all the other suppositions which must be made, can be derived with small labor. The bridge truss treated was a remarkable case in point.

### WHEEL WITH TENSION-ROD SPOKES.

A very interesting example is found in the wheel represented in Fig. 4, in which the spokes are tension rods, and the rim is under compression. Let the greatest weight which the wheel ever sustains be applied at the hub of the wheel on the left, and let this weight be represented by the force *aa'* on the right, which is also equal to the reaction of the point of support upon which the wheel stands; hence *aa'* represents the force acting between two joints of this

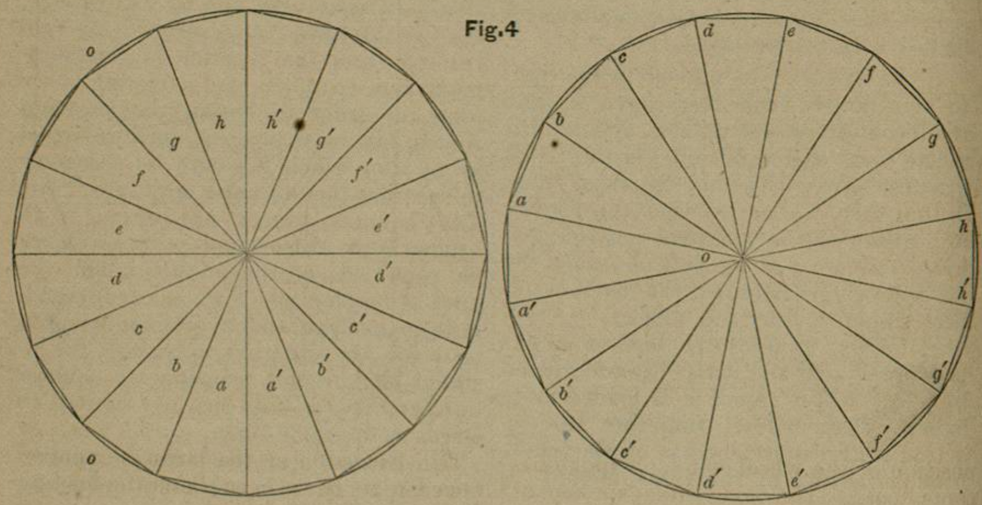


Fig.4

frame. The same effect would be caused upon the other members of the frame by "keying" the rod *aa'* sufficiently to cause this force to act between the hub and the lowest joint.

It should be noticed in passing, that the weights of the parts of the wheel itself are not here considered; their effect will be considered in Fig. 5. Also, the construction is based upon the supposition that there is a flexible joint at the extremity of each spoke. This is not an incorrect supposition when the flexibility of the rim is considerable compared with the extensibility of the spokes, a condition which is fulfilled in practice.

A similar statement holds in the case of the roof truss with continuous rafters, or a bridge truss with a continuous upper chord. The flexibility of the rafters or the upper chord is sufficiently great in comparison with the extensibility of the bracing, to render the stresses practically the same as if pin joints existed at the extremities of the braces.

Furthermore, the extremities of the spokes are supposed to be joined by straight pieces, since the forces between the joints of the rim act in those directions. Such forces will cause small bending moments in the arcs of the rim joining the extremities of the spokes. Each arc of the rim is an arch subjected to a force along its chord or span, and it can be treated by the method applicable to arches. This discussion is unimportant in the present case and will be omitted.

Upon completing the force polygon in the manner previously described, it is found that the stress on every spoke is the same in amount, and is represented by a side of the regular polygon *abcd*, etc. upon the left, while the compression of the pieces of the rim are represented by the radii *oa ob*, etc.

As previously explained these diagrams are mutually reciprocal, and it happens in this case that they are also similar figures.

We then conclude that in designing such a wheel each spoke ought to be proportioned to sustain the total load, and that the maker should key the spokes until each spoke sustains a stress at least equal to that load. Then in no



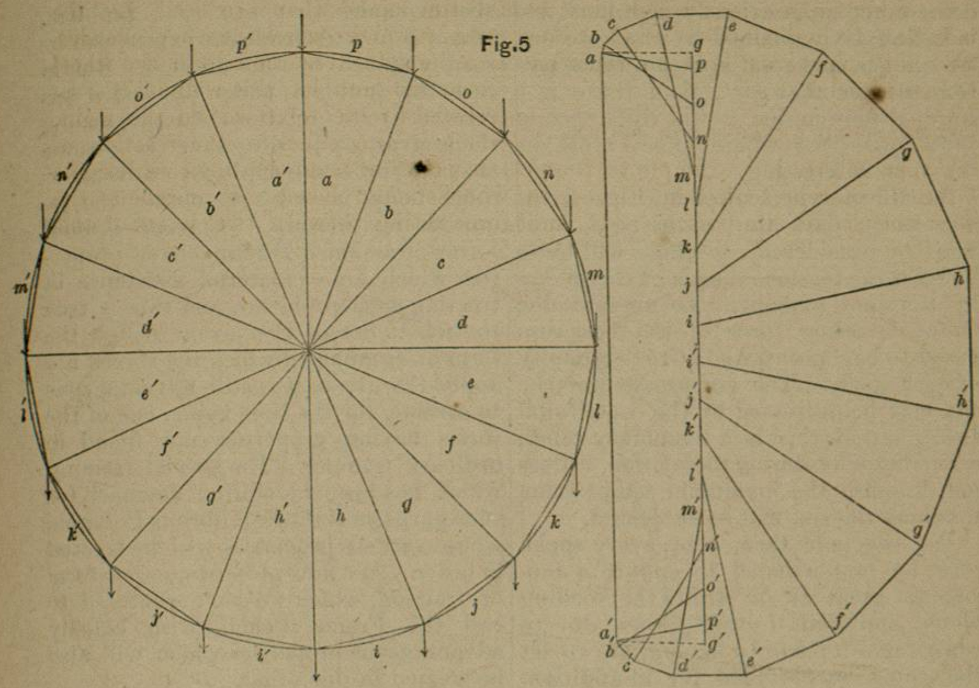


Fig.5

position of the wheel can any spoke become loose. The load here spoken of includes, of course, the effect of the most severe blow to which the wheel may be subjected while in motion.

