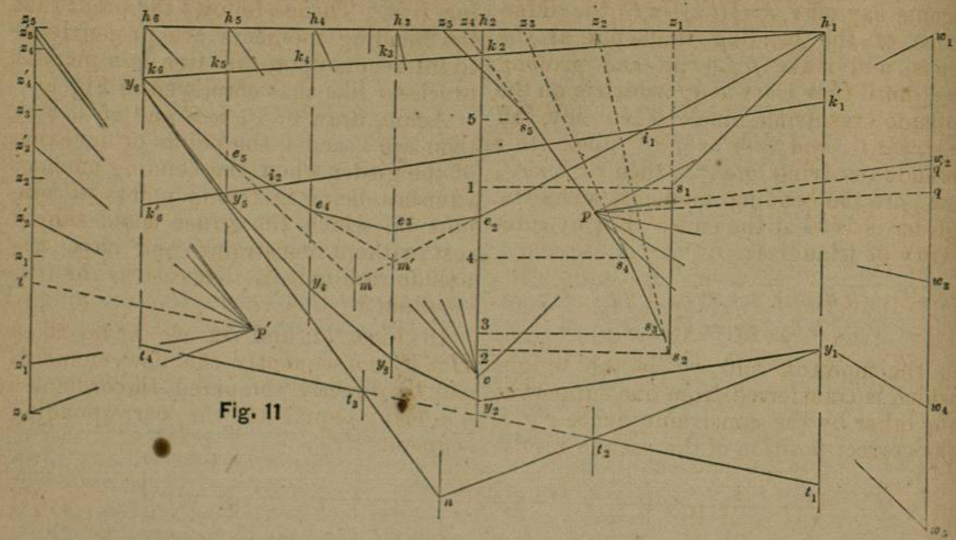

Fig. 11

shall have $h_2v.h_1z_5$ = twice the area of the moment area. We have used the sum of the two parallel sides of each trapezoid instead of half that quantity for greater accuracy.

Now lay off from z_0, $z_0z_1=h_1z_1$, $z_0z_2=h_1z_2$, etc., as a weight line and assume the pole p'.

Of the triangle $h_1h_2e_2$, one-third rests at h_1 and two-thirds at h_2; make $z_0z_1'=\frac{1}{3}z_0z_1$, it is the part of the area applied at h_1. Of the area $h_2e_2e_3h_3$, one half, approximately, rests at h_2 and one half at h_3. Bisect z_1z_2 at z_2', then $z_1'z_2'$ rests at h_2. Bisect each of the other quantities z_2z_3, etc. except z_4z_5, in which make $z_6z_5'=\frac{1}{3}$ z_6z_4. With the weights $z'z'$ so obtained, construct the second equilibrium polygon yy, which shows that the center of gravity of the moment area is in the vertical through n. There is a balancing of errors in this approximation which renders the position of n quite exact; if, however, greater precision is desired, determine the centers of gravity of the trapezoids forming the moment area, and use new verticals through them as weight lines, with the weights zz instead of the weights $z'z'$.

Draw verticals which divide the span into three equal parts,—they cut ny_1 and ny_6 at t_2 and t_3, and draw $p't' \parallel t_2t_3$. Then is $t_1t_2nt_3t_4$ an equilibrium polygon due to the force z_5z_0 applied at n, and to the forces z_0t', and $t'z_5$ applied at t_2 and t_3 respectively. As explained when treating this matter in the New Constructions in Graphical Statics, z_0t' and $t'z_5$ are proportional to the bending moments at the extremities of the fixed girder. In this case, since we have taken $h_2v=\frac{1}{2}h_1h_6$, we find that $h_1k_1'=\frac{1}{2}z_0t'$, and $k_6k_6'=\frac{1}{2}$ $t'z_6$ are the end moments, and they fix the position of the pseudo closing line. Draw $pq' \parallel k'k'$ then are w_1q' and $q'w_6$ the reactions of the piers. The pseudo resultant is at m'.

To obtain the same result by help of a frame pencil, let Fig. 12 represent the same weights applied in the same manner as in Fig. 11. Choose the vertex v, and draw the equilibrating polygon dd, etc. as in Fig. 8. Make $h_2 1=r_1r_2$, $h_2 2=r_1r_2+r_1r_3$, etc., since these quantities are proportional to the bending moments as previously shown. With v as the common point of the rays of a pencil, find h_1z_6 by the help of the summation polygon ss just as in Fig. 11.

Lay off the second weight line z_0z_1', etc., just as in Fig. 11, and with v as vertex construct the second equilibrating polygon xx. Then as readily appears $vn \parallel z_0x_6$ determines n the center of gravity of the moment area. Make z_0x_0 $\parallel vt_2$ and $x_6x_0 \parallel vt_3$; if t_2 and t_3 divide the span into three equal parts, then the horizontal through x_0 fixes t' corresponding to t' in Fig. 11.

To find the position of the pseudo resolving line and its segments proportional to the new bending moments, lay off $r_1j=\frac{1}{2}(t'z_5-z_0t')$ the difference of the bending moments at the ends, and make $jr_6' \parallel r_1w_1$ and prolong u_5r_6 until they meet at r_6' which is on the pseudo resolving line. Then lay off $r_1r'=\frac{1}{2}z_0t'$ and $r_6'r'=\frac{1}{2}t'z_5'$ upon this pseudo resolving line $r'q'$, then $r'r_2'$, $r'r_3'$, etc., are the bending moments when the girder is fixed at the ends. For by similarity of triangles

$$h_1h_6 : V :: r_1'r_6' : qq',$$
$$\therefore h_1h_6 \cdot qq_1 = V \cdot r_1'r_6',$$

is the moment, and qq' is the weight which is transferred from one support to the other by the constraint, hence $r'q'$ is the correct position of the pseudo resolving line. Thence follows the proof that the bending moments are proportional to intercepts upon this line in a manner precisely like that employed in Fig. 9.

Again, draw $vi_1 \parallel w_1r'$ and $vi_2 \parallel u_5r'$, then are i_1 and i_2 the points of inflexion of the girder when the bending moment vanishes, being in reality points of support on which the girder could simply rest without constraint and have the pseudo resultant in that case as the true resultant.

In Figs. 11 and 12 we have taken $H=V$, consequently the new moments can be directly compared, the ordinates $k'e$ being equal to the corresponding segments $r'r$.

