NOTE A.

ADDENDUM TO PAGE 12, CHAPTER I.

The truth of Proposition IV is, perhaps, not sufficiently established in the demonstration heretofore given. As it is a fundamental proposition in the graphical treatment of arches, and as it is desirable that no doubt exist as to its validity, we now offer a second proof of it, which, it is thought, avoids the difficulties of the former demonstration.

Prop. IV. If in any arch that equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from considerations of the conditions imposed by the supports, etc.; and if, furthermore, the curve of the arch itself be regarded as another equilibrium polygon due to some system of loading not given, and its closing line be also found from the same considerations respecting supports, etc.; then when these two polygons are so placed that their closing lines coincide, and their areas partially cover each other, the ordinates intercepted between these two polygons are proportional to the real bending moments acting in the arch.

The bending moments at every point of an arch are due to the applied forces and to the shape of the arch itself.

The applied forces are these: the vertical forces, which comprise the loading and the vertical reactions of the piers; the horizontal thrust; and the bending moments at the piers, caused by the constraint at these points of support. The loading may cause all the other applied forces or it may not: in any case the bending moments are unaffected by the dependence or want of dependence of the thrust, etc., upon the loading.

Now, so far as the loading and the moments due to the constraint at the piers are concerned, they cause the same bending moments at any point of the arch as they would when applied to a straight girder of the same span, for neither are the forces nor their arms different in the two cases.

But the horizontal thrust, which is the same at every point of the arch, causes a bending moment proportional to its arm, which is the distance of its line of application from the curve of the arch. This line of application is known to be the closing line; hence the ordinates which represent the bending moments due to the horizontal thrust, are included between the curve of the arch and a closing line drawn in such a manner as to fulfill the conditions imposed by the joints or kind of support at the piers, hence the curved neutral axis of the arch is the equilibrium or moment polygon due to the horizontal thrust.

But the same conditions fix both the closing line of the equilibrium polygon which represents the bending moments due to the loading and to the constraint at the piers, and the closing line of the equilibrium polygon due to the horizontal thrust. Hence the resultant bending moment is found by taking the difference of the ordinates at each point, or by laying them off from one and the same closing line exactly as described in the statement of our proposition.

NOTE B.

ADDENDUM TO PAGE 10, CHAPTER I.

Attention should be directed to the two senses in which M is used in our fundamental formulae.

In equation (3) the primary signification of M is this: it is the numerical amount of the bending moment at the point O; and if this magnitude be laid off as an ordinate, y_m is the fraction or multiple of it found by equation (3).

Now M assumes, in the equations (3), (4), (5) and (3'), (4'), (5'), a slightly different and secondary signification; viz., the intensity of the bending moment at O. The intensity of the bending moment is the amount distributed along a unit in length of a girder, and may be exactly obtained as follows:

$$M = \int_x^{x+1} M dx, \quad \therefore \; \Sigma_0^x (M) = \int_0^x M dx.$$

In this secondary sense M is geometrically represented by an area one unit wide, and having for its height the average value which ordinate M, as first found, has along the unit considered.

Thus the M used in the equations of curvature, bending and deflection is one dimension higher than that used in the equation expressing the moment of the applied forces; but the double sense need cause no confusion, and is well suited to express in the shortest manner the quantities dealt with in our investigation.

Furthermore, in case of an inclined girder such as is treated in Prop. V, if the bending moment M, which causes the deflection there treated, be represented, it must appear as an area between two normals to the girder which are at the distance of one unit apart.

In order to apply Prop. V to inclined and curved girders, such as constitute the arch, with entire exactness, one more proposition is needed.

Prop. If weights be sustained by an inclined girder, and the amount of the deflection of this girder, which is caused by the weights, be compared with the deflection of an horizontal girder of the same cross section, and of the same horizontal span, and deflected by the same weights applied in the same verticals; the vertical component of the deflection of the inclined girder, at any point O, is equal to the corresponding vertical deflection of the horizontal girder, multiplied by the secant of the inclination.

For the bending moment of both the inclined girder and the horizontal girder is the same in the same vertical, but the distance along the inclined girder exceeds that along the horizontal girder in the ratio of the secant of the inclination to unity; hence the respective moment areas have this same ratio; therefore the deflections at right angles to the respective girders of their corresponding points are in the ratio of the square of the secant to unity; and the vertical components of the deflections are therefore in the ratio of the secant of the inclination to unity.