### WATER WHEEL WITH TENSION-ROD SPOKES.

The effect of a load distributed uniformly around the circumference of such a wheel as that just treated is represented in Fig. 5. Should it be desirable to compute the effect of both sets of forces upon the same wheel, it will be sufficient to take the sum of the separate effects upon each piece for the total effect upon that piece, though it is perfectly possible to construct both at once.

We shall suppose a uniform distribution of the loading along the circumference in the case of the Water Wheel, because in wheels of this kind such is practically the case so far as the spokes are concerned, since the power is transmitted, not through them to the axis, but, instead, to a cog wheel situated near the center of gravity of the "water arc." This arrangement so diminishes the necessary weight of the wheel, and the consequent friction of the gudgeons, as to render its adoption very desirable.

The discussion of the stresses appears however, to have been heretofore erroneously made.*

Let the weight *pp'*, at the highest joint of the wheel, be sustained by the rim alone, since the spoke *aa'* cannot assist in sustaining *pp'*, as *aa'* is suited to resist tension only. Conceive, for the moment, that two equal and opposite horizontal forces are introduced at the highest joint such as the two parts of the rim exert against each other, then $\frac{1}{2}pp'=pq=p'q'$ being sustained by each of the pieces *ap*, *a'p'* respectively we have *apq* and *a'p'q'* as the triangles which together represent the forces at the highest joint. The force *aa'* on the right is the upward force at the axis, equal and opposed to the resultant of the total load upon the wheel, and the apparent peculiarity of the diagram is due to this;—the direction of the reaction or sustaining force of the axis passes through the highest joint of the wheel and yet it is not a force acting between those joints and could not be replaced by keying the tie connecting those joints. In other particulars the force diagram is

* "A Manual of the Steam Engine, etc.," by W. J. M Rankine. Page 182, 7th Ed.

constructed as previously described and is sufficiently explained by the lettering. Should the spoke $aa'$ have an initial tension greater than $pp'$, then there is a residual tension due to the difference of those quantities whose effect must be found as in Fig. 4.

Should the wheel revolve with so great a velocity that the centrifugal force must be considered, its effect will be to increase the tension on each of the spokes by the same amount,—the amount due to the deviating force of the mass supposed to be concentrated at the extremity of each spoke. The compression of the rim may be decreased by the centrifugal force, but as this is a temporary relief, occurring only during the motion, it does not diminish the maximum compression to which the rim will be subjected.

We conclude then, that every spoke must be proportioned to endure a tension as great as $hh'$ from the loading alone; and that if other forces, due to centrifugal force or to keying, are to act they must be provided for in addition. Furthermore, we see that the rim must be proportioned to bear a compression as great as $hi$, due to the loading alone, and that the centrifugal force will not increase this, but any keying of the spokes beyond that sufficient to produce an initial tension on each spoke as great as $pp'$ must be provided for in addition.

The diagram could have been constructed with the same facility in case the applied weights had been supposed unequal.

It can be readily shown that the differential equation of the curve circumscribing the polygon $abcd$, etc. of Fig. 5 is

$$y+x\frac{dx}{dy}+c\ \tan.^{-1}\left(\frac{dx}{dy}\right)=0$$

which equation is not readily integrable. When, however, the number of spokes is indefinitely increased, it appears from simple geometrical considerations that this curve becomes a cycloid having its cusps at $q$ and $q'$.

### ASSUMED FRAMING.

Thus far, we have treated the effect of known external forces upon a given form of framing, and it is evident from the previous discussions and the illustrative examples that any such problem, which is of a determinate nature, can be readily solved by this method. But in case the problem under discussion has reference to the relations of forces among themselves, it is necessary to assume that the forces are applied to a frame or other body, in order to obtain the required relationship. Certain general forms of assumed framing have properties which are of material assistance in treating such problems, and this is true to such an extent that even though the form of framing to which the forces are applied is given, it is still advantageous to assume, for the time being, one of the forms having properties not found in ordinary framing. The special framing which has been heretofore assumed for such purposes is the Equilibrium Polygon, whose various properties will be treated in order. We now propose another form of framing, which we have ventured to call the Frame Pencil, with equally advantageous properties which will also be treated in due order.

It may be mentioned here, that the particular case of parallel forces is that most frequently met with in practice. In case of parallel forces the properties of the equilibrium polygon and frame pencil are more numerous and important than those belonging to the general case alone. We shall first treat the general case, and afterwards derive the additional properties belonging to parallel forces.

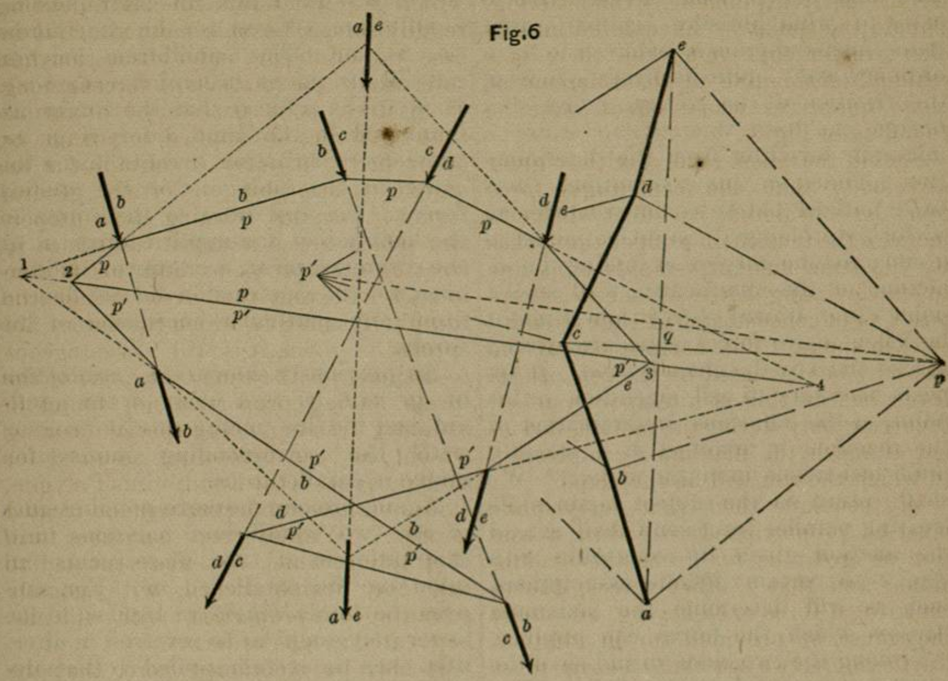
### THE EQUILIBRIUM POLYGON FOR ANY FORCES IN ONE PLANE.