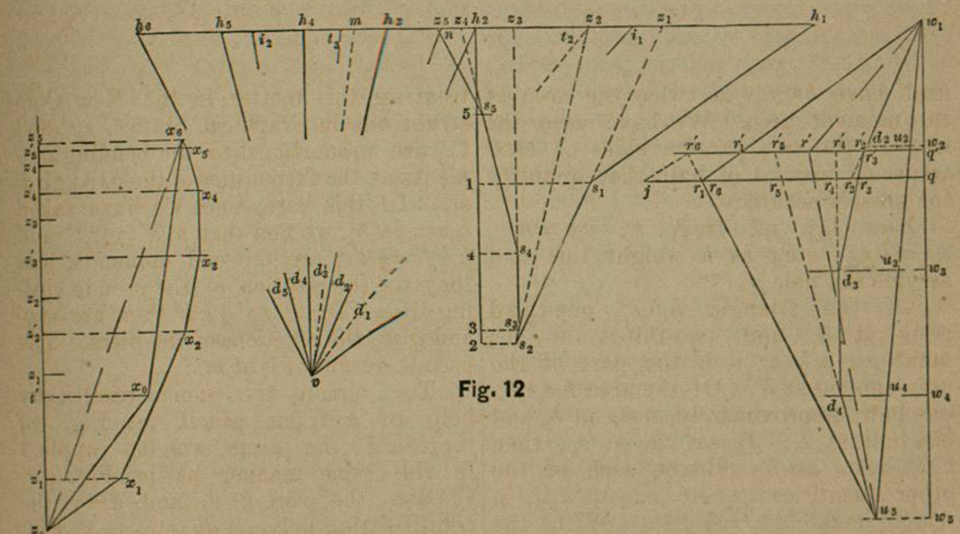

Fig. 12

Apparently in this example Fig. 12 presents a construction somewhat more compact than that of Fig. 11, it is certainly equally good.

It remains to remark before proceeding to further considerations of a slightly different character, that we owe to the genius of Culmann* the establishment of the generality of the method of the equilibrium polygon.

He adopted the funicular polygon, some of whose properties had long been known, and upon it founded the general processes and methods of systematic work which are now employed by all.

Furthermore it should be stated that parallelograms of forces were compounded and applied in such a way as to give rise to a frame pencil and equilibrating polygon by the illustrious Poncelet* who by their use determined the centers of gravity of portions of the stone arch. Whether he recognized other properties besides the simple determination of the resultant of parallel forces, I am not informed, as my knowledge of Poncelet's memorial is derived from so much of his work as Woodbury† has incorporated in his graphical construction for the stone arch.

So far as known, the method has been advanced by no one of the numerous recent writers upon Graphical Statics

* Graphische Statik. Zurich, 1866.

* Memorial de l' officier du Genie. No. 12.

† Treatise on the Stability of the Arch. D. P. Woodbury, New York, 1858.

which would certainly have been the case had Poncelet established its claim to be regarded as a general method. I think the method of the frame pencil may now fairly claim an equal generality and importance with that of the equilibrium polygon.

ANY FORCES LYING IN ONE PLANE, AND APPLIED AT GIVEN POINTS.

We have previously referred to this problem, having treated a particular case of it in Fig. 2; and subsequently certain statements were made respecting the indeterminateness of the process for finding the reactions of supports in case the applied forces were not vertical.

The case most frequently encountered in practice is wind-pressure combined with weight, and we can take this case as being sufficiently general in its nature; so that we are supposed to know the precise points of application of each of the forces, and its direction. Now it may be that the reaction of the supports cannot be exactly determined, but in all cases an extreme supposition can be made which will determine stresses in the framework which are on the safe side.

For example, if it is known that one of the reactions must be vertical, or normal to the bed plate of a set of supporting rollers, this will fix the direction of one reaction and the other may then be found by a process, like that employed in Fig. 2, of which the steps are as follows:

Resolve each of the forces at its point of application into components parallel and perpendicular to the known direction of the reaction, which we will call vertical for convenience, since the process is the same whatever the direction may be. By means of an equilibrium polygon or frame pencil find the line of action of the resultant of the horizontal components, whose sum is known. Then this horizontal resultant, can be treated precisely as was the single horizontal force in Fig. 2, which will determine the alteration of the vertical components of the reactions due to the couple caused by the horizontal components.

Also, find by an equilibrium polygon, or frame pencil, the vertical reactions due to the vertical components. Correct the point of division q of the weight line as found from the vertical components by the amount of alteration already found to be due to the horizontal components. Call this point q', then the polygon of the applied forces must be closed by two lines representing the reactions, which must meet on a horizontal through q'; but one of them has a known direction, hence the other is completely determined.

This determination causes the entire horizontal component to be included in a single one of the reactions, and it is usually one of the suppositions to be made when it is not known that the reaction of a support is normal to the plane of the bed joint.

Another supposition in these circumstances is that the horizontal component is entirely included in the other reaction; and a third supposition is that the horizontal component is so divided between the reactions that they have the same direction. These suppositions will usually enable us to find the greatest possible stress on any given piece of the frame by taking that stress for each piece which is the greatest of the three.

In every supposition care must be taken to find the alteration of the vertical components due to the horizontal components. This is the point which has been usually overlooked heretofore.

KERNEL, MOMENTS OF RESISTANCE AND INERTIA: EQUILIBRIUM POLYGON METHOD.

The accepted theory respecting the flexure of elastic girders assumes that the stress induced in any cross section by a bending moment increases uniformly from the neutral axis to the extreme fiber.

The cross section considered, is supposed to be at right angles to the plane of action or solicitation of the bending moment, and the line of intersection of this plane with that of the cross section is called the axis of solicitation of the cross section.

The radius of gyration of the cross section about any neutral axis is in the direction of the axis of solicitation.

It is well known that these two axes intersect at the center of gravity of the cross section, and have directions which are conjugate to each other in the ellipse which is the locus of the extremities of the radii of gyration.

We shall assume the known relation

$$M=SI\div y$$

in which M is the magnitude of the bending moment, or moment of resistance of the cross section, S is the stress on the extreme fiber, I is the moment of inertia about any neutral axis x, and y is the distance of the extreme fiber in the direction of the axis of solicitation, *i. e.* the distance between the neutral axis x and that tangent to the cross section which is parallel to x and most remote from it, the distance being measured along the axis of solicitation.

Let $M=Sm$ in which m is called the "specific moment of resistance" of the cross section; it is, in fact, the bending moment which will induce a stress of unity on the extreme fiber.

Now $I=k^2A$

in which k is the radius of gyration and A is the area of the cross section.

Let $k^2\div y=r$, $\therefore m=rA$,

is the specific moment of resistance about x, and when the direction of x varies, r varies in magnitude: r is called the "radius of resistance" of the cross section. The locus of the extremity of r, taken as a radius vector along the axis of solicitation, is called the "kernel."

The kernel is usually defined to be the locus of the center of action of a stress uniformly increasing from the tangent to the cross section at the extreme fiber. It was first pointed out by Jung,* and subsequently by Sayno, that the radius vector of the kernel is the radius of resistance of the cross section measured on the axis of solicitation. This will also appear from our construction by a method somewhat different from that heretofore employed.

Jung has also proposed to determine values of k, by first finding r; and has given methods for finding r. We shall obtain r by a new method which renders the proposal of Jung in the highest degree useful.

