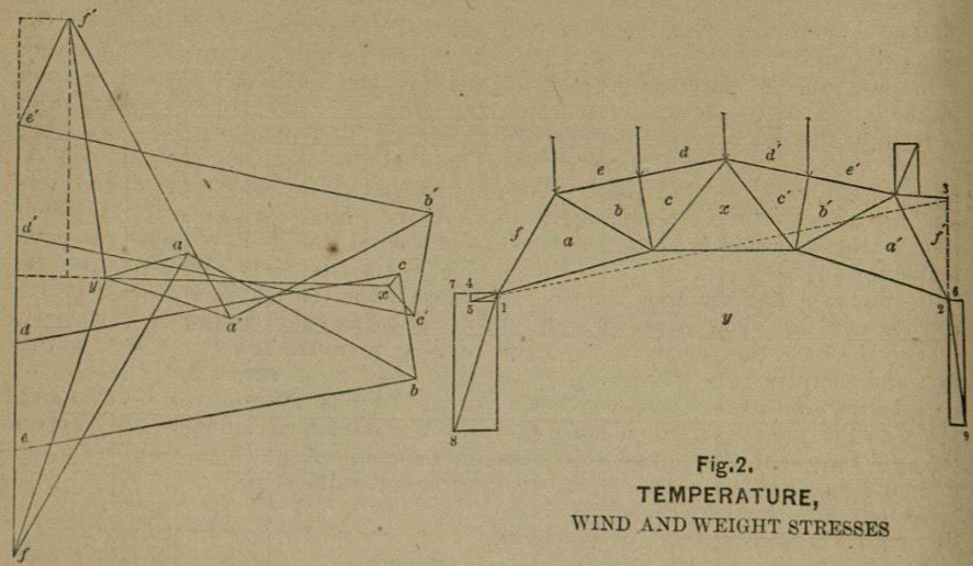


Fig.2.
TEMPERATURE,
WIND AND WEIGHT STRESSES

upon the left abutment decreased. For, let $k.\overline{14}=\overline{12}$. then $k.\overline{45}=\overline{23}$. Now the couple due to the wind $=\overline{23}\cdot\overline{14}$ but $k.\overline{23}.\overline{14}=\overline{12}.\overline{23}=k.\overline{12}.\overline{45}$, $\therefore \overline{23}.\overline{14}=\overline{12}.\overline{45}$. The right hand side of this last equation is the couple equivalent to the wind couple, having the arm 12 and a pair of equal and opposite forces represented by 45. Let 45 be added to half the weight of the symmetrical loading upon the roof to obtain the vertical reaction of the left abutment, and subtracted from the same quantity for the vertical reaction of the right abutment.

If any doubt occurs as to the manner in which the wind pressure is distributed between the abutments that distribution should be adopted which will cause the greatest stresses upon the pieces, or, as it may be stated in better terms, each piece should be proportioned to bear the greatest stress which any distribution of that pressure can cause.

Let us suppose that a horizontal compression is exerted upon the truss due to temperature or other cause, and represented by the width 26 of the rectangle at the right abutment, then the reaction at that point is the resultant 92 of this compression and the vertical reaction; while at the left abutment the total horizontal reaction 71 is the sum of this compression and the resistance called into action by the wind, giving 81 as the resultant reaction at the left abutment.

Now, using a scale of force twice that just employed, for the sake of greater convenience and accuracy, construct *defyf″e′d′* the polygon of the applied forces; and proceed to construct as in Fig. 1 the polygons of forces for each of the joints. The accuracy of the construction will be tested by the closing of the figure at the completion of the process.

The force diagram at the left is the reciprocal figure of the diagram of the frame and applied forces at the right, but the figure at the right is not the reciprocal of that at the left since it is not a closed figure with at least three lines meeting at each intersection.

BRIDGE TRUSS.

As a further example take the bridge truss shown in Fig. 3, which is represented as of disproportionate depth in order to fit the diagram to the size of the page. The method employed is a simplification of that given by Mr. Charles H. Tutton on page 385, vol. XVII of this Magazine.

Let us suppose the dead load of the bridge itself to consist of a series of equal weights w, applied at the upper joints x_1, x_2, etc., of the bridge. Let each of these weights when laid off to scale be represented by the length of $zy'''=w$, then the horizontal lines xx and $y'''o$ include between them ordinates which represent these weights.