Let $ab$, $bc$, $cd$, $de$ Fig. 6 be the diagram of any forces lying in the plane of the paper, and $abcde$ their force polygon, then, as previously shown, $ae$ the closing side of the polygon of the applied forces represents the resultant of the given forces in amount and direction. Assume any point $p$ as a pole, and draw the force pencil $p-abcde$. The object in view in so doing, is to use this force pencil and polygon of the applied forces together in order to determine a figure of which it is the reciprocal.

From any convenient point as 2 draw the side $ap$ parallel to the ray $ap$ until it intersects the line of action of the force $ab$, and from that intersection draw the side $bp$ parallel to the ray $bp$, etc., etc.; then the polygon $p$ will have its sides



EQUILIBRIUM POLYGON.




RECIPROCAL FIGURES.

| | | | | |
|---|---|---|---|---|
| *Direction and Position.* | *Force Diagram,* | *abcde,* | *Force Polygon.* | *Direction and Magnitude.* |
| | *Equilibrium Polygon,* | *ap, bp, cp, dp, ep,* | *Force Pencil.* | |
| | *Equilibrium Polygon,* | *ap', bp', cp', dp', ep',* | *Force Pencil.* | |
| | *Closing Line,* | 23 ‖ *pq,* | *Closing Ray.* | |
| | *Resultant Force,* | *ae,* | *Resultant Force.* | |

parallel respectively to the rays of the pencil $p$.

The polygon $p$ and the given forces $ab$, $bc$, etc, then form a force and frame diagram to which the pencil $p-abcde$ is reciprocal, and of which it is the force diagram. It is seen that no internal bracing is needed in the polygon $p$, and hence it is called an equilibrium (frame) polygon: it is the form which a funicular polygon, catenary, or equilibrated arch, would assume if occupying this position and acted upon by the given forces.

As represented in Fig. 6 the sides of the polygon $p$ are all in compression so that $p$ represents an ideal arch. If the line 23 be drawn cutting the sides $ap$, $ep$ so that it be considered to be the span of the arch having the points of support 2 and 3, then this arch exerts a thrust in the direction 23 which may be borne either by a tie 23 or by fixed abutments 2 and 3: the force in either case is the same and is represented by $pq \parallel 23$. It is usual to call 23 a closing line of the polygon $p$. The point $q$ divides the resultant $ae$ into two parts such that $qapq$ and $epqe$ are triangles whose sides represent forces in equilibrium, *i.e.*, the forces at the points 2 and 3; hence, $qa$ and $eq$ are the parts of the total resultant which would be applied at 2 and 3 respectively.

This method is frequently employed to find the forces acting at the abutments of a bridge or roof truss such as that in Fig. 2. But it appears that it has often been erroneously employed. It must be first ascertained whether the reaction at the abutments is really in the direction $ae$ for the forces considered. It may often happen far otherwise. If the surfaces upon which the truss rests without friction are perpendicular to $ae$, then this assumption is probably correct; as, for instance, when one end is mounted on rollers devoid of friction, running on a plate perpendicular to *ae*. But in cases of wind pressure against a roof truss the assumption is believed to be in ordinary cases quite incorrect. Indeed, the friction of the rollers at end of a bridge has been thought to cause a material deviation from the determination founded on this assumption. It is to be noticed that any point whatever on *pq* (or *pq* prolonged) might be joined to *a* and *e* for the purpose of finding the reactions of the abutments. Call such a point *x* (not drawn), then *ax* and *ex* might be taken as two forces which are exerted at two and 3 by the given system. It appears necessary to call attention to this point, as the fallacious determination of the reactions is involved in a recently published article upon this subject.* We shall return to the subject again while treating parallel forces and shall extend the method given in connection with Fig. 2 to certain definite assumptions, such as will determine the maximum stresses which the forces can produce.

Prolong the two sides *ap* and *ep* of the polygon *p* until they meet. It is evident that if a force equal to the resultant *ae* be applied at this intersection of *ap* and *ep* prolonged, then the triangles *apq* and *epq* will represent the stresses produced at 2 and 3 by the resultant. But as these are the stresses actually produced by the forces, and as the resultant should cause the same effects at 2 and 3 as the forces, it follows that the intersection of *ap* and *ep* must be a point of the resultant *ae*; and if, through this intersection, a line be drawn parallel to the resultant *ae*, it will be a diagram of the resultant, showing it in its true position and direction.

This is in reality a geometric relationship and can be proved from geometric considerations alone. It is sufficient for our purposes, however, to have established its truth from the above mentioned static considerations which may be regarded as mechanical proof of the geometric proposition.

The pole *p* was taken at random: let any other point *p′* be taken as a pole. To avoid multiplying lines *p′* has been taken upon *pq*. Now draw the force pencil *p′—abcde* and the corresponding equilibrium polygon for the same forces *ab*, *bc*, etc. This equilibrium polygon has all its pieces in tension except *p′c*. It is to be noticed that the forces are employed in the same order as in the previous construction, because that is the order in the polygon of the applied forces: but the order of the forces in the polygon of the applied forces is, at the commencement, a matter of indifference, for the construction did not depend upon any particular succession of the forces.

As previously shown, the intersection of *ap′* with *ep′* is a point of the resultant, and the line joining this intersection with the corresponding intersection above is parallel to *ae*.