The method heretofore employed by Culmann and other investigators has been to find values of k first, and then having drawn the ellipse of inertia to construct the kernel as the locus of the antipole of the tangent at the extreme fiber. The method now proposed is the reverse of this, as it constructs several radii of the kernel first, then the corresponding radii of gyration, and from them the ellipse, and finally completes the kernel. In the old process there are inconvenient restrictions in the choice of pole distances which are entirely avoided in the new process.

Let the cross section treated be that of the T rail represented in Fig. 13, which is $4\frac{1}{2}\times2\frac{1}{2}$ inches and $\frac{1}{2}$ inch thick. We have selected a rail of uniform thickness in order to avoid in this small figure the numerous lines needed in the summation polygon for determining the area; but any cross section can be treated with ease by using a summation polygon for finding the area.

To find the center of gravity, let the weights w_1w_2 and w_2w_3, which are proportional to the areas between the verticals at b_1b_2 and b_2b_3 be applied at their centers of gravity a_1 and a_2 respectively; then the equilibrium polygon c_1c_2, having the pole p_1, shows that o is the required center of gravity.

Let the area b_2b_3 be divided into two parts at o, then w_2w_0 and w_0w_3 are weights proportional to the areas b_2o and ob_3 respectively; and $c_1c_3c_4$ is the equilibrium polygon for these weights applied at their centers of gravity a_3 and a_4.

The intercepts mm have been previously shown to be proportional to the products of the applied weights by their distances from the center of gravity o.

We have heretofore spoken of these products as the moments of the weights about their common center of gravity o. But the weights in this case are areas and the product of an area by a distance is a volume. Let us for convenience call volumes so generated "stress solids." The elementary stress solids obtained by multiplying each elementary area by its distance from the neutral axis will correctly represent the stresses on the different parts of the cross section, and they will be contained between the cross section and a plane intersecting the cross section along the neutral axis and making an angle of 45° with the cross section.

If b_1b_3 is the ground line, b_1b_3 and d_1d_3 are the traces of the planes between

* "Rappresentazioni grafische dei momenti resistenti di una sezione piana." G. Jung, Rendiconti dell' Instituto Lombardo, Ser. 2, t, IX, 1876, No. XV. "Complemento alla nota precedente." No. XVI.

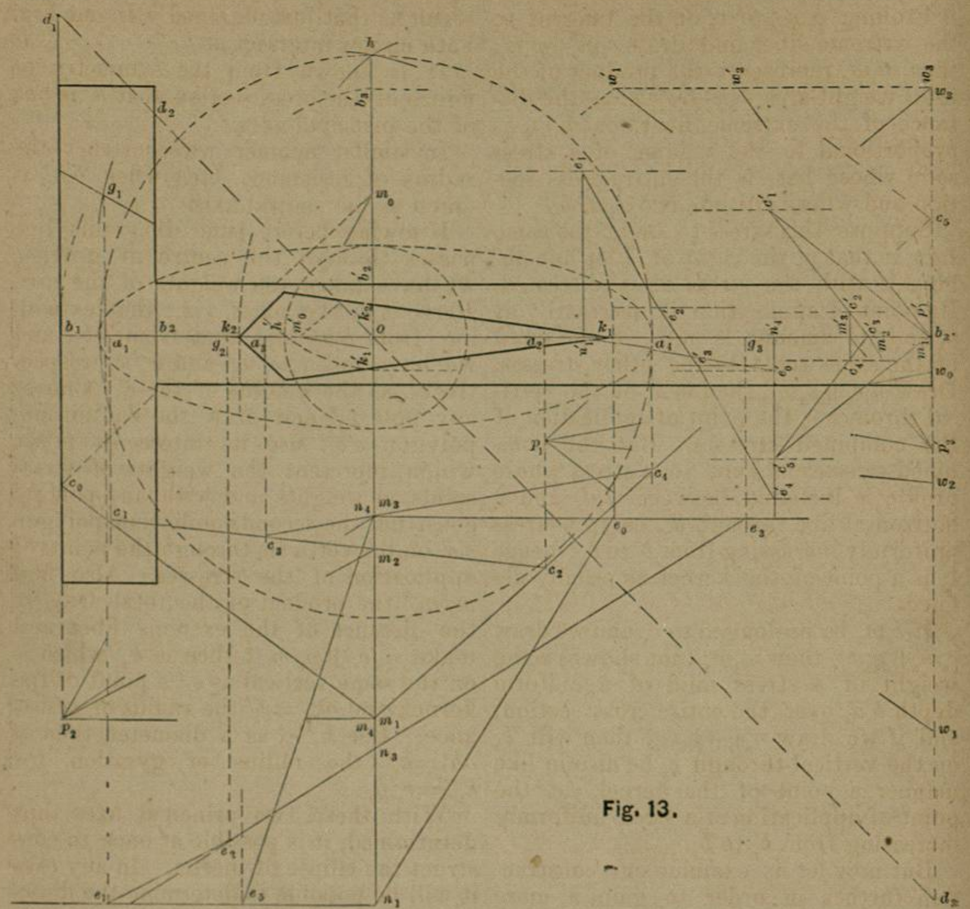

Fig. 13.

which the stress solid lies on a plane at right angles to the neutral axis.

The distances of the centers of gravity of the stress solids from o are also the distances of the points of application of the resultant stresses, and the magnitude of the resultant stresses are are proportional to the stress solids. The stress solids may be considered to be some kind of homogeneous loading whose weight produces the stress upon the cross section. The moment of inertia I is the moment of this stress with respect to o.

Now the intercept m_3m_1 represents the weight of the stress solid whose profile is ob_3d_3. Its point of application is g_3, if $og_3=\frac{2}{3}ob_3$. Similarly the weight m_2m_3 has its point of application at g_2 if $og_2=\frac{2}{3}ob_2$. And the weight m_1m_2 is applied in the vertical through g_1; for the profile of this stress solid is the trapezoid $b_1b_2d_2d_1$, and g_1 is its center of gravity found geometrically. In case the area is divided into narrow bands parallel to the neutral axis the points of application coincide sensibly with the centers of gravity of the bands.

Now take any pole p_2 and construct a second equilibrium polygon ee due to the stress solids applied in the verticals through $g_1g_2g_3$.

The last two sides e_1n_1 and e_3n_4 are necessarily parallel and have their intersection at infinity, for the total stress is a couple.

The intercept n_1n_4 is not drawn through the common center of gravity of the stress solids, *i.e.*, it is not an intercept on the line of the resultant stress, but since parallels are everywhere equidistant this intercept is proportional to the moment of the stresses about their center of gravity; in other words n_1n_4 when multiplied successively by the two pole distances would be I. We shall not need to effect the multiplication.