In applying this proposition to the graphical construction for the arch, it will be necessary to increase the ordinate of the moment polygon at each point by multiplying by the secant of the inclination of the arch at that point. This is easily effected when the ordinates are vertical by drawing normals at each point of the arch; then the distance along the normal whose vertical component is the bending moment is the value of M to be used in determining the deflection.

In the arches which we have treated the rise is so small a fraction of the span that the secant of the inclination at any point does not greatly exceed unity; or, to state it otherwise, the length of the arch differs by a comparatively small quantity from the actual span. It is a close approximation under such circumstances to use the moments themselves in determining the deflections; and we have so used them in our constructions. A more accurate result can be obtained by multiplying each ordinate by the secant of the inclination of the arch at that point to the horizon.

THE THEORY OF INTERNAL STRESS

IN

GRAPHICAL STATICS.

THE THEORY OF INTERNAL STRESS

IN

GRAPHICAL STATICS.

Stress includes all action and reaction of bodies and parts of bodies by attraction of gravitation, cohesion, electric repulsion, contact, etc., viewed especially as distributed among the particles composing the body or bodies. Since action and reaction are necessarily equal, stress is included under the head of Statics, and it may be defined to be the equilibrium of distributed forces.

Internal stress may be defined as the action and reaction of molecular forces. Its treatment by analytic methods is necessarily encumbered by a mass of formulæ which is perplexing to any except an expert mathematician. It is necessarily so encumbered, because the treatment consists in a comparison of the stresses acting upon planes in various directions, and such a comparison involves transformation of quadratic functions of two or three variables, so that the final expressions contain such a tedious array of direction cosines that even the mathematician dislikes to employ them.

Now, since the whole difficulty really lies in the unsuitability of Cartesian coordinates for expressing relations which are dependent upon the parallelogram of forces, and does not lie in the relations themselves, which are quite simple, and, which no doubt, can be made to appear so in quaternion or other suitable notation; it has been thought by the writer that a presentation of the subject from a graphical stand point would put the entire investigation within the reach of any one who might wish to understand it, and would also be of assistance to those who might wish to read the analytic investigation.

The treatment consists of two principal parts: in the first part the inherent properties of stress are set forth and proved by a general line of reasoning which entirely avoids analysis, and which, it is hoped, will make them well understood; the second part deals with the problems which arise in treating stress. These problems are solved graphically, and if analytic expressions are given for these solutions, such expressions will result from elementary considerations appearing in the graphical solutions. The constructions by which the solutions are obtained are many of them taken from the works of the late Professor Rankine, who employed them principally as illustrations, and as auxiliary to his analytic investigations.

It is thus proposed to render the treatment of stress exclusively graphical, and by so doing to add a branch to the science of Graphical Statics, which has not heretofore been recognized as susceptible of graphical treatment. It seems unnecessary to add a word as to the importance, not to say necessity, to the engineer of a knowledge of the theory of combined internal stress, since all correct designing presupposes such knowledge.

STRESS ON A PLANE.—"If a body be conceived to be divided into two parts by an ideal plane traversing it in any direction, the force exerted between those two parts at the plane of division is *an internal stress*."—*Rankine*.

A STATE OF INTERNAL STRESS is such a state that an internal stress is or may be exerted upon every plane passing through a point at which such a state exists.

It is assumed as a physical axiom that the stress upon an ideal plane of division which traverses any given point of a body, cannot change suddenly, either as to direction or magnitude, while that plane is gradually turned in any way about the given point. It is also assumed as axiomatic that the stress at any point upon a moving plane of division which undergoes no sudden changes of motion, cannot change suddenly either as to direction or amount. A sudden variation can only take place at a surface where there is a change of material.

GENERAL PROPERTIES OF PLANE STRESS.

We shall call that stress *a plane stress* which is parallel to a plane; *e.g.*, let the plane of the paper be this plane and let the stress acting upon every ideal plane which is at right angles to the plane of the paper be parallel to the plane of the paper, then is such a stress a plane stress.

The *obliquity* of a stress is the angle included between the direction of the stress and a line perpendicular to the ideal plane it acts upon. This last plane we shall for brevity call the *plane of action* of the stress, and any line perpendicular to it, its *normal*. In plane stress, the planes of action are shown by their traces on the plane of the paper, and then their normals, as well as their directions, the magnitudes of the stresses, and their obliquities are correctly represented by lines in the plane of the paper.

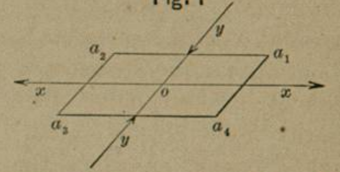
The definition of stress which has been given is equivalent to the statement that stress is *force* distributed over an area in such wise as to be in equilibrium.