Again, prolong the corresponding sides of the two equilibrium polygons until they intersect at 1234, these points fall upon one line parallel to *pp′*. For, suppose the forces which are applied to the lower polygon *p′* to be reversed in direction, then the system applied to the polygons *p* and *p′* must together be in equilibrium; and the only bracing needed is a piece 23 || *pp′*, since the upper forces produce a tension *pq* along it, and the lower forces a tension *qp′*, while the parts *aq* and *qe* of the resultant which are applied at 2 and 3 are in equilibrium. The same result can be shown to hold for each of the forces separately; *e.g.* the opposite forces *ab* may be considered as if applied at opposite joints of a quadrilateral whose remaining joints are 1 and 2: the force polygon corresponding to this quadrilateral is *apbp′*, hence 12 || *pp′*. Hence 1234 is a straight line. The intersection of *pc* and *p′c* does not fall within the limits of the figure.

It is to be noticed that the proposition just proved respecting the collinearity of the intersections of the corresponding sides of these equilibrium polygons is one of a geometric nature and is susceptible of a purely geometric proof.

### THE FRAME PENCIL FOR ANY FORCES IN ONE PLANE.

Let *ab*, *bc*, *cd*, *de* in Fig. 7 represent a system of forces, of which *abcde* is the force polygon. Choose any single point upon the line of action of each of these

* See paper No. 71 of the Civil Engineers' Club of the Northwest. Applications of the Equilibrium Polygon to determine the Reactions at the Supports of Roof Trusses. By James R. Willett, Architect. Chicago.



FRAME PENCILS.

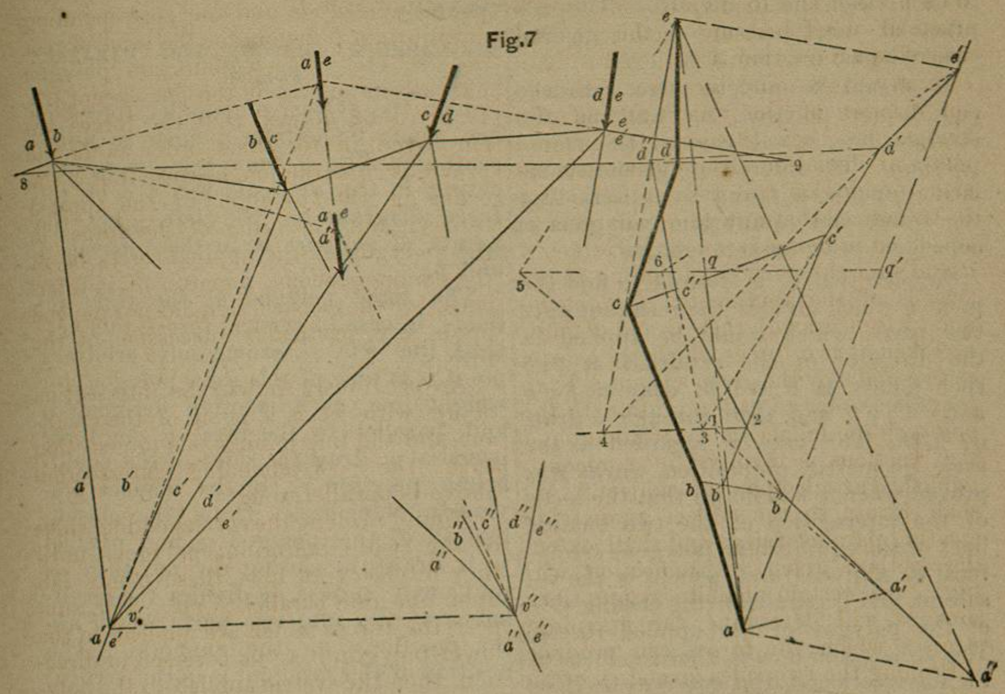


Fig. 7

RECIPROCAL FIGURES.

| | | | | |
|---|---|---|---|---|
| *Direction and Position.* | *Force Diagram,* | *a b c d e,* | *Force Polygon.* | *Direction and Magnitude.* |
| | *Frame Pencil,* | *a′ b′ c′ d′ e′,* | *Equilibrating Polygon.* | |
| | *Frame Pencil,* | *a″ b″ c″ d″ e″,* | *Equilibrating Polygon.* | |
| | *Frame Polygon,* | *bb′, cc′, dd′, ee′,* | *Force Lines.* | |
| | *Resultant Force,* | *a e,* | *Resultant Force.* | |
| | *Resultant Ray,* | *a′ e′.* | *Resultant Side.* | |

forces, and join these points to any assumed vertex *v′* by the rays of the frame pencil *a′b′c′d′e′*. Also join the successive points chosen by the lines *bb′*, *cc′*, *dd′* which form sides of what we shall call the frame polygon. Now consider the given forces to be borne by the frame pencil and frame polygon as a system of bracing, which system exerts a force at the vertex *v′* in some direction not yet known, and also exerts a force along some assumed piece *ee′*, which may be regarded as forming a part of the frame polygon. The stresses upon the rays of the frame pencil will be represented by the sides of *ab′c′d′e′* which we shall call the equilibrating (force) polygon; while the stresses in the frame polygon are given by the force lines *bb′*, *cc′*, etc. If a resultant ray *a′e′* be drawn from *v′* parallel to the resultant side *ae′* of the equilibrating polygon it will intersect *ee′* at a point of the resultant of the system of forces; for that is a point at which if the resultant be applied it will cause the same stresses along the pieces *a′e′* and *ee′* which support it as do the forces themselves.

If the point *e′* in the force polygon be moved along *e′d′*, the locus of the intersection of the corresponding positions of the resultant ray *a′e′* and the last side *ee′* will be the resultant *ae*. It would have been unnecessary to commence the equilibrating polygon at *a* had the direction of *aa′* been known. Having obtained the direction of *aa′* as shown at 8, the equilibrating polygon could be drawn by commencing at any point of *aa*, || *aa′*.

In cases like that in the Fig., where there is no reason for choosing the points which determine the sides of the frame polygon otherwise, it is simpler to make the frame polygon a straight line, which may in that case be called the frame line. Then the force lines are parallel to each other and to $aa'$ also. This is a practical simplification of the general case of much convenience.