Prolong c_1m_1 to c_0 on the tangent to the extreme fiber and draw $c_0m_0 \parallel p_1w_3$, then m_1m_2 represents the product of the total weight-area w_1w_3 by $ob_1=y$ the distance of the extreme fiber, or m_1m_0 is proportional to the volume of a stress solid whose base is the entire cross section and whose altitude is $b_1d_1=ob_1$.

Suppose this stress to be of the same sign as that at the right of o, let us combine it with the stress already treated. Its point of application is necessarily at o, and its amount is m_1m_0 if measured on the same scale as the other stresses. Draw $n_1e_0 \parallel p_2m_0$, then is k_1 on the vertical through e_0 the point of application of the combined stresses. But the combined stresses amount to a stress whose profile is included between d_1d_3 and a horizontal line through d_1, *i.e.* to a stress uniformly increasing from b_1 to b_3; hence k_1 is a point of the kernel as usually defined.

If c_4m_4 be prolonged to c_5 and we draw $c_5m_5 \parallel p_1w_1$, then m_4m_5 (not shown) is the weight of a stress solid of a uniform depth b_3d_3 over the entire cross section; and if we draw $n_4e_5 \parallel p_2m_5$, then will k_2 on the vertical through e_5 be also in like manner a point of the kernel, *i.e.* the point of application of a stress uniformly increasing from b_3 to b_1.

But now let us examine our construction further in order to gain a more exact understanding of what the distances $r_1=ok_1$ and $r_2=ok_2$ signify.

We have shown that m_1m_0 represents the product of the area of the cross section by the distance ob_1 of the extreme fiber, *i.e.* the quantity Ay_1; but n_1n_4 represents the moment of this weight when applied at k_1, *i.e.* the product Ay_1r_1. Also as previously shown n_1n_4 represented I on the same scale, hence

$$I=Ay_1r_1, \text{ but } I=Ak_1^2 \therefore r_1=k_1^2 \div y_1$$

and r_1 is the radius of resistance previously mentioned.

In order to determine the radius of gyration k_1, which is a mean proportional between r_1 and y_1, describe a circle on b_1k_1 as a diameter intersecting mm at h then $oh=k_1$ the semi-axis of the ellipse of inertia conjugate to mm as a neutral axis. The accuracy of the construction is tested by using b_3k_2 as a diameter and finding the mean proportional between ok_2 and ob_3. It should give the same result as that just obtained. In our Fig. both circles intersect at h.

It is known from the symmetry of figure of the cross section that k_1 is one of the principal axes.

In similar manner we construct the radius of resistance, etc., when b_1b_3 is taken as the neutral axis.

Knowing before hand that this line passes through the centre of gravity, we have taken the weights of the area above it in two parts, viz.: that extending from b_1b_2, and that from b_2b_3, and we have taken $w_1'w_2'$ and $w_2'w_3'$ respectively, as the weights of these. Choose any pole p_1' and draw the equilibrium polygon $c'c'$: use its intercepts $m'm'$, which represent the weights of stress solids, as weights and with any pole p_2' construct the second equilibrium polygon $e'e'$ on the verticals through the points of application of the stresses. Also find $m_1'm_0'$ the product of the total area by the distance of the extreme fiber and make $n_1'e_0' \parallel p_2'm_0'$; then is k_1' which is on the same vertical as e_0' a point of the kernel, and $ok_1'=r_1'$ the radius of resistance. Use $k_1'b_3'$ as a diameter, then is $oh'=k_1'$ the radius of gyration, for $k_1'^2=r_1'y_1'$.

With these two principal axes thus determined, it is possible at once to construct the ellipse of inertia. In any case it will be possible to determine the direction of the axis of solicitation corresponding to any assumed neutral axis by actual construction, it being simply necessary to find the line through o upon which lie the points of application of the positive and negative stresses considered separately. These axes being conjugate directions in the ellipse of inertia, when we have found the radii of resistance in those two directions we can at once obtain the corresponding radii of gyration which are conjugate semi-diameters, and so draw the ellipse.

After the ellipse is drawn the kernel can be readily completed by making r in every direction a third proportional to the distance of the extreme fiber and the radius of gyration.

We are assisted in drawing the kernel by noticing that to each straight side of the cross section there corresponds a single point in the kernel, and to each non-re-entrant angular point a side of the kernel, these standing in the mutual relation of polar and anti-pole with respect to the ellipse of inertia, as shown by the equation $k^2=ry$.

In Fig. 13 the point k_1 corresponds to the left hand vertical side, the point k_2 to the right hand vertical side, and the sides k_1k_1', k_2k_2' to the angular points at the upper and lower extremities of the left side respectively, while the points $k_1'k_2'$ at the very obtuse angular points of the kernel correspond to the upper and lower horizontal sides of the flange. The two remaining angular points of the kernel correspond to tangent lines when they just touch the corners of the flange and web, while the intermediate sides correspond to the angles at the extremities of these lines.

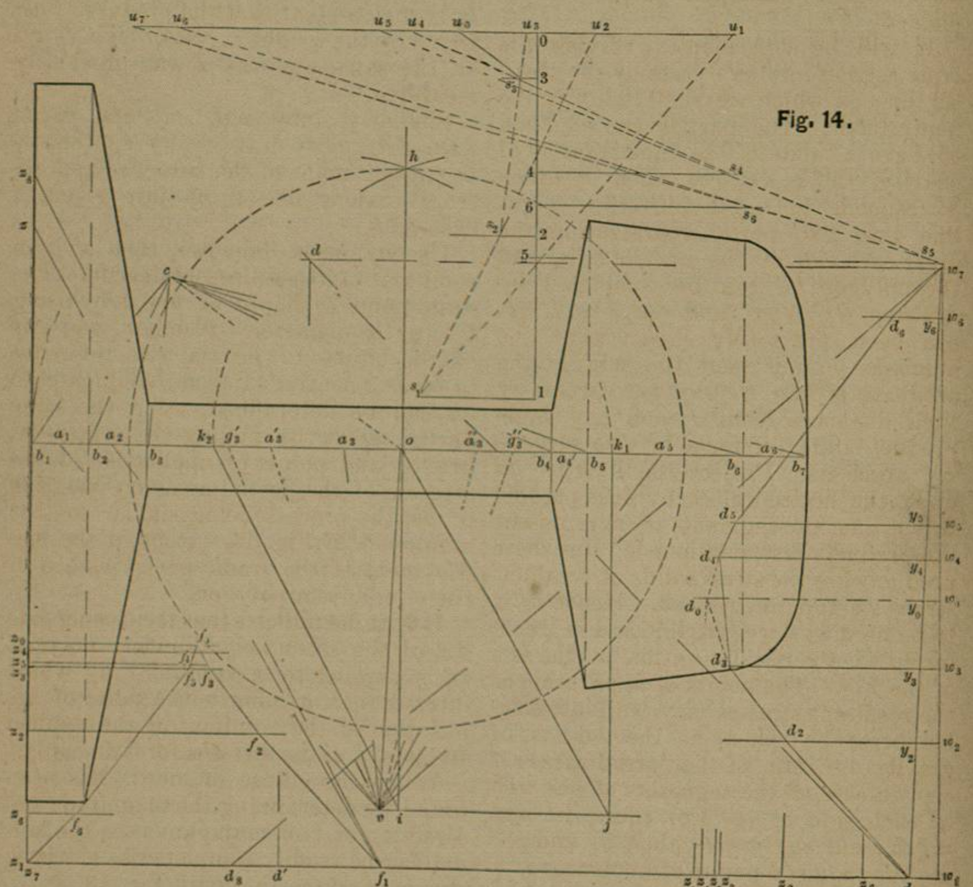

Fig. 14.