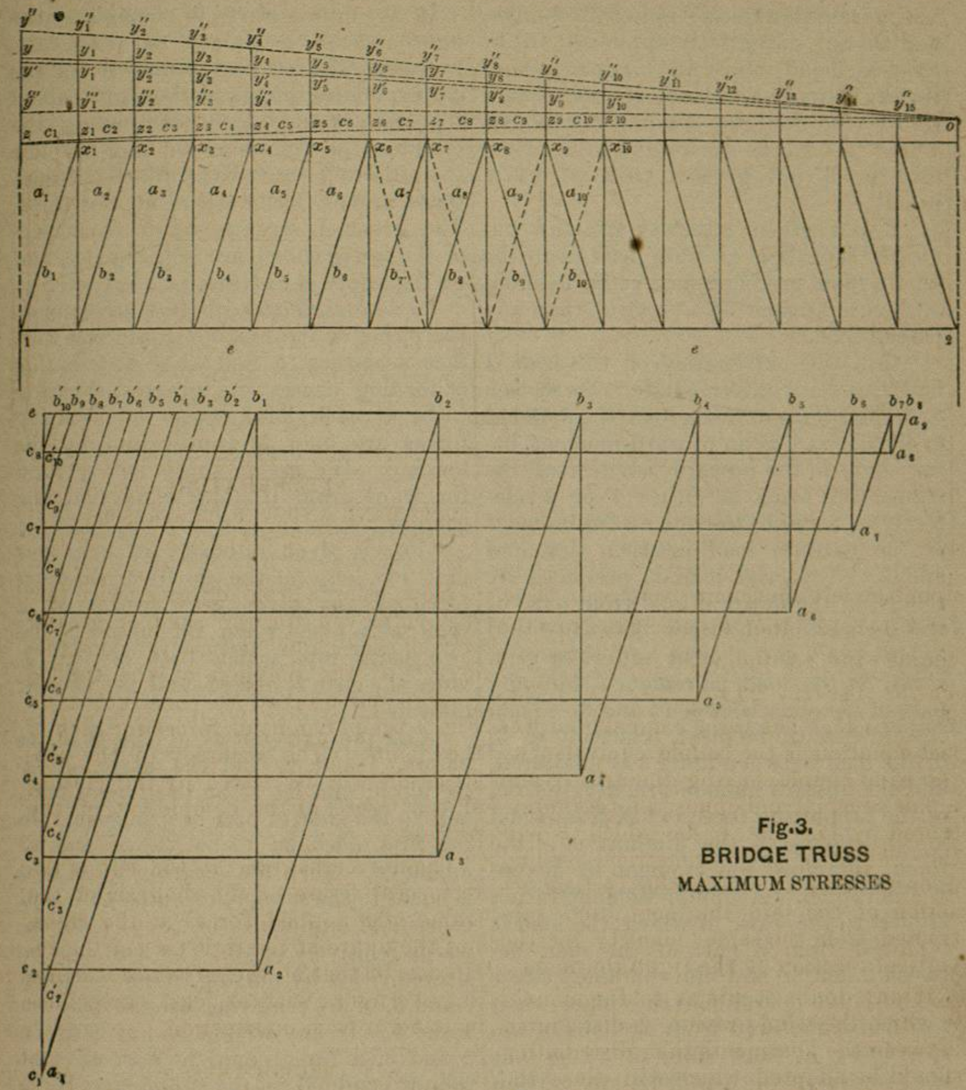


Fig.3.
BRIDGE TRUSS
MAXIMUM STRESSES

Let the live load consist of one or more locomotives which stand at the joints x_1 and x_2, and a uniform train of cars which covers the remaining joints. Let the load at each joint due to the cars be represented by $y'''y'=w'$, and the excess above this of the load at each of the joints covered by the locomotives be represented by $y'y''=w''$, $\therefore$ $w+w'+w''=c_1c_2=zy''=c_2c_3$ is the load at x_1 and at x_2, and $w+w'=c_3c_4=zy'$ is the load at x_3 and at each of the remaining joints.