It should be noticed here that the equilibrium polygon, as well as the straight line, is one case of the frame polygon. The interesting geometric relationships to be found by constructing the frame and equilibrium polygons as coincident must be here omitted.

Suppose that it is desired to find the point $q$ which divides the resultant into two parts, which would be applied in the direction of the resultant at two such points as 8 and 9: draw $a6 \parallel v'8$ and $\epsilon'6 \parallel v'9$ and then through 6 draw $qq' \parallel 89$. This may be regarded as the same geometric proposition, which was proved when it was shown that the locus of the intersection of the two outside lines of the equilibrium polygons (reciprocal to a given force pencil) is the resultant, and is parallel to the closing side of the polygon of the applied forces. The proposition now is, that the locus of the intersection of the two outside lines of the equilibrating polygon (reciprocal to a given frame pencil) is the resolving line, and is parallel to the abutment line: for these two statements are geometrically equivalent.

Assume a different vertex $v''$, and draw the frame pencil and its corresponding equilibrating polygon $a''b''c''d''e$. If $a_1$ 5 and $e$ 5 be drawn parallel to $v''$ 8 and $v''$ 9 respectively their intersection is upon $qq'$ as before proven.

Again, the corresponding sides of these two equilibrating polygons intersect at 1 2 3 4 upon a line parallel to $v'v''$, for this is the same geometric proposition respecting two vertices and their equilibrating polygons which was previously proved respecting two poles and their equilibrium polygons.

It would be interesting to trace the geometric relations involved in different but related frame polygons, as for example, those whose corresponding sides intersect upon the same straight line, but as our present object is to set forth the essentials of the method, a consideration of these matters is omitted. Enough has been proven, however, to show that we have in the frame pencil an independent method equally general and fruitful with that of the equilibrium polygon.

EQUILIBRIUM POLYGON FOR PARALLEL FORCES.

LET the system of parallel forces in one plane be four in number as represented in Fig. 8, viz: $w_1w_2$, $w_2w_3$, etc., acting in the verticals 2 3 4 5 of the force diagram on the left. Let the points of support be in the verticals 1 and 6.

The force polygon at the right reduces, in case of vertical forces, to a vertical line $ww$. Assume any arbitrary point $p$ as pole of this force polygon, (or weight line, as it is often designated) and, parallel to the rays of the force pencil at $p$, draw the sides of the equilibrium polygon $ee$, in the manner previously described. Draw the closing line $kk$ of this polygon $ee$, and parallel to it draw the closing ray $pq$; then, as previously shown, $pq$ divides the resultant $w_1w_5$ at $q$ into two parts which are the reactions of the supports. The position of the resultant is in the vertical $mm$ which passes through the intersection of the first and last sides of the polygon $ee$, as was also previously shown.

Designate the horizontal distance from $p$ to the weight line by the letter H. It happens in Fig. 8 that $pw_1$=H, but in any case the pole distance H is the horizontal component of the force $pq$ acting along the closing line.

Now by similarity of triangles

$$k_1e_2(=h_1h_2) : k_2e_2 :: pw_1 : qw_1$$
$$\therefore \text{H}.k_2e_2 = qw_1.h_1h_2 = \text{M}_2,$$

the moment of flexure, or bending moment at the vertical 2, which would be caused in a simple straight beam or girder under the action of the four given forces and resting upon supports in the verticals 1 and 6.

Again, from similarity of triangles,

$$h_1h_3 \ (=k_1f_3) : k_3f_3 :: H : qw_1$$
$$h_2h_3 \ (=e_2f_3) : e_3f_3 :: H : w_1w_2$$
$$\therefore H(k_3f_3 - e_3f_3) = H.k_3e_3$$
$$= qw_1.h_1h_3 - w_1w_2.h_2h_3 = M_3$$

the moment of flexure of the simple girder at the vertical 3.

Similarly it can be shown in general that

$$H.ke = M,$$



EQUILIBRIUM POLYGON.

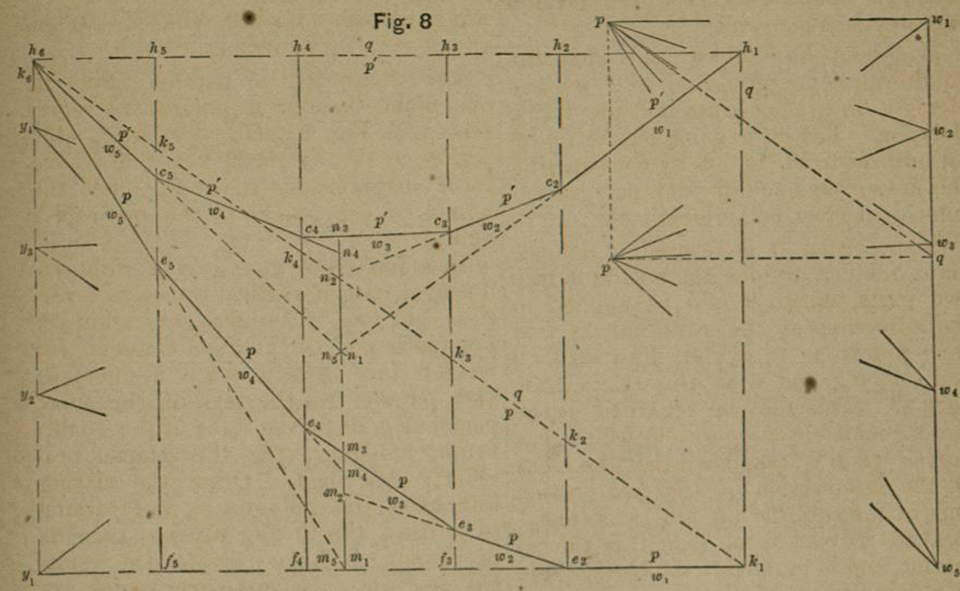


Fig. 8

*i.e.* that the moment of flexure at any vertical whatever (be it one of the verticals 2 3 4, etc., or not) is equal to the product of the assumed pole distance $H$ multiplied by the vertical ordinate $ke$ included between the equilibrium polygon $ee$ and the closing line $kk$ at that vertical.