KERNEL, MOMENTS OF RESISTANCE AND INERTIA: FRAME PENCIL METHOD.

Let the cross section treated be that shown in Fig. 14, which is nearly that of a 56 lb. steel rail, the difference consisting only in a slight rounding at the angles.

Let the cross section be divided by lines perpendicular to the axis of symmetry bb at b_2, b_3, etc., then the partial areas and the total area may be found by a summation polygon.

Take c as the common point of the rays through b_1b_2, etc., and make 01, 02, etc., proportional to the mean ordinates of the areas standing on the bases b_1b_2, b_2b_3, etc. respectively. Draw $s_1u_1 \parallel cb_1$, $s_2u_2 \parallel cb_2$, etc., then will the segments of the line uu represent the respective partial areas, and u_1u_7 will represent the total area.

Divide the vertical line ww into segments equal to those of the line nn, then is ww the weight line for finding the center of gravity, etc., of the cross section. Let a_1, a_2, a_3, etc., be the centers

of gravity of the partial areas, and let v be the vertex of a frame pencil whose rays pass through these centers of gravity. Draw the equilibrating polygon dd with its sides parallel to the rays of this frame pencil, then the ray vo parallel to the closing side yy of the equilibrating polygon determines the center of gravity o of the cross section, according to principles previously explained.

It will be convenient to divide the cross section into two parts by the vertical line oi, which we shall take as the neutral axis. The partial areas b_3o and ob_4 have a_3' and a_3'' as their centers of gravity. Make $s_3u_0 \parallel co$, then w_0 which corresponds to u_0, divides the weight line into two parts, representing the areas each side of the neutral axis, and the polygon dd can be completed by drawing $d_0d_3 \parallel va_3'$ and $d_0d_4 \parallel va_3''$. It has been previously shown that the abscissas yd represent the sum of the products of the weights (*i.e.* areas) by their distances from o; and any single product is the difference of two successive abscissas. Project the lengths yd upon the horizontal zz by lines parallel to yy, then the segments of zz represent the products just mentioned. But these products are the stress solids or resultant stresses before mentioned. Hence zz is to be used as a weight line and is transferred to a vertical position at the left of the Fig. The points of application of the resultant stresses may without sensible error be taken at the centers of gravity a_1a_2, etc., of the partial areas except in case of the segments of the web on each side of o. For these, let $og_3' = \frac{2}{3}ob_3$, and $og_3'' = \frac{2}{3}ob_4$, then g_3' and g_3'' are the required points of application.

Now with the weight line zz, which consists partly of negative loads, and with the same vertex v construct the second equilibrating polygon ff, then z_1f_1 represents the moment of inertia of the cross section, it being proportional the moment of the resultant stresses about o. It is seen that the sides f_3f_0 and f_0f_4 are so short that any small deviation in their directions would not greatly affect the result, and that there would therefore have been little error if the resultant stresses in the web had been applied at a_3' and a_3''.

Again, draw $dd_1 \parallel vb_1$, then the horizontal line dw_7 $(=d_1d')$ represents Ay_1, the product of the total weight w_1w_7 (*i. e.* the total area of the cross section), by the distance of the extreme fiber $ob_1=y_1$. Use this as a stress solid or resultant stress applied at o and having a weight $zz_1=d_1d'$, and draw $oj \parallel zf_1$, j being at the same vertical distance from bb as v is; then is k_1, which on the same vertical at j, a point of the kernel. For k_1 is such a point that the product of ok_1 $(=r_1)$ by the weight $zz_1(=Ay_1)$ is $z_1f_1=I$ on the same scale as I was previously measured.

Similarly draw $w_7d_8 \parallel vb_7$ and make $z_8z_1=d_1d_8$; also draw $ik_2 \parallel f_1z_8$: then is k_2 another point of the kernel as appears from reasons like those just given in case of k_1.

Use b_1k_1 as a diameter, then oh is a semi-axis of the ellipse of inertia. The same point h should be found by using k_2b_7 as a diameter. Another semi-axis of the ellipse of inertia with reference to bb as a neutral axis, and conjugate to oh can be determined, using the same partial areas, by finding the centers of gravity and points of application of the stresses of the partial areas on one side of bb, the process being similar to that employed in Fig. 13, except in the employment of the frame pencil instead of the equilibrium polygon.

It is to be noticed that the closing side f_1z_1 of the second equilibrating polygon ff is parallel to a resultant ray which intersects bb at infinity, the point of application of the resultant of the applied stresses, *i. e.* the stresses form a couple.

When the ellipse of inertia has been found by determining the magnitude and direction of two conjugate axes, the kernel can be readily completed as has been shown in connection with Fig. 13.

UNIFORMLY VARYING STRESS IN GENERAL.

The methods employed in Figs. 13 and 14 are applicable also to any uniformly varying stress, for a stress which uniformly increases from any neutral axis x through the center of gravity of the cross section can be changed into a stress which uniformly increases from same parallel axis x' at a distance y_0 from x by simply combining with the former a stress uniformly distributed over the cross-section and of such intensity as to make the resultant intensity zero along x'.

In the construction given in Figs. 13 and 14 it is only necessary to use the proposed line x' at a distance y_0 from o, instead of the tangent to the extreme fiber at a distance y_1 or y_2 from o, when we wish to determine the weight or volume of the resultant stress solid, its moment about o, and its center of gravity or application.

Since the locus of the center of application of the resultant stress is the antipole of x' with respect to the ellipse of inertia, it is evident that when the proposed axis x' lies partly within the cross section the center of application of the resultant stress is without the kernel, and that when x' is entirely without the cross section its center of application is within the kernel.

It is frequently more convenient to determine the center of application from the kernel itself than from the ellipse of inertia. This can be readily found from the equation which we are now to state

$$Ar_0 y_0 = Ar_1 y_1 = I,$$

in which equation Ay_0 and Ay_1 are the volumes of the stress solids which if uniformly distributed and compounded with the stress whose neutral axis is x, will cause the resultant stresses to vanish at distances y_0 and y_1, respectively; while r_0 and r_1 are the distances from o of the respective centers of application of these stresses.

