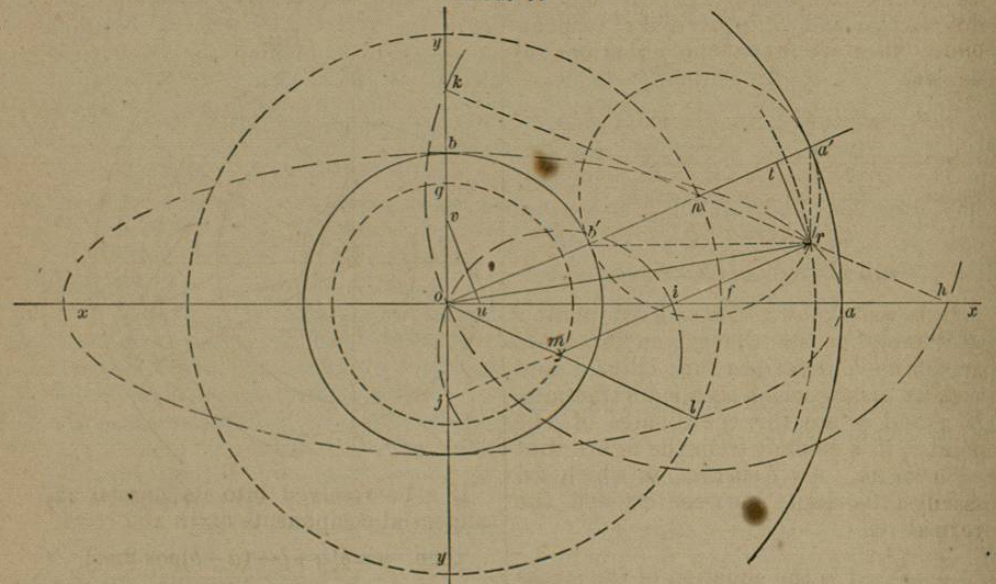
Hence any state of stress can be separated into component stresses one of which is a right shearing stress on any two planes at right angles and a stress having those planes for its principal planes.

The fact of the existence of conjugate stresses points to still another kind of separation into component stresses.

PROBLEMS IN PLANE STRESS.

PROBLEM 1.—When a state of stress is defined by principal stresses which are of unequal intensity and like sign, *i.e.*, in a state of oblique stress, to find the intensity and obliquity of the stress at o on any assumed plane in the direction uv.

FIG. 7.



In Fig. 7 let the principal stresses at o be a on yy and b on xx; and on some convenient scale of intensities let $oa=a$ and $ob=b$. Let uv show the direction of the plane through o on which we are to find the stress, and make on perpendicular uv. Make $oa'=oa$ and $ob'=ob$. Bisect $a'b'$ at n, then $on=\frac{1}{2}(a+b)$ and $na'=\frac{1}{2}(a-b)$. Make $xol=xon$ and complete the paralellogram $nomr$; then is the diagonal $or=r$ the resultant stress on the given plane in direction and intensity.

The point r can also be obtained more simply by drawing $b'r \parallel xx$ and $a'r \parallel yy$.

We now proceed to show the correctness of the constructions given and to discuss several interesting geometrical properties of the figure which give to it a somewhat complicated appearance, which complexity is, however, quite unnecessary in actual construction, as will be seen hereafter. It has been shown that a state of stress defined by its two principal stresses a and b can be separated into a fluid stress having a normal intensity $\frac{1}{2}(a+b)$ on every plane, and a right shearing stress whose principal stresses are $+\frac{1}{2}(a-b)$ and $-\frac{1}{2}(a-b)$ respectively.

Since the fluid stress causes a normal stress on any given plane, its intensity is rightly represented by $on=\frac{1}{2}(a+b)$, which is the amount of force distributed over one unit of the given plane. Since, further, it was shown that a right shearing stress causes on any plane a stress with an obliquity such that the principal stress bisects the angle between its direction and the normal to the plane, and causes a stress of the same intensity on every plane, we see that $om=\frac{1}{2}(a-b)$ represents, in direction and amount, the force distributed over one unit of the given plane which is due to the right shearing stress.

To find the resultant stress we have only to compound the forces on and om, which give the resultant $or=r$.

The obliquity nor is always toward the greater principal stress, which is here assumed to be a.

It is seen that in finding r by this method it is convenient to describe one circle about o with a radius $of=\frac{1}{2}(a+b)$ and another with a radius $og=\frac{1}{2}(a-b)$, after which any parallelogram mn can be readily completed. Let nr and mr intersect xx and yy in hk and ij respectively; then we have the equations of angles,

$$noh=nho=\tfrac{1}{2}kno,\quad nok=nko=\tfrac{1}{2}hno,$$

$$moi=mio=\tfrac{1}{2}jmo,\quad moj=mjo=\tfrac{1}{2}imo,$$

hence $$hn=kn=on=\tfrac{1}{2}(a+b)$$

$$\therefore hk=a+b,$$

$$\text{and } rk=rj=a,\ rh=ri=b.$$

It is well known that a fixed point r on a line of constant length as $hk=a+b$, or $ij=a-b$ describes an ellipse, and such an arrangement is called a trammel. If x and y are the coordinates of the point r, it is evident from the figure that $x=a\cos xn$, $y=b\sin xn$, in which xn signifies the angle between xx and the normal on.

$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of the stress ellipse which is the locus of r; and xn is then the eccentric angle of r. Also, since $noh=nho$, $nb'r=nrb'$; hence $b'r \parallel xx$ and $a'r \parallel yy$ determine r.