Draw $y'o$, $y''o$ and zo, then is $z_1y_1''=\frac{15}{16}zy''$ that part of the load at x_1 which is sustained at the left abutment, as appears from the principle of the lever. Again $z_2y_2''=\frac{14}{16}zy''$ is that part of the load at x_2 sustained by the same abutment, and $z_3y_3'=\frac{13}{16}zy'$ is a similar part of load at x_3. Let the sum of these weights sustained by the left abutment be obtained; it is c_1e upon the lower figure. Upon c_1e lay off $c_1c_2=w+w'+w''$, $c_2c_3=w+w'+w''$, $c_3c_4=w+w'$, etc., equal to the loads applied at x_1, x_2, etc. We are now prepared to construct a diagram of forces which shall give the stresses in the various pieces under this assumed loading. Before constructing such a diagram, we wish to show that the assumed position of the load causes greater stresses in the chords of the bridge than any other possible position. The demonstration is quoted nearly verbatim from Rankine's Applied Mechanics, and though not strictly applicable to the case in hand, since it refers to a uniformly distributed load, it is substantially true for the loading supposed, when the excess of weight in the locomotives is not greater than occurs in practice.

"For a given intensity of load per unit of length, a uniform load over the whole span produces a greater moment of flexure at each cross section than any partial load."

"Call the extremities of the span 1 and 2, and any intermediate cross section 3. Then for a uniform load, the moment of flexure at 3 is an upward moment, being equal to the upward moment of the supporting force at either 1 or 2 relatively to 3, minus the downward moment of the uniform load between that end and 3. A partial load is produced by removing the uniform load from part of the span, situated either between 1 and 3, between 2 and 3, or at both sides of 3. First, let the load be removed from any part of the span between 1 and 3. Then the downward moment, relatively to 3, of the load between 2 and 3 is unaltered, and the upward moment, relatively to 3, of the supporting force at 2 is diminished in consequence of the diminution of the force; therefore the moment of flexure is diminished. A similar demonstration applies to the case in which the load is removed from a part of the span between 2 and 3; and the combined effect of those two operations takes place when the load is removed from portions of the span lying at both sides of 3; so that the removal of the load from any portion of the beam diminishes the moment of flexure at each point."

The stress upon a chord multiplied by the height of the truss is equal to the moment of flexure; hence in a truss of uniform height the stresses upon the chords are proportional to the moments of flexure, and when one has its greatest value the other has also.

The sides of the triangle c_1eb_1 represents the forces in equilibrium at the joint c_1eb_1 at the left abutment 1. The polygon $c_2c_1b_1a_2c_2$ represents the forces in equilibrium at the joint of the same name, *i.e.*, at the joint x_1. The forces at the other joints are found in a similar manner.

It is unnecessary to complete the figure above e unless to check the process. The stresses obtained for the corresponding pieces in the right half of the truss would, upon completing the diagram, be found to be slightly less than those already determined because there are no locomotives at the right. The greatest stresses upon the pieces of the lower chord are eb_8, eb_7, etc., and on the upper chord are a_8c_8, a_7c_7, etc.

To determine the greatest stress upon the pieces of the bracing (posts and ties) it is necessary to find what distribution of loading causes the greatest shearing force at each joint, since the shearing forces are held in equilibrium by the bracing. We again quote nearly word for word from Rankine's Applied Mechanics.

"For a given intensity of load per unit of length, the greatest shearing force at any given cross-section in a span takes place when the longer of the two parts into which that section divides the span is loaded, and the shorter unloaded."

"Call the extremities of the span, as before, 1 and 2, and the given cross-section 3; and let 13 be the longer part, and 23 the shorter part of the span. In the first place, let 13 be loaded and 23 unloaded. Then the shearing force at 3 is equal to the supporting force at 2, and consists of a tendency of 23 to slide upwards relatively to 13. The load may be altered either by putting weight between 2 and 3, or by removing weight between 1 and 3. If any weight be put between 2 and 3, a force equal to *part* of that weight is added to the supporting force at 2, and, therefore, to the shearing force at 3; but at the same time a force equal to the *whole* of that weight is taken away from that shearing force; therefore the shearing force at 3 is diminished by this alteration of the load. If weight be removed from the load between 1 and 2, the shearing force at 3 is diminished also, because of the diminution of the supporting force at 2. Therefore any alteration from that distribution of load in which the longer segment 13 is loaded, and the shorter segment 23 is unloaded, diminishes the shearing force at 3."