From this it is evident that the equilibrium polygon is a moment curve, *i.e.* its vertical ordinate at any point of the span is proportional to the bending moment at that point of a girder sustaining the given weights and supported by simply resting without constraint upon piers at its extremities.

From this demonstration it appears that $H.e_3f_3 = w_1w_2.h_2h_3$ is the moment of the force $w_1w_2$ with respect to the vertical 3; and similarly $H.m_1m_2 = w_1w_2.e_2m_1$ is the moment of the same force with respect to the vertical through the center of gravity. Also, $H.y_1y_2 = w_1w_2.h_2h_6$ is the moment of the same force with respect to the vertical 6.

Similarly $m_1m_3$ is proportional to the moment of all forces at the right, and $m_3m_5$ to all the forces left of the center of gravity, but $m_1m_3 + m_3m_6 = 0$, as should be the case at the center of gravity, about which the moment vanishes. From these considerations it appears that the segments $mm$ of the resultant are proportional to the bending moments of a girder supporting the given weights and resting without constraint upon a single support at their center of gravity.

Let us move the pole to a new position $p'$ having the same pole distance $H$ as $p$, and in such a position that the new closing line will be horizontal, *i.e.* $p'q$ must be horizontal.

One object in doing this is to furnish a sufficient test of the correctness of the drawing in a manner which will be immediately explained; and another is to transfer the moment curve to a new position $cc$ such that its ordinates may be measured from an assumed horizontal position $hh$ of the girder to which the forces are applied, so that the girder itself forms the closing line.

The polygon $cc$ must have its ordinates $hc$ equal to the corresponding ordinates $ke$, for

$$M = H.ke = H.hc$$

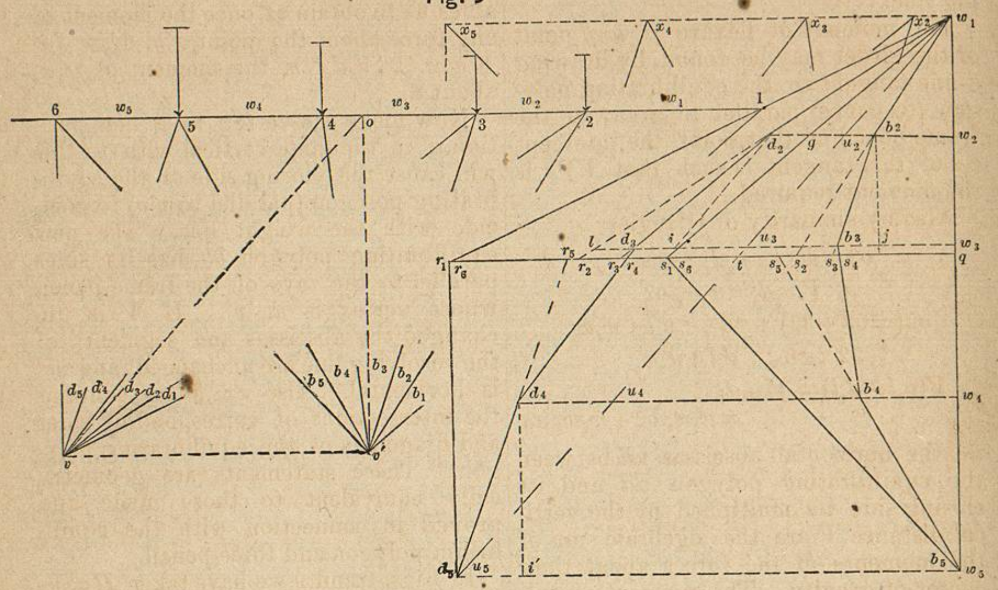
Also the segments of the line $mm$ are equal to the corresponding segments of the line $nn$ for similar reasons.

Again, as has been previously shown, the corresponding sides (and diagonals as well) of the polygons $ee$ and $cc$ intersect upon the line $yy \parallel pp'$.

These equalities and intersections furnish a complete test of the correctness of the entire construction.

FRAME PENCIL.

Fig. 9



FRAME PENCIL FOR PARALLEL FORCES.

Let the same four parallel forces in one plane which were treated in Fig. 8 be also treated in Fig. 9, and let them be applied at 2, 3, 4, 5 to a horizontal girder resting upon supports at 1 and 6.

Use 16 as the frame line and choose any vertex $v$ at pleasure from which to draw the frame pencil $dd$. Draw the force lines $wd$ parallel to the horizontal frame line 16, and then draw the equilibrating polygon $dd$ with its sides parallel to the rays of the frame pencil $dd$.

As has been previously shown, if a resultant ray $vo$ of the frame pencil $dd$ be drawn from $v$, as represented in Fig. 9, parallel to the closing side $uu$ of the equilibrating polygon, this ray intersects 16 at the point $o$ where the resultant of the four given forces cuts 16.

Furthermore, the lines $w_1r_1$ and $d_5r_6$ parallel to the abutment rays $v1$ and $v6$ of the frame pencil intersect on $rr$ the resolving line, which determines the point of division $q$ of the reactions of the two supports, as was before shown.

Let the vertical distance between the vertex and the frame line be denoted by $V$.

In Fig. 9 it happens that $v6=V$. If the frame polygon is not straight, or being straight is inclined to the horizon, $V$ has different values at the different joints of the frame polygon: in every case $V$ is the vertical distance of the joint under consideration above or below the vertex. It will be found in the sequel that this possible variation of $V$ may in certain constructions be of considerable use.

By similarity of triangles we have

$$12 : v6 : : r_1r_2 : w_1q$$
$$\therefore V.r_1r_2=w_1q.12=M_2,$$

the bending moment of the girder at the point 2.

Draw a line through $w_1$ parallel to $v3$; this line by chance coincides so nearly with $w_1s_1$ that we will consider that it is the line required, though it was drawn for another purpose. Again, by similarity of triangles

$$13 : v6 : : r_1s_1 : w_1q$$
$$23 : v6 : : d_2g(=r_3s_1) : w_1w_2$$
$$\therefore V(r_1s_1-r_3s_1)=V.r_1r_3$$
$$=w_1q.13-w_1w_2.23=M_3$$

the bending moment at 3.