The truth of the equation is evident from the fact that the moment about o of any stress solid uniformly distributed is zero, hence the composition of such a stress with that previously acting will leave its moment unchanged.

From the equation just stated we have

$$y_0 : y_1 :: r_1 : r_0,$$

from which r_0 can be found by an elementary construction, since y_0, y_1 and r_1 are known quantities. When it is desired to express these results in terms of the intensities of the actual stresses,

let $p_0 = ny_0$ be the mean stress;
and let $p_1' = n\,(y_0 + y_1)$ be the greatest,
and let $p_2' = n\,(y_0 - y_2)$ be the least intensity at the extreme fiber:

then $\quad ny_1 = p_1' - ny_0 = p_1' - p_0$

or $\quad ny_2 = ny_0 - p_2' = p_0 - p_2'$

$\therefore \quad p_0 : p_1' - p_0 :: r_1 : r_0$

or $\quad p_0 : p_0 - p_2' :: r_2 : r_0$

in which r_1 and r_2 are the two radii of the kernel.

DISTRIBUTION OF SHEARING STRESS.

It is well known that the equation $dM = Tdz$, expresses the relation of the total shearing stress T sustained at any cross section of a girder to the variation dM of the bending moment M at a parallel cross-section situated at the small distance dz from the first mentioned cross section.

We have already treated the normal components of the stress caused by the bending moment M: we shall now treat the tangential component or shear which accompanies any variation of the bending moment.

We shall assume as already proved the following equation* which expresses the intensity q of the shearing stress at any point of the cross section:

$$Iqx = TV$$

in which x is the width of the girder measured parallel to the neutral axis at any distance y from the neutral axis, and q is the intensity of the shearing stress at the same distance, I is the moment of inertia of the cross section about the neutral axis, T is the total shear at this cross section, and V is the volume of that part of one of the stress solids used in finding the moment of inertia which is situated at a greater distance than y from the neutral axis, *i.e.* in Fig. 13 if we were finding the value of q at b_2, with respect to om_2 as the neutral axis, then V would signify the stress solid whose profile is $d_1 d_2 b_2 b_1$. It, however, makes no difference whether we define V as the stress solid situated at the left or at the right of b_2; for, since the total stress solid, positive and negative, is zero, that on either side of any assumed plane is the same.

The first step in our process is to find the intensity of the shear at the neutral axis, which we denote by q_0; and if we also call x_0 the width here and V_0 the volume of either of the two equal stress

* See Rankine's Applied Mechanics. Eighth Edition, Art. 309, p. 338.

solids between this axis and the extreme fiber, we have

$$Iq_0 x_0 = T\,V_0, \text{ but } I = V_0 d$$

when d is the distance between the centers of application of the equal stress solids, *i.e.*, d is the arm of the couple of the resultant stresses. Also $T = A\bar{q}$ when A is the total area of the cross section and $\bar{q}$ is the mean intensity of the shearing stress. Hence at the neutral axis we have the equation

$$q_0 x_0 d = A\bar{q} = T$$

Now the length of the arm d is found in Fig. 13 by prolonging the middle side (*i.e.* the side through n_3) of the second equilibrium polygon until it intersects the first side and the last. These intersections will give the position of the centers of gravity of the stress solids on either side of o.

In Fig. 14 the same points are found by drawing rays from v parallel respectively to $z_1 f_0$ and $f_1 f_0$ until they intersect aa.

In Fig. 15 the points f_1 and f_2 are found by either of these methods and $f_1 f_2 = d$ is the required distance.

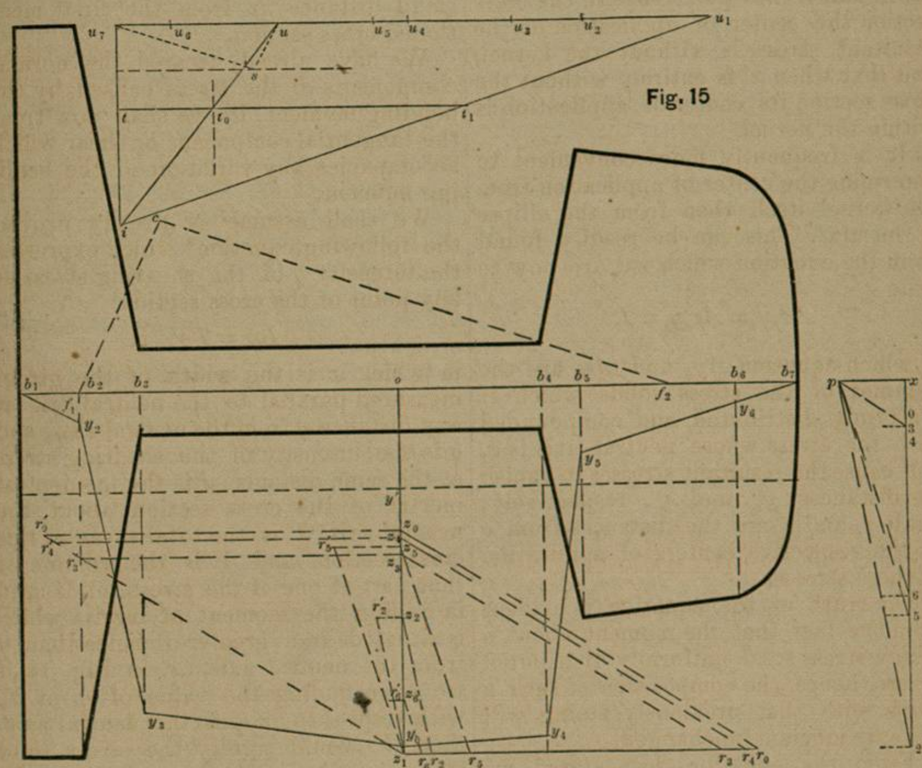

Fig. 15

Now in Fig. 15 let the segments uu of the summation polygon be obtained just as in Fig. 14, and parallel to uu draw a line through s representing the width of the cross section x_0 on the same scale as before used in constructing the summation polygon. Also make $su_7 \parallel cf_2$ and $su \parallel cf_1$, c being the common point in the rays of the pencil of the summation polygon for finding the area.

Then uu_7 represents the product $x_0 d$ on same scale that $u_1 u_7$ represents A.

Now draw from any point i rays to u_1, u and u_7, and also a parallel to iu_7 at a distance $\bar{q}$ and intersecting iu at some point t_0 such that $tt_0 = \bar{q}$ to such a scale as may be convenient. The mean intensity $\bar{q}$ is supposed to be a known quantity, and $tt_0 \parallel uu$. Then from the proposed equation we have the proportion

$$x_0 d : A :: \bar{q} : q_0$$

or $\quad uu_7 : u_1 u_7 :: tt_0 : tt_1$

Hence tt_1 represents the intensity of the shearing stress at the neutral axis on the same scale that tt_0 represents the mean intensity.