The shearing force at any point is the resultant vertical force at that point, and can be computed by subtracting

from the weight which rests upon either abutment the sum of all the weights between that point and the abutment, *i.e.*, by taking the algebraic sum of all the external forces acting upon the truss from either extremity to the point in question; the reaction of the abutment is, of course, one of these external forces.

The greatest stress upon the brace a_1b_1 is that already found, while x_1 is loaded with the live load.

If the live load be moved to the right so that no live load rests upon x_1, and the locomotives rest upon x_2 and x_3, the pieces b_1a_2 and a_2b_2 will sustain their greatest stress. To find the shear at x_2 in that case, we notice that the change in position of the live load has changed the reaction c_1e of the left abutment by the following amounts: the reaction has been diminished by the quantity $y_1'''y_1'' = \frac{15}{16}(w'+w'')$, since the load at x_1 has been removed, and it has been increased by $y_3'y_3'' = \frac{13}{16}w''$, since x_3 is loaded more heavily than before, therefore the reaction of the abutment has on the whole been decreased by the total amount $\frac{1}{16}(15w'+2w'')$.

Now the shear at x_2 is this reaction diminished by the load w at x_2. In order to construct it, draw yy_{14}'' parallel to $y'o$, then $yy' = \frac{2}{16}w''$. $\therefore$ Shear at $x_2 = ec_1 - w - \frac{1}{16}(15w'+2w'') = ec_1 - x_1y_1$. Lay off $c_1c_2' = x_1y_1$, then the shear at $x_2 = ec_2' =$ the greatest stress in the brace b_1a_2; and $b_2'c_2' =$ the greatest stress in a_2b_2.

Again, to find the greatest shear at x_3 when the live load has moved one panel further to the right, we have the equation: Shear at $x_3 = ec_2' - w - \frac{14}{16}(w'+w'') + \frac{12}{16}w'' = ec_2' - w - \frac{1}{16}(14w'+2w'') = ec_2' - x_2y_2$. Lay off $c_2'c_3' = x_2y_2$, then the shear at $x_3 = ec_3'$, which is the greatest stress in the piece b_2a_3, while $b_3'c_3'$ is the greatest stress in a_3b_3.

In similar manner lay off, $c_3'c_4' = x_3y_3$, $c_4'c_5' = x_4y_4$, etc., until the whole of the original reaction ec_1 of the abutment is exhausted, then are ec_1, ec_2', ec_3', ec_4', etc., the successive shearing stresses at the end of the load, *i.e.* the greatest shearing stresses, and consequently these stresses are the greatest stresses on the successive vertical members of the bracing, while c_1b_1, $c_2'b_2'$, $c_3'b_3'$, etc., are the greatest stresses on the successive inclined members of the bracing.

Had the greater load, such as the locomotives, extended over a larger number of panels, the line $y_1y_2y_3$ would have cut off a larger fraction of $y'y''$. Suppose, for instance, that the locomotives had covered the joints x_1x_8 inclusive, then the line y_1y_2 would have passed through y_8'', and been parallel to its present position. In that case the ordinates x_1y_1, x_2y_2 would have been successively subtracted from the reaction of the abutment due to a live load covering every joint, in order to obtain the shearing forces, just as at present, until we arrive at x_8, after which it would be necessary to subtract the ordinates x_8y_8'', x_9y_9'', etc. The counter braces are drawn with broken lines. Two counters are necessary on each side of the middle under the kind of loading which we have supposed. It is convenient, and avoids confusion in lettering the diagram to let a_9b_9, for instance, denote the principal or counter indifferently, as both are not subject to stress at the same time.

The devices here used can be applied to a variety of cases in which the loading is not distributed in so simple a manner as in this case.

IN GENERAL.

This method permits the determination of the stresses in any frame when we know the relative position of its pieces and the applied forces, provided the disposition of the pieces is such as to admit of a determination of the stresses.

The determination of what the applied forces are in case of a continuous girder or arch is a matter of some complexity, depending upon the elasticity of the materials employed, and the method in its present form affords little assistance in finding them.