In order to measure stress it is necessary to express its amount per unit of area: this is called the *intensity* of the stress.

Stress, like force, can be resolved into components. An oblique stress can be resolved into a component perpendicular to its plane of action called the *normal component*, and a component along the plane called the *tangential component* or *shear*.

When the obliquity is zero, the entire stress is normal stress, and may be either a compression or a tension, *i.e.*, a thrust or a pull. When the obliquity is $\pm 90°$, the stress consists entirely of a tangential stress or shear. If a compression be considered as a positive normal stress, it is possible to consider a normal tension as a stress whose obliquity is $\pm 180°$, and the obliquities of two shears having opposite signs, also differ by 180°.

Fig. 1



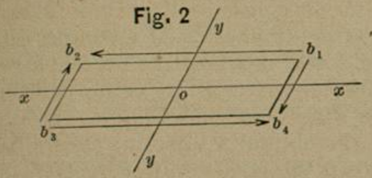
CONJUGATE STRESSES.—If in Fig. 1 any state of stress whatever exists at *o*, and *xx* be the direction of the stress on a plane of action whose trace is *yy*, then is *yy* the direction of the stress at *o* on the plane whose trace is *xx*. Stresses so related are said to be *conjugate stresses*.

For consider the effect of the stress upon a small prism of the body of which $a_1a_2a_3a_4$ is a right section. If the stress is uniform that acting upon a_1a_4 is equal and opposed to that acting upon a_2a_3, and therefore the stress upon these faces of the prism are a pair of forces in equilibrium. Again, the stresses upon the four faces form a system of forces which are in equilibrium, because the prism is unmoved by the forces acting upon it. But when a system of forces in equilibrium is removed from a system in equilibrium, the remaining forces are in equilibrium. Therefore the removal of the pair of stresses in equilibrium acting upon a_1a_4 and a_2a_3 from the system of stresses acting upon the four faces, which are also in equilibrium, leaves the stresses upon a_1a_2 and a_3a_4 in equilibrium. But if the stress is uniform, the stresses on a_1a_2 and a_3a_4 must

be parallel to *yy*, as otherwise a couple must result from these equal but not directly opposed stresses, which is inconsistent with equilibrium.

This proves the fact of conjugate stresses when the state of stress is uniform: in case it varies, the prism can be taken so small that the stress is sensibly uniform in the space occupied by it, and the proposition is true for varying stress in case the prism be indefinitely diminished, as may always be done.

Fig. 2



TANGENTIAL STRESSES.—If in Fig. 2 the stress at *o* on the plane *xx* is in the direction *xx*, *i.e.* the stress at *o* on *xx* consists of a shear only; then there necessarily exists some other plane through *o*, as *yy*, on which the stress consists of a shear only, and the shear upon each of the planes *xx* and *yy* is of the same intensity, but of opposite sign.

For let a plane which initially coincides with *xx* revolve continuously through 180° about *o*, until it again coincides with *xx*, the obliquity of the stress upon this revolving plane has changed gradually during the revolution through an angle of 360°, as we shall show.

Since the obliquity is the same in its final as in its initial position, the total change of obliquity during the revolution is 0° or some multiple of 360°. It cannot be 0°, for suppose the shear to be due to a couple of forces parallel to *xx*, having a positive moment; then if the plane be slightly revolved from its initial position in a plus direction, the stress upon it has a small normal component which would be of opposite sign, if the pair of forces which cause it were reversed or changed in sign; or, what is equivalent to that, the sign of the small normal component would be reversed if the plane be slightly revolved from its initial position in a minus direction. Hence the plane *xx*, on which the stress is a shear alone, separates those planes through *o* on which the obliquity of the stress is greater than 90° from those on which it is less than 90°, *i.e.*, those having a plus normal component from those having a minus normal component.

Since in revolving through +180° the plane must coincide, before it reaches its final position, with a plane which has made a slight minus rotation, it is evident that the sign of the normal component changes at least once during a revolution of 180°. But a quantity can change sign only at zero or infinity, and since an infinite normal component is inadmissible, the normal component must vanish at least once during the proposed revolution. Hence the obliquity is changed by 360° or some multiple of 360° while the plane revolves 180°. In fact the normal component vanishes but once, and the obliquity changes by once 360° only, during the revolution.

It is not in every state of stress that there is a plane on which there is no stress except shear, but, as just shown, when there is one such plane *xx* there is necessarily another *yy*, and all planes through *o* and cutting the angles in which are b_1 and b_3 have normal components of opposite sign from planes through *o* and cutting the angles in which are b_2 and b_4.