Similarly it may be shown that

$$V.\ r_1r_n=M_n,$$

*i.e.* that the moment of flexure at any point of application of a force to the girder is the product of the assumed vertical distance $V$ multiplied by the corresponding segment $rr$ of the resolving line.

The moment of flexure at *any* point of the girder may be found by drawing a line tangent to the equilibrating polygon (or curve) parallel to a ray of the frame pencil at that point, the intercept $r_1r$ of this tangent is such that $V.r_1r$ is the moment required.

Also by similarity of triangles

$$o2 : v6 : : u_2d_2 : w_1w_2$$
$$\therefore V.u_2d_2=w_1w_2.o2$$
$$o2(=o3+32) : v6 : : u_3l : w_1w_3$$
$$32 : v6 : : d_3l : w_2w_3$$
$$\because V(u_3l-d_3l)=V.u_3d_3$$
$$=w_1w_2.o2+w_2w_3.o3,$$

*i.e.* the horizontal abscissas $ud$ between the equilibrating polygon $dd$ and its closing side $uu$ multiplied by the vertical distance $V$ are the algebraic sum of the moments of the forces about their center of gravity. The moment of any single force about the center of gravity being the difference between two successive algebraic sums may be found thus: draw $d_2i$ || $uu$, then is $V.d_2i$ the moment of $w_1w_2$ about the center of gravity, as may be also proved by similarity of triangles.

Again by proportions derived from similar triangles, precisely like those already employed, it appears that

$$V.w_2d_2=w_1w_2.26$$

is the moment of the force $w_1w_2$ about the point 6. And similarly it may be shown that

$$V.w_3d_3=w_1w_2.26+w_2w_3.36$$

is the moment of $w_1w_2$ and $w_2w_3$ about 6.

Furthermore, as this point 6 was not specially related to the points of application 1 2 3 4, we have thus proved the following property of the equilibrating polygon: if a pseudo resultant ray of the frame pencil be drawn to any point of the frame line, then the horizontal abscissas between the equilibrating polygon and a side of it parallel to that ray, (which may be called a pseudo closing side), are proportional to the sum total of the moments about that point of those forces which are found between that abscissa and the end of the weight line from which this pseudo side was drawn. The difference between two successive sum totals being the moment of a single force, a parallel to the pseudo side enables us to obtain at once the moment of any force about the point, *e.g.* draw $d_4i'$ || $ww$ $\therefore$ $V.d_5i'$ is the moment of $w_4w_5$ about 6.

Now move the vertex to a new position $v'$ in the same vertical with $o$ : this will cause the closing side of the equilibrating polygon (parallel to $v'o$) to coincide with the weight line. The new equilibrating polygon $bb$ has its sides parallel to the rays of the frame pencil whose vertex is at $v'$. If $V$ is unchanged the abscissas and segments of the resolving line are unchanged, and $vv'$ is horizontal. Also $xx$ || $vv'$ contains the intersections of corresponding sides and diagonals of the equilibrating polygon. These statements are geometrically equivalent to those made and proved in connection with the equilibrium polygon and force pencil.

In Figs. 8 and 9 we have taken $H=V$, hence the following equations will be found to hold,

$$k_2e_2=r_1r_2,\ k_3e_3=r_1r_3,\ k_4e_4=r_1r_4, \text{ etc.}$$
$$m_1m_2=u_2d_2,\ m_1m_3=u_3d_3,\ m_1m_4=u_4d_4, \text{ etc.}$$
$$y_1y_2=w_2d_2,\ y_1y_3=w_3d_3,\ y_1y_4=w_4d_4, \text{ etc.}$$
$$m_2m_3=d_3i, \text{ etc., } y_4k_6=d_5i', \text{ etc.}$$

By the use of *etc.* we refer to the more general case of many forces. From these equations the nature of the relationship existing between the force and frame pencils and their equilibrium and equilibrating polygons becomes clear. Let us state it in words.

The height of the vertex (a vertical distance), and the pole distance (a horizontal force) stand as the type of the reciprocity or correspondence to be found between the various parts of the figures.

The ordinates of the equilibrium polygon (vertical distanees) correspond to the segments of the resolving line (horizontal forces), each of these being proportional to the bending moments of a simple girder sustaining the given weights, and resting without constraint upon supports at its two extremities.

The segments of the resultant line (vertical distances) correspond to the abscissas of the equilibrating polygon (horizontal forces) each of these being proportional to the bending moments of a simple girder sustaining the given weights and resting without constraint upon a support at their center of gravity. The segments of any pseudo resultant line, parallel to the resultant, which are cut off by the sides of the equilibrium polygon, are proportional to the bending moments of a girder supporting the given weights and rigidly built in and supported at the point where the line intersects the girder; to these segments correspond the abscissas between the equilibrating polygon and a pseudo side of it parallel to the pseudo resultant ray.