Some authors have applied the method to find the stresses induced in the various pieces of a frame by a single force first applied at one joint, and then at another, and so on, and, finally, to find the stresses induced by the action of several simultaneous forces, by taking the algebraic sum of their separate effects. This is theoretically correct but laborious in practice in ordinary cases. Usually, some supposition respecting the applied forces can be made from which the results of all the other suppositions which must be made, can be derived with small labor. The bridge truss treated was a remarkable case in point.

WHEEL WITH TENSION-ROD SPOKES.

A very interesting example is found in the wheel represented in Fig. 4, in which the spokes are tension rods, and the rim is under compression. Let the greatest weight which the wheel ever sustains be applied at the hub of the wheel on the left, and let this weight be represented by the force *aa′* on the right, which is also equal to the reaction of the point of support upon which the wheel stands; hence *aa′* represents the force acting between two joints of this

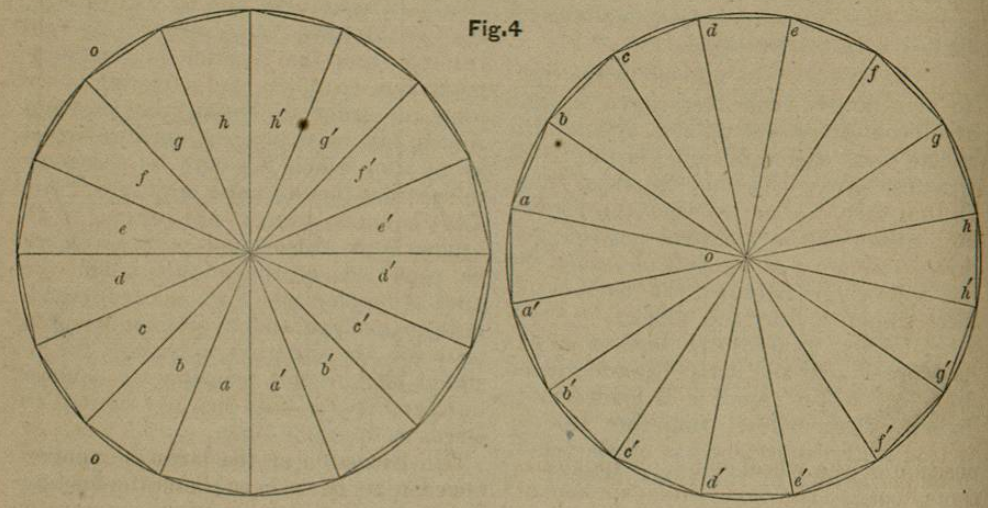


Fig.4

frame. The same effect would be caused upon the other members of the frame by "keying" the rod *aa′* sufficiently to cause this force to act between the hub and the lowest joint.

It should be noticed in passing, that the weights of the parts of the wheel itself are not here considered; their effect will be considered in Fig. 5. Also, the construction is based upon the supposition that there is a flexible joint at the extremity of each spoke. This is not an incorrect supposition when the flexibility of the rim is considerable compared with the extensibility of the spokes, a condition which is fulfilled in practice.

A similar statement holds in the case of the roof truss with continuous rafters, or a bridge truss with a continuous upper chord. The flexibility of the rafters or the upper chord is sufficiently great in comparison with the extensibility of the bracing, to render the stresses practically the same as if pin joints existed at the extremities of the braces.

Furthermore, the extremities of the spokes are supposed to be joined by straight pieces, since the forces between the joints of the rim act in those directions. Such forces will cause small bending moments in the arcs of the rim joining the extremities of the spokes. Each arc of the rim is an arch subjected to a force along its chord or span, and it can be treated by the method applicable to arches. This discussion is unimportant in the present case and will be omitted.

Upon completing the force polygon in the manner previously described, it is found that the stress on every spoke is the same in amount, and is represented by a side of the regular polygon *abcd*, etc. upon the left, while the compression of the pieces of the rim are represented by the radii *oa ob*, etc.

As previously explained these diagrams are mutually reciprocal, and it happens in this case that they are also similar figures.