To show that the intensity of the shear on *xx* is the same as that on *yy*, consider a prism one unit long and having the indefinitely small right section $b_1b_2b_3b_4$. Let the area of its upper or lower face be $a_1 = b_1b_2$, that of its right or left face be $a_2 = b_2b_3$, then a_1s_1 and a_2s_2 are the total stresses on these respective faces if s_1 and s_2 are the intensities of the respective shears per square unit. Let the angle $xoy = i$, then

$$a_1s_1 \cdot a_2 \sin. i$$

is the moment of the stresses on the upper and lower faces of the prism, and

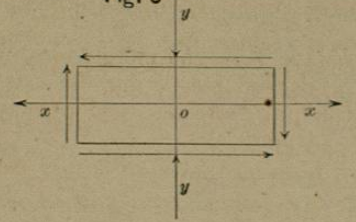
$$a_2s_2 \cdot a_1 \sin. i$$

is the moment of the stresses on the right and left faces; but since the prism is unmoved these moments are equal.

$$\therefore s_1 = s_2$$

These stresses are at once seen to be of opposite sign.

Fig. 3



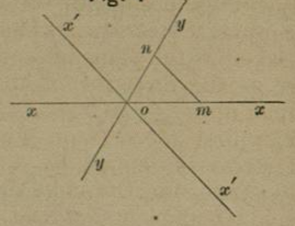
Tangential Components.—In Fig. 3 if *xx* and *yy* are any two planes at right angles to each other, then the intensity at *o* of the tangential component of the stress upon the plane *xx* is necessarily the same as that upon the plane *yy*, but these components are of opposite sign.

For the normal components acting upon the opposite faces of a right prism are necessarily in equilibrium, and by a demonstration precisely like that just employed in connection with Fig. 2 it is seen that for equilibrium it is necessary and sufficient that the intensity of the tangential component on *xx* be numerically equal to that on *yy*, but of opposite sign.

State of Stress.—In a state of plane stress, the state at any point, as *o*, is completely defined, so that the intensity and obliquity of the stress on any plane traversing *o* can be determined, when the intensity and obliquity of the stress on any two given planes traversing that point are known.

For suppose in Fig. 4 that the intensity and obliquity of the stress on the given planes *xx* and *yy* are known, to find that on any plane *x'x'* draw *mn* || *x'x'* then the indefinitely small prism one unit in length whose right section is *mno*, is held in equilibrium by the forces acting upon its three faces. The forces acting upon the faces *om* and *on* are known in direction from the obliquities of the stresses, and, if p_x and p_y are the respective intensities of the known stresses, then the forces are $om.p_x$ and $on.p_y$ respectively. The resultant of these forces and the reaction which holds it in equilibrium, together constitute the stress acting on the face *mn*: this resultant divided by *mn* is the intensity of the stress on *mn* and its direction is that of the stress on *mn* or *x'x'*.

Fig. 4



It should be noticed that the stress at *o* on two planes as *xx* and *yy* cannot be assumed at random, for such assumption would in general be inconsistent with the properties which we have shown every state of stress to possess. For instance we are not at liberty to assume the obliquities and intensities of the stresses on *xx* and *yy* such that when we compute these quantities for any plane *x'x'* and another plane *y'y'* at right angles to *x'x'* in the manner just indicated, it shall then appear that the tangential components are of unequal intensity or of the same sign. Or, again, we are not at liberty to so assume these stresses as to violate the principle of conjugate stresses.

But in case the stresses assumed are conjugate, or consist of a pair of shears of equal intensity and different sign on any pair of planes, or in case any stresses are assumed on a pair of planes at right angles such that their tangential components are of equal intensity but different sign, we know that we have made a consistent assumption and the state of stress is possible and completely defined.

The state of stress is not completely defined when the stress upon a single plane is known, because there may be any amount of simple tension or compression along that plane added to the state of stress without changing either the intensity or obliquity of the stress on that plane.

Principal Stresses.—In any state of stress there is one pair of conjugate stresses at right angles to each other, *i.e.* there are two planes at right angles on which the stresses are normal only. Stresses so related are said to be *principal stresses.*

It has been previously shown that if a plane be taken in any direction, and the direction of the stress acting on it be found, then these are the directions of a pair of conjugate stresses of which either may be taken as the plane of action and the other as the direction of the stress acting upon it.