This first step of our process has determined the intensity of the stress at the neutral axis relatively to the mean stress; the second step will determine the intensity of the stress at any other point relatively to the stress at the neutral axis. When this last point is all that is desired the first step may be omitted.

The equation $Ixq=TV$ may be written $xq=cV$, in which $c=T\div V$ is a constant. At the neutral axis this equation is

$$x_0q_0=cV_0 \text{ or } V_0 : q_0 :: x_0 : c$$

In Fig. 15 lay off the segments of the line zz just as in Fig. 14; then z_1z_0 represents the weight or volume V_0; also make $x0$, $x2$, $x3$, etc., proportional to width of the girder at o, b_2, b_3, etc., and lay off $z_1r_0=z_0r_0'=tt_1$.

Draw $p0 \parallel r_0z_0$, then by similar triangles

$$z_1z_0 : z_1r_0 :: x0 : xp$$

or

$$V_0 : q_0 :: x_0 : c$$

$\therefore$ px represents the constant c.

Now the several segments z_1z_2, z_1z_3, z_1z_4, etc., represent respectively the values of V_2, V_3, V_4, or the stress solids between one extreme fiber and b_2, b_3, b_4, etc.; it is of no consequence which extreme fiber is taken as the stress solid is the same in either case.

Now using p as a pole draw rays to 2 3 4 5 etc., and make $z_2r_2 \parallel p2$, $z_3r_3 \parallel p3$, etc., then by similar triangles

$$z_1z_2 : z_1r_2 :: x2 : c, \text{ or } x_2q_2=cV_2$$

and $z_1z_3 : z_1r_3 :: x3 : c$, or $x_3q_3=cV_3$

etc., etc., and z_1r_2, z_1r_3, etc., represent the intensity of the shearing stresses at b_2, b_3, etc. These can be constructed equally well by drawing rays from z_1 parallel to the rays at p, from which we obtain

$$z_2r_2'=z_1r_2,\ z_3r_3'=z_1r_3, \text{ etc.}$$

Now lay off $b_2y_2=z_1r_2$, $b_3y_3=z_1r_3$, etc., then the ordinates by of the polygon yy represent the intensity of the shearing stress on the same scale that $tt_1=z_1r_0$ represents the intensity q_0 at the neutral axis, and on the same scale that $tt_0=oy'$ represents the mean intensity $\bar{q}$. The lines joining y_2, y_3, etc., should be slightly curved, but when they are straight the representation is quite exact.

RELATIVE STRESSES.

It is proposed here to develop a new construction which will exhibit the relative magnitude of the normal components of the stresses produced by a given system of loading in the various cross-sections of a girder having a variable cross section. The value of such a construction is evident, as it shows graphically the weakest section, and investigates the fitness of the assumed disposition of the material for sustaining the given system of loading.

The constructions heretofore given for the kernel and moments of resistance at any given cross section admit of the immediate comparison of the normal components of the stresses produced in that single cross section when different neutral axes are assumed, but by this proposed construction, a comparison is effected between these stresses at any different cross sections of the same girder or truss.

In the equation previously used

$$M=SI\div y=SAk^2\div y=SAr$$

in which M is the moment of flexure which produces the stress S in the extreme fiber of a cross section whose area is A and whose radius of resistance is r, we see, since the specific moment of resistance $m=Ar$ is the product of two factors, that the same product can result from other and very different factors.

For example, let $m=A_0r'$ in which A_0 is the area of some cross section which is assumed as the standard of comparison, and $r'=Ar\div A_0=ar$, when $a=A\div A_0$. Then is A_0r' the specific moment of resistance of a cross section of an assumed area A_0 which has a different disposition of material from that whose specific moment of resistance is Ar, but the cross sections A and A_0 are equivalent to each other in this sense, that they have the same specific resistance, and consequently the same bending moment will produce equal stresses in the extreme fiber in each.

The two cross sections do not have the same moment of inertia, and so the deflections of the girder would be

changed by substituting one cross section for the other. We shall then speak of them as equivalent only in the former sense, and on the basis of this definition, state the result at which we have arrived thus: Equivalent cross sections under the action of the same bending moment, have the same stresses at the extreme fiber (though they are not equally stiff); hence in comparing stresses equivalent cross sections may be substituted for each other (but they may not be so substituted in comparing deflections).

It is proposed to utilize this result by substituting for any girder or truss having a variable cross section A or a variable specific moment of resistance whose magnitude is expressed by the variable quantity Ar, a different one having a cross section everywhere of constant area A_0, but of such disposition of material that its specific moment of resistance is $A_0r'=Ar$ at corresponding cross sections.

The proposed substitution is especially easy in case of a truss, for in it the value of r varies almost exactly as its depth, as may be seen when we compute the value of $m=Ak^2\div y=Ar$ in this case.

Since the material which resists bending is situated in the chords alone and is all approximately at the same distance from the neutral axis we have $k=y=r=\frac{1}{2}h$ very nearly when h is the distance between the chords, $\therefore$ $m=\frac{1}{2}Ah$ nearly. Even when the two chords are of unequal cross section and the neutral axis not midway between them the same result holds when the ratio of the two cross sections is constant.