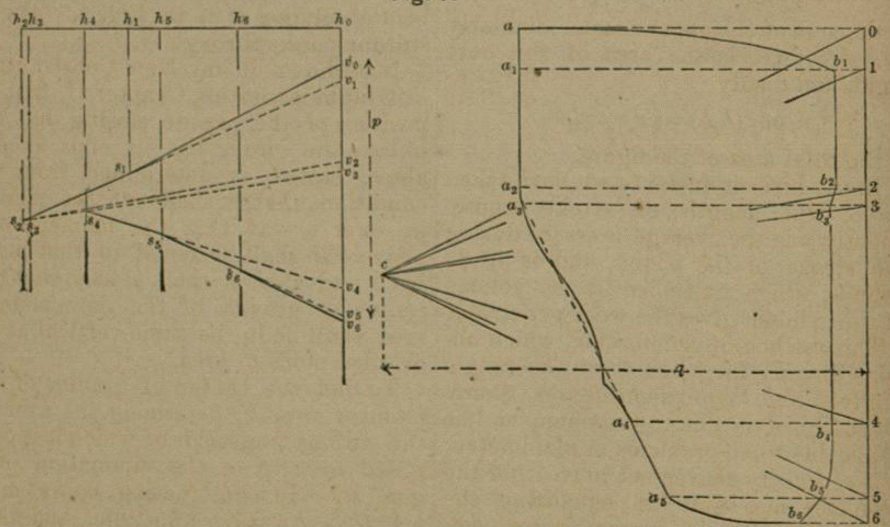
The two different kinds of support which we have supposed, viz. support without constraint and support with constraint, can be treated in a somewhat more general manner, as appears when we consider that at any point of support there may be, besides the reaction of the support, a bending moment, such as would be induced, for instance, when the span in question forms part of a continuous girder, or when it is fixed at the support in a particular direction. In such a case the closing line of the equilibrium polygon is said to be moved to a new position. It seems better to call it in its new position a pseudo closing line. The ordinates between the pseudo closing line and the equilibrium polygon are proportional to the bending moments of the girder, so supported. It is possible to induce such a moment at one point of support as to entirely remove the weight from the other, and cause it to exert no reaction whatever; and any intermediate case may occur in which the total weight in the span is divided between the supports in any manner whatever. When the weight is entirely supported at $h_6$ then $y_1e_2$ is the pseudo closing line of the polygon *ee*. In that case *xx* becomes the pseudo resolving line, and in general the ordinates between the pseudo closing line and the equilibrium polygon correspond to the segments of the pseudo resolving line, and are proportional to the bending moments of the girder. This general case is not represented in Figs. 8 and 9; but the particular case shown, in which the total weight is borne by the left pier, gives the equations

$$e_3f_3=w_1x_2,\ e_4f_4=w_1x_3,\ e_5f_5=w_1x_4, \text{ etc.}$$

In order to represent the general case in which the weights, supported by the piers, are not the same as in the case of the simple girder, by reason of some kind of constraint, we propose to treat the case of the straight girder, fixed horizontally at its extremities; but it is necessary first to discuss the following auxiliary construction.

SUMMATION POLYGON.

Fig. 10



### THE SUMMATION POLYGON.

In Fig. 10 let *aabb* be any closed figure of which we wish to determine the area. The example which we have



chosen is that of an indicator card taken from page 12 of Porter's Treatise on Richard's Steam Indicator, it being a card taken from the cylinder of an old-fashioned paddle-wheel Cunarder, the Africa. The scale is such that $a_1b_1$ is 26.9 pounds per square inch and 06 parallel to the atmospheric line is the length of the stroke.

Divide the figure by parallel lines $a_1b_1$, $a_2b_2$, etc. into a series of bands which are approximately trapezoidal. A sufficient number of divisions will cause this approximation to be as close as may be desired. The upper and lower bands may in the present case be taken as approximating sufficiently to parabolic areas. Let 06 be perpendicular to $a_1b_1$, etc., then will 01, 12, etc., be the heights of the partial areas. Lay off

$$h_0h_1=\tfrac{2}{3}\,a_1b_1,\quad h_0h_2=\tfrac{1}{2}(a_1b_1+a_2b_2),$$
$$h_0h_3=\tfrac{1}{2}(a_2b_2+a_3b_3), \text{ etc.}$$

then will these distances be the bases of the partial areas. Assume any point $c$ at a distance $l$ from 06 as the common point of the rays of a pencil passing through 0, 1, 2, etc.; and draw the parallels *hs*: then from any point $v_0$ of the first of these make $v_0s_1 \parallel c0$, and $s_1s_2 \parallel c1$, $s_2s_3 \parallel c2$, etc.

The polygon *ss* is called the summation polygon, and has the following properties.

By similarity of triangles

$$l : 01 :: h_0h_1 : v_0v_1, \quad \therefore 01.h_0h_1=l.v_0v_1$$

is the area of the upper band. Similarly $12.h_0h_2=l.v_1v_2$ is the area of the next band, and finally

$$06\Sigma(h_0h)=l.v_0v_6=lp$$

is the total area of the figure.

In the present instance we have taken $l=06$, the length of stroke, consequently $p$ is the average pressure during the stroke of the piston, and is 21.25 pounds, which multiplied by the volume of the cylinder gives the work per stroke.

This method of summation, which obtains directly the height $p$ of a rectangle of given base $l$ equivalent in area to any given figure, is due to Culmann, and is applicable to all problems in planimetry; it is especially convenient in treating the problems met with in equalizing the areas of profiles of excavation and embankment, and is frequently of use in dividing land. It is much more expeditious in application than the method of triangles founded on Euclid, and is also, in general, superior to the method of equidistant ordinates, whether the partial areas are then computed as trapezoids or by Simpson's Rule; for it reduces the number of ordinates and permits them to be placed at such points as to make the bands approximate much more closely to true trapezoids than does the method of equidistant ordinates.

### GIRDER WITH FIXED ENDS.

It is to be understood that by a girder with fixed ends, we mean one from which if the loading were entirely removed, without removing the constraint at its ends, there would be no bending moment at any point of it, and, when the loading is applied to it the supports constrain the extremities to maintain their original direction unchanged, but furnish no horizontal resistance. Under those circumstances the girder may not be straight, and may not have its supports on the same level, but it will be more convenient to think of the girder as straight and level, as the moments, etc., are the same in both cases.

Suppose in Fig. 11 that any weights $w_1w_2$, etc. are applied at $h_2$, $h_3$, $h_4$, $h_5$, to a girder which is supported and fixed horizontally at $h_1$ and $h_6$. With $p$ as the pole of a force pencil draw the equilibrium polygon *ee* as in Fig. 8. The resultant passes through $m$.

It is shown in my New Constructions in Graphical Statics, Chapter II, that the position of the pseudo closing line $k'k'$, in case the girder has its ends fixed as above stated, is determined from the conditions that it shall cut the curve *ee* in such a way that the moment area above $k'k'$ shall be equal to that below $k'k'$, and also in such a way that the center of gravity of the new moment area shall be in the same vertical as the original moment area.

To find the center of gravity of the moment area *ek*; determine the areas of the various trapezoids of which it is composed by help of the summation polygon *ss*. In constructing *ss* we make $h_21=k_2e_2$, $h_22=k_2e_2+k_3e_3$, etc., and using $v$ as the common point of the pencil we