Consider first the case in which the state of stress is defined by a pair of conjugate stresses of the same sign; *i.e.*, the normal components of this pair of conjugate stresses are both compressions or both tensions.

It is seen that they are of opposite obliquities, and if a plane which initially coincides with one of these conjugate planes of action be continuously revolved until it finally coincides with the other, the obliquity must pass through all intermediate values, one of which is 0°, and when the obliquity is 0° the tangential component of the stress vanishes. But as has been previously shown there is another plane at right angles to this which has the same tangential component; hence the stress is normal on this plane also.

Consider next the case in which the pair of conjugate stresses which define the state of stress are of opposite sign, *i.e.*, the normal component on one plane is a compression and that on the other a tension.

In this case there is a plane in some intermediate position on which the stress is tangential only, for the normal component cannot change sign except at zero. It has been previously shown that in case there is one plane on which the stress is a shear only, there is another plane also on which the stress is a shear only, and that this second shear is of equal intensity with the first but of opposite sign. Let us consider then that the state of stress, in the case we are now treating, is defined by these opposite shears instead of the conjugate stresses at first considered.

Now let a plane which initially coincides with one of the planes of equal shear revolve continuously until it finally coincides with the other. The obliquity gradually changes from +90° to −90°, during the revolution, hence at some intermediate point the obliquity is 0°; and since the tangential component has the same intensity on a plane at right angles to this, that is another plane on which the obliquity of the stress is also 0°.

We have now completely established the proposition respecting the existence of principal stresses which may be restated thus:

Any possible state of stress can be completely defined by a pair of normal stresses on two planes at right angles to each other.

As to the direction of these principal planes and stresses, it is easily seen from considerations of symmetry that in case the state of stress can be defined by equal and opposite shears on a pair of planes, that the principal planes bisect the angles between the planes of equal shear, for there is no reason why they should incline more to one than to the other. We have before shown that the planes of equal shear are planes of separation between those whose stresses have normal components of opposite sign: hence it appears that the principal stresses are of opposite sign in any state of stress which can be defined by a pair of equal and opposite shears on two planes.

It will be hereafter shown how the direction and magnitude of the principal stresses are related to any pair of conjugate stresses.

For convenience of notation in discussing plane stress let us denote *compression* by the sign +, and *tension* by the sign −.

Let us also call that state of stress which is defined by equal principal stresses of the same sign *a fluid stress.* A material fluid can actually sustain only a + fluid stress, but it is convenient to include both compression and tension under one head as fluid stress, the properties of which we shall soon discuss.

Let us call a state of stress which is defined by unequal principal stresses of the same sign *an oblique stress.* This may be taken to include fluid stress as the particular case in which the inequality is infinitesimal. In this state of stress there is no plane on which the stress is a shear only, and the normal component of the stress on any plane whatever has the same sign as that of the principal stresses.

Furthermore let us call that state

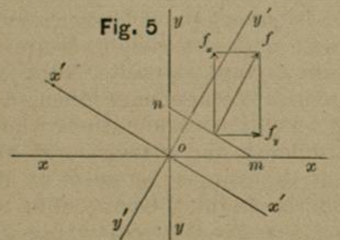
of stress which is defined by a pair of shearing stresses of equal intensity and different sign on two planes at right angles to each other *a right shearing stress.* We shall have occasion immediately to discuss the properties of this kind of stress, but we may advantageously notice one of its properties in this connection. It has been seen previously from considerations of symmetry that the principal stresses and planes which may be used to define this state of stress, bisect the angles between the planes of equal shear. Hence in right shearing stress the principal stresses make angles of 45° with the planes of equal shear. We can advance one step further by considering the symmetrical position of the planes of equal shear with respect to the principal stresses and show that the principal stresses in a state of right shearing stress are equal but of opposite sign.

We wish to call particular attention to fluid stress and to right shearing stress, as with them our subsequent discussions are to be chiefly concerned: they are the special cases in which the principal stresses are of equal intensities, in one case of the same sign, in the other case of different sign.

Let us call a state of stress which is defined by a pair of equal shearing stresses of opposite sign on planes not at right angles *an oblique shearing stress.* The principal stresses, which in this case are of unequal intensity and bisect the angles between the planes of equal shear, are of opposite sign. A right shearing stress may be taken as the particular case of oblique shearing in which the obliquity is infinitesimal.

We may denote a state of stress as + or − according to the sign of its larger principal stress.

Fig. 5



Fluid Stress.—In Fig. 5 let xx and be yy two planes at right angles, on which the stress at o is normal, of equal intensity and of the same sign; then the stress on any plane, as $x'x'$, traversing o is normal, of the same intensity and same sign as that on xx or yy.