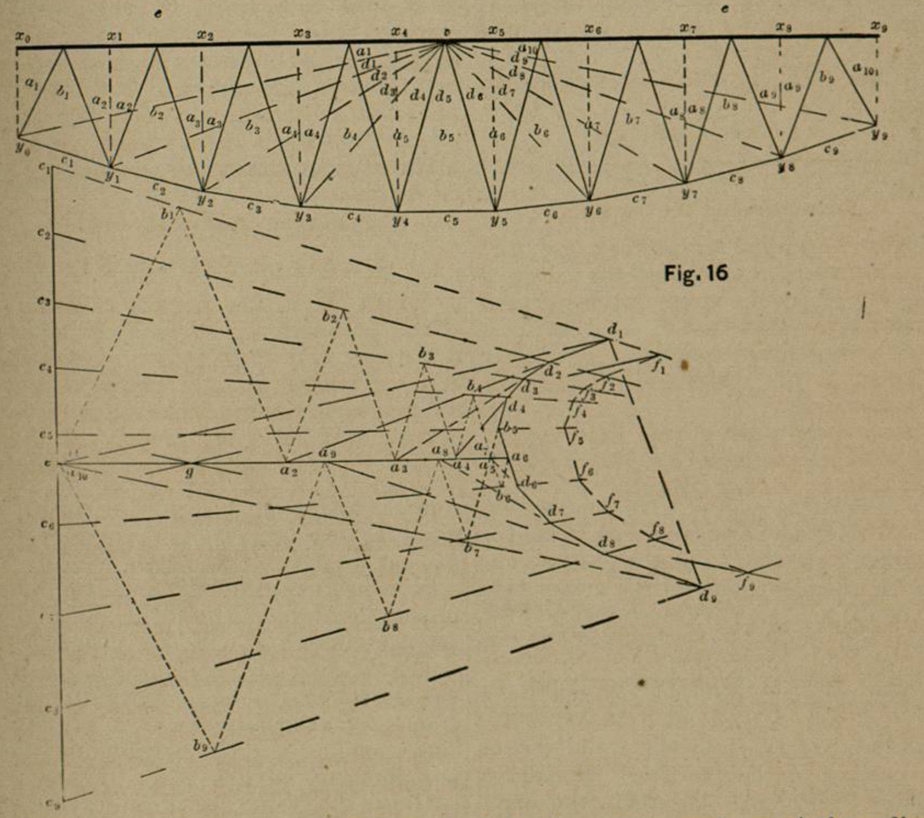

Fig. 16

In Fig. 16 let xx be the axis of a girder sustaining at the points x_1, x_2, etc., the weights c_1c_2, c_2c_3, etc. Lay off the ordinates xy at each of the points at which weights are applied, so that $xy=Ar$ on some assumed scale; then since $A_0r'=Ar=xy$, xy varies as r', the radius of resistance of a girder having at every point a cross section A_0 so disposed as to be equivalent to that of the given girder xx.

Assume some form of framing connecting the points xy as shown in the Fig., and suppose the weights applied at the points yy of the lower chord, the points of support being at y_0 and y_9. Then by a method like that employed in Fig. 3, we obtain the total stresses ea_2, ea_3, ea_4, etc., in the segments of the upper chord which are opposite to y_1, y_2, y_3, etc. Now these total stresses are resisted by a cross section of constant area A_0, consequently they have the same ratio to one another as the intensities per square unit; or further, they represent, as we have just shown, the relative intensities of the stresses on the extreme fiber of the given girder.

It is well known from mechanical considerations, that the stress in the several segments of the upper chord is dependent upon the loading and upon the position of y_1, y_2, etc., and is not dependent upon the position of the joints in the upper chord. Of this fact we offer the following geometrical proof derived from the known relations between the frame and force polygons.

We know, if any joint of the upper chord, such as ea_2b_1 for example, be removed to a new position, such as v, that so long as the weights c_1c_2, c_2c_3, etc., are unchanged, that the vertex b_1 of the triangle ea_2b_1 in the force polygon must be found on the force line $c_1f_1 \parallel y_0y_1$. We shall show that while the side ea_2 is unchanged, the locus of b_1 is the force line c_1f_1; hence conversely, so long as c_1f_1 is the locus of b_1, ea_2 is unchanged, since there can be but one such triangle.

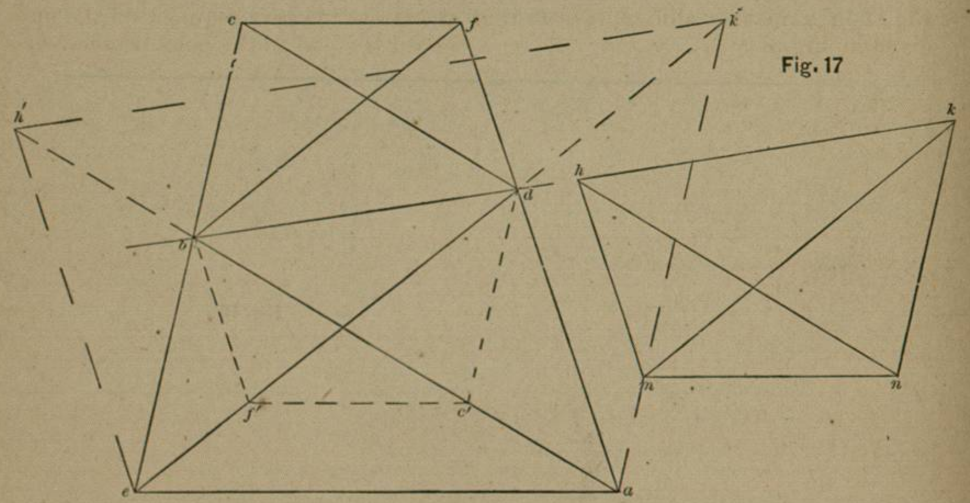

Fig. 17

In Fig. 17 let the two triangles abe, hnk, have the sides meeting at b and n mutually parallel. Let the bases ae and hk be invariable but let the vertex b be removed to any point d such that $bd \parallel hk$, then will the vertex n be removed to a point m such that $mn \parallel ae$.

For, prolong ad and eb, and draw $bf \parallel ed$ and $dc \parallel ab$, then is $abfcdea$ a hexagon inscribed in the conic section consisting of the two lines af and ec, hence by Pascal's Theorem, the opposite diagonals ea and cf intersect on the same line as the remaining pairs of opposite diagonals, $ab \parallel dc$ and $ed \parallel bf$. But this line is at infinity, hence $cf \parallel ae$. Also $c'f' \parallel cf$, from elementary considerations; and $c'f' \parallel mn$ from similarity of figures, hence $mn \parallel ae$. There are two cases, according as mn is above or below bk, but we have proved them both.

Now in Fig. 16 let all the joints in the upper chord be removed to v, then the segments ea_2, a_2a_3, etc., are unchanged, hence ea_3, ea_4, etc. are unchanged, and the assumed framing reduces to the frame pencil whose vertex is v. The corresponding force polygon is the equilibrating polygon dd.

Hence the frame pencil can be used as the assumed framing just as well as any other form of framing, and it is unnecessary to use any construction except that of the frame pencil and equilibrating polygon for finding the relative stresses ea_2, ea_3, etc.

STRESSES IN A HORIZONTAL CHORD.

If Fig. 16 be regarded as representing an actual bridge truss, whose chords are not of uniform cross section; it is seen that the total stresses on the horizontal chord are given by the segments ea_2, ea_3, etc., which are found from the equilibrating polygon alone without regard to the kind of bracing in the truss, which it is unnecessary to consider; and this method can be used to take the place of that given in connection with Fig. 3 for finding the maximum stresses on the chords.

The equilibrating polygon ff was constructed to determine the reactions of the piers by finding the point e. The outer sides of the polygon ff intersect at g which determines e as explained in Fig. 7 in a manner different from that given in Fig. 3.

This construction sheds new light upon the significance of the frame pencil and equilibrating polygon. The frame pencil is the limiting case of a truss when the joints along one chord are removed to a single point, so that each ray may be regarded as compounded of a tension member and a compression member, having the same direction, *e.g.*, the tension member of which y_1v is compounded has the stress d_1a_2, and the compression member the stress d_2a_2, but if the two be combined, the resultant tension is d_1d_2.

In case yy is the equilibrium curve due to the applied weights, and v falls upon the closing line, the force lines cd meet at the pole and the lines ed_1, ed_2, coincide with aa, so that the polygon dd is at the pole and infinitely small, and the stress in every segment of the upper chord is equal to the pole distance de.