In this method of finding r it is convenient to describe circles about o with radii a and b, and from a' and b' where the normal of the given plane intersects them find r.

We shall continue to use the notation employed in this problem, so far as applicable, so that future constructions may be readily compared with this. It will be convenient to speak of the angle xon as xn, nor as nr, etc.

PROBLEM 2.—When a state of stress is defined by principal stresses of unequal intensity and unlike sign, *i.e.* in a state of oblique shearing stress, to find the intensity and obliquity of the stress at o on any assumed plane having the direction uv.

In Fig. 8 the construction is effected according to both the methods detailed in Problem 1, and it will be at once apprehended from the identity of notation.

Since a and b are of unlike signs $a+b=on$ is numerically less than $a-b=a'b'$.

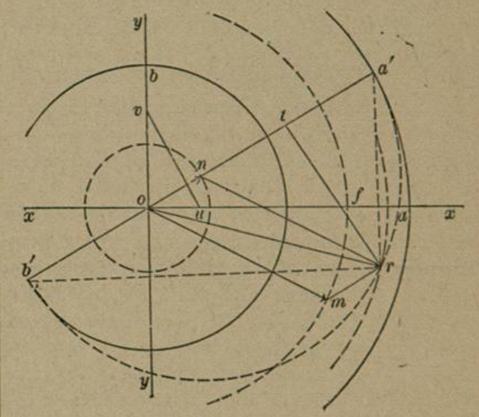
The results of these two problems are expressed algebraically thus:

$$r^2=\tfrac{1}{4}(a+b)^2+\tfrac{1}{4}(a-b)^2+\tfrac{1}{2}(a^2-b^2)\cos 2xn$$

$$\therefore r^2=\tfrac{1}{2}[a^2+b^2+(a^2-b^2)\cos 2xn]$$

$$\text{or, } r^2=a^2\cos^2 xn+b^2\sin^2 xn.$$

FIG. 8.



If r be resolved into its normal and tangential components $ot=n$ and $rt=t$

$$\text{then, } n=\tfrac{1}{2}[a+b+(a-b)\cos 2xn],$$

$$\text{or, } n=a\cos^2 xn+b\sin^2 xn,$$

and,

$$t=\tfrac{1}{2}(a-b)\sin 2xn=(a-b)\sin xn\cos xn.$$

It is evident from the value of the normal component n, that the sum of the normal components on any two planes at right angles to each other is the same and its amount is $a+b$: this is also a general property of stress in addition to those previously enumerated.

Also $\tan nr=\frac{t}{n}=\frac{a-b}{a\cot xn+b\tan xn}$

The obliquity nr can also be found from the proportion

$$\sin nr : \tfrac{1}{2}(a-b) :: \sin 2xn : r.$$

In the case of fluid stress the equations reduce to the more simple forms:

$$a=b=r=n,\ t=0$$

For right shearing stress they are:

$$a=-b=\pm r,\ n=\pm a\cos rn,$$

$$t=\pm a\sin rn,\qquad rn=2\,xn.$$

And for simple stress they become:

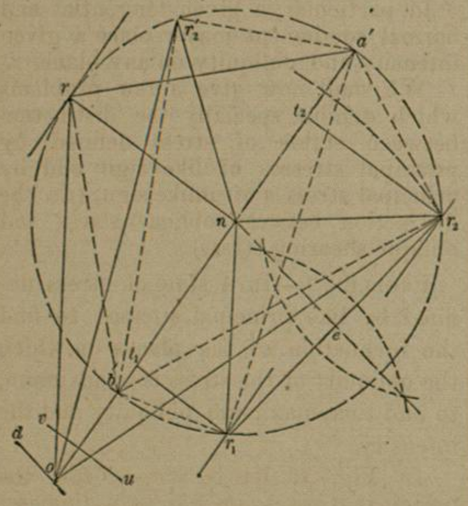
$$b=0,\ r=a\cos rn,\ n=a\cos^2 rn,$$
$$t=a\sin rn\cos rn,\ rn=xn.$$

PROBLEM 3.—In any state of stress defined by its principal stresses, a and b, to find the obliquity and plane of action of the stress having a given intensity r intermediate between the intensities of the principal stresses.

To find the obliquity nr and the direction uv let Fig. 7 or 8 be constructed as follows: assume the direction uv and its normal on, and proceed to determine the position of the principal axes with respect to it. Lay off $oa'=a$, $ob'=b$, in the same direction if the intensities are of like sign, in opposite directions if unlike. Bisect $a'b'$ at n, and on $a'b'$ as a diameter draw the circle $a'rb'$. Also, about o as a center and with a radius $or=r$ draw a circle intersecting that previously drawn at r; then is nr the required obliquity; and $xx \parallel b'r$, $yy \parallel a'r$ are the directions of the principal stresses with respect to the normal on.

PROBLEM 4.—In a state of stress defined by two given obliquities and intensities, to find the principal stresses, and the relative position of their planes of action to each other and to the principal stresses.

FIG. 9.



In Fig. 9 let nr_1, nr_2 be the given obliquities measured from the same normal on, and $or_1=r_1$, $or_2=r_2$ the given intensities. As represented in the figure these intensities are of the same sign, but should they have different signs, it will be necessary to measure one of them from o in the opposite direction, for a change of sign is equivalent to increasing the obliquity by 180°, as was previously shown.

Join r_1r_2 and bisect it by a perpendicular which intersects the common