For consider a prism a unit long and of infinitesimal cross section having the face $mn \parallel x'x'$, then the forces f_x and f_y, acting on the faces om and on are such that

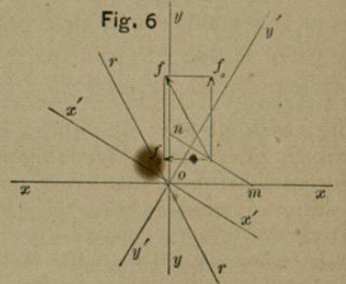
$$f_x : f_y : : om : on.$$

Now $nm = \sqrt{om^2 + on^2}$, and the resultant force which the prism exerts against nm is

$$f = \sqrt{f_x^2 + f_y^2}, \quad \therefore \; f_x : f : : om : mn.$$

But $f_x \div om$ is the intensity of the stress on xx and $f \div mn$ is the intensity of the stress on $x'x'$, and these are equal. Also by similarity of triangles the resultant f is perpendicular to mn.

Fig. 6



Right Shearing Stress.—In Fig. 6, let xx and yy be two planes at right angles to each other, on which the stress is normal, of equal intensity, but of opposite sign; then the stress on any plane, as $x'x'$, traversing o is of the same intensity as that on xx and yy, but its obliquity is such that xx and yy respectively, bisect the angles between the direction rr of the resultant stress, and the normal $y'y'$ to its plane of action.

For, if the intensity of the stress on $x'x'$ be computed in the same manner as in Fig. 5, the intensity is found to be the same as that on xx or yy; for the stresses to be combined are at right angles and are both of the same magnitude. The only difference between this case and that in Fig. 5 is this, that one of the component stresses, that one normal to yy say, has its sign the opposite of that in Fig. 5. In Fig. 5 the stress on $x'x'$ was in the direction $y'y'$, making a certain angle yoy' with yy. In Fig. 6 the resultant stress on $x'x'$ must then make an equal negative angle with yy, so that $yor = yoy'$. Hence the statement which has been made respecting right shearing stress is seen to be thus established.

Combination and Separation.—Any states of stress which coexist at the same point and have their principal stresses in the same directions xx and yy combine to form a single state of stress whose principal stresses are the sums of the respective principal stresses lying in the same directions xx and yy: and conversely any state of stress can be separated into several coexistent stresses by separating each of its two principal stresses into the same number of parts in any manner, and then grouping these parts as pairs of principal stresses in any manner whatever.

The truth of this statement is necessarily involved in the fact that stresses are forces distributed over areas, and that as a state of stress is only the grouping together of two necessarily related stresses, they must then necessarily follow the laws of the composition and resolution of forces.

For the sake of brevity, we shall use the following nomenclature of which the meaning will appear without further explanation.

The terms applied to forces and stresses are:	The terms applied to states of stress are:
Compound,	*Combine,*
Composition,	*Combination,*
Component,	*Component state,*
Resolve,	*Separate,*
Resolution,	*Separation,*
Resultant.	*Resultant state.*

Other states of stress can be combined besides those whose principal stresses coincide in direction, but the law of combination is less simple than that of the composition of forces; such combinations will be treated subsequently.

Component Stresses.—Any possible state of stress defined by principal stresses whose intensities are p_x and p_y on the planes xx and yy respectively is equivalent to a combination of the fluid stress whose intensity is $+\frac{1}{2}(p_x + p_y)$ on each of the planes xx and yy respectively, and the right shearing stress whose intensity is $+\frac{1}{2}(p_x - p_y)$ on xx and $-\frac{1}{2}(p_x - p_y)$ on yy.

For as has been shown, the resultant stress due to combining the fluid stress with the right shearing stress is found by compounding their principal stresses. Now the stress on xx is

$$\tfrac{1}{2}(p_x + p\ ) + \tfrac{1}{2}(p_x - p_y) = p_x$$

and that on yy is

$$\tfrac{1}{2}(p_x + p_y) - \tfrac{1}{2}(p_x - p_y) = p_y$$

and hence these systems of principal stresses are mutually equivalent

In case $p_y = 0$, the stress is completely defined by the single principal stress p_x, which is a simple normal compression or tension on xx. Such a stress has been called a *simple stress.*

A fluid stress and a right shearing stress which have equal intensities combine to form a simple stress.

It is seen that the definition of a state of stress by its principal stresses, is a definition of it as a combination of two simple stresses which are perpendicular to each other.

There are many other ways in which any state of stress can be separated into component stresses, though the separation into a fluid stress and a right shearing stress has thus far proved more useful than any other, hence most of our graphical treatment will depend upon it. It may be noticed as an instance of a different separation, that it was shown that the tangential components of the stresses on any pair of planes xx and yy at right angles to each other are of equal intensity but opposite sign. These tangential components, then, together form a right shearing stress whose principal planes and stresses $x'x'$ and $y'y'$ bisect the angles between xx and yy, while the normal components together define a state of stress whose principal stresses are, in general, of unequal intensity.

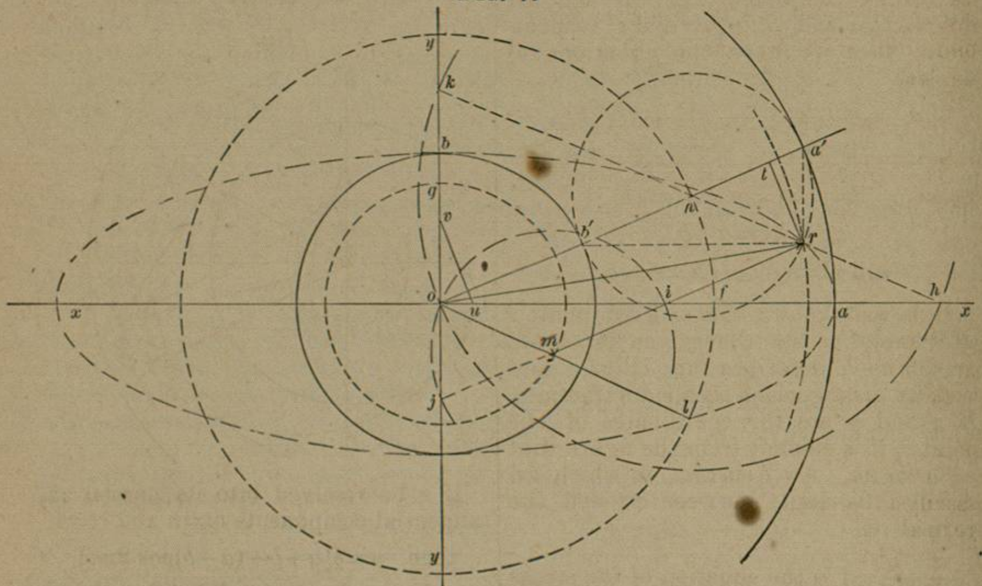
Hence any state of stress can be separated into component stresses one of which is a right shearing stress on any two planes at right angles and a stress having those planes for its principal planes.

The fact of the existence of conjugate stresses points to still another kind of separation into component stresses.

PROBLEMS IN PLANE STRESS.

PROBLEM 1.—When a state of stress is defined by principal stresses which are of unequal intensity and like sign, *i.e.*, in a state of oblique stress, to find the intensity and obliquity of the stress at o on any assumed plane in the direction uv.

FIG. 7.



In Fig. 7 let the principal stresses at o be a on yy and b on xx; and on some convenient scale of intensities let $oa=a$ and $ob=b$. Let uv show the direction of the plane through o on which we are to find the stress, and make on perpendicular uv. Make $oa'=oa$ and $ob'=ob$. Bisect $a'b'$ at n, then $on=\frac{1}{2}(a+b)$ and $na'=\frac{1}{2}(a-b)$. Make $xol=xon$ and complete the paralellogram $nomr$; then is the diagonal $or=r$ the resultant stress on the given plane in direction and intensity.

The point r can also be obtained more simply by drawing $b'r \parallel xx$ and $a'r \parallel yy$.

We now proceed to show the correctness of the constructions given and to discuss several interesting geometrical properties of the figure which give to it a somewhat complicated appearance, which complexity is, however, quite unnecessary in actual construction, as will be seen hereafter. It has been shown that a state of stress defined by its two principal stresses a and b can be separated into a fluid stress having a normal intensity $\frac{1}{2}(a+b)$ on every plane, and a right shearing stress whose principal stresses are $+\frac{1}{2}(a-b)$ and $-\frac{1}{2}(a-b)$ respectively.

Since the fluid stress causes a normal stress on any given plane, its intensity is rightly represented by $on=\frac{1}{2}(a+b)$, which is the amount of force distributed over one unit of the given plane. Since, further, it was shown that a right shearing stress causes on any plane a stress with an obliquity such that the principal stress bisects the angle between its direction and the normal to the plane, and causes a stress of the same intensity on every plane, we see that $om=\frac{1}{2}(a-b)$ represents, in direction and amount, the force distributed over one unit of the given plane which is due to the right shearing stress.

To find the resultant stress we have only to compound the forces on and om, which give the resultant $or=r$.

The obliquity nor is always toward the greater principal stress, which is here assumed to be a.

It is seen that in finding r by this method it is convenient to describe one circle about o with a radius $of=\frac{1}{2}(a+b)$ and another with a radius $og=\frac{1}{2}(a-b)$, after which any parallelogram mn can be readily completed. Let nr and mr intersect xx and yy in hk and ij respectively; then we have the equations of angles,

$$noh=nho=\tfrac{1}{2}kno,\quad nok=nko=\tfrac{1}{2}hno,$$
$$moi=mio=\tfrac{1}{2}jmo,\quad moj=mjo=\tfrac{1}{2}imo,$$

hence $$hn=kn=on=\tfrac{1}{2}(a+b)$$
$$\therefore hk=a+b,$$
$$\text{and } rk=rj=a,\ rh=ri=b.$$

It is well known that a fixed point r on a line of constant length as $hk=a+b$, or $ij=a-b$ describes an ellipse, and such an arrangement is called a trammel. If x and y are the coordinates of the point r, it is evident from the figure that $x=a\cos xn$, $y=b\sin xn$, in which xn signifies the angle between xx and the normal on.

$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of the stress ellipse which is the locus of r; and xn is then the eccentric angle of r. Also, since $noh=nho$, $nb'r=nrb'$; hence $b'r \parallel xx$ and $a'r \parallel yy$ determine r.

In this method of finding r it is convenient to describe circles about o with radii a and b, and from a' and b' where the normal of the given plane intersects them find r.

We shall continue to use the notation employed in this problem, so far as applicable, so that future constructions may be readily compared with this. It will be convenient to speak of the angle xon as xn, nor as nr, etc.

PROBLEM 2.—When a state of stress is defined by principal stresses of unequal intensity and unlike sign, *i.e.* in a state of oblique shearing stress, to find the intensity and obliquity of the stress at o on any assumed plane having the direction uv.

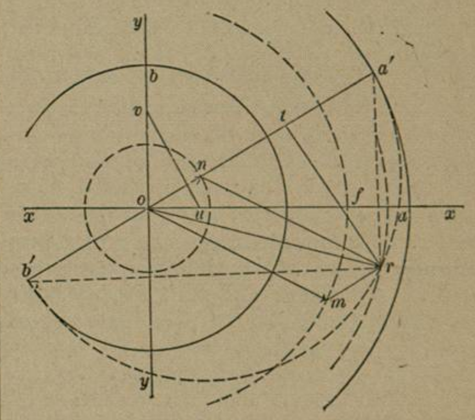
In Fig. 8 the construction is effected according to both the methods detailed in Problem 1, and it will be at once apprehended from the identity of notation.

Since a and b are of unlike signs $a+b=on$ is numerically less than $a-b=a'b'$.

The results of these two problems are expressed algebraically thus:

$$r^2=\tfrac{1}{4}(a+b)^2+\tfrac{1}{4}(a-b)^2+\tfrac{1}{2}(a^2-b^2)\cos 2xn$$
$$\therefore r^2=\tfrac{1}{2}[a^2+b^2+(a^2-b^2)\cos 2xn]$$
$$\text{or, } r^2=a^2\cos^2 xn+b^2\sin^2 xn.$$

FIG. 8.



If r be resolved into its normal and tangential components $ot=n$ and $rt=t$

$$\text{then, } n=\tfrac{1}{2}[a+b+(a-b)\cos 2xn],$$
$$\text{or, } n=a\cos^2 xn+b\sin^2 xn,$$

and,

$$t=\tfrac{1}{2}(a-b)\sin 2xn=(a-b)\sin xn\cos xn.$$

It is evident from the value of the normal component n, that the sum of the normal components on any two planes at right angles to each other is the same and its amount is $a+b$: this is also a general property of stress in addition to those previously enumerated.

Also $\tan nr=\frac{t}{n}=\frac{a-b}{a\cot xn+b\tan xn}$

The obliquity nr can also be found from the proportion

$$\sin nr : \tfrac{1}{2}(a-b) :: \sin 2xn : r.$$

In the case of fluid stress the equations reduce to the more simple forms:

$$a=b=r=n,\ t=0$$

For right shearing stress they are:

$$a=-b=\pm r,\ n=\pm a\cos rn,$$
$$t=\pm a\sin rn,\qquad rn=2\,xn.$$