We then conclude that in designing such a wheel each spoke ought to be proportioned to sustain the total load, and that the maker should key the spokes until each spoke sustains a stress at least equal to that load. Then in no

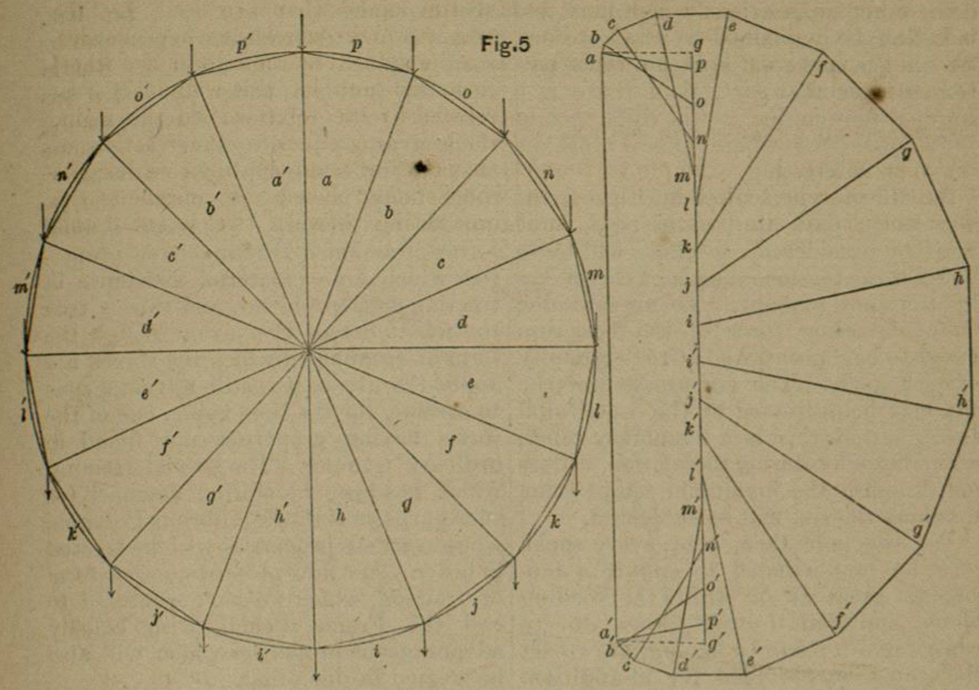


Fig.5

position of the wheel can any spoke become loose. The load here spoken of includes, of course, the effect of the most severe blow to which the wheel may be subjected while in motion.

WATER WHEEL WITH TENSION-ROD SPOKES.

The effect of a load distributed uniformly around the circumference of such a wheel as that just treated is represented in Fig. 5. Should it be desirable to compute the effect of both sets of forces upon the same wheel, it will be sufficient to take the sum of the separate effects upon each piece for the total effect upon that piece, though it is perfectly possible to construct both at once.

We shall suppose a uniform distribution of the loading along the circumference in the case of the Water Wheel, because in wheels of this kind such is practically the case so far as the spokes are concerned, since the power is transmitted, not through them to the axis, but, instead, to a cog wheel situated near the center of gravity of the "water arc." This arrangement so diminishes the necessary weight of the wheel, and the consequent friction of the gudgeons, as to render its adoption very desirable.

The discussion of the stresses appears however, to have been heretofore erroneously made.*

Let the weight *pp′*, at the highest joint of the wheel, be sustained by the rim alone, since the spoke *aa′* cannot assist in sustaining *pp′*, as *aa′* is suited to resist tension only. Conceive, for the moment, that two equal and opposite horizontal forces are introduced at the highest joint such as the two parts of the rim exert against each other, then $\frac{1}{2}pp'=pq=p'q'$ being sustained by each of the pieces *ap*, *a′p′* respectively we have *apq* and *a′p′q′* as the triangles which together represent the forces at the highest joint. The force *aa′* on the right is the upward force at the axis, equal and opposed to the resultant of the total load upon the wheel, and the apparent peculiarity of the diagram is due to this;—the direction of the reaction or sustaining force of the axis passes through the highest joint of the wheel and yet it is not a force acting between those joints and could not be replaced by keying the tie connecting those joints. In other particulars the force diagram is

* "A Manual of the Steam Engine, etc.," by W. J. M Rankine. Page 182, 7th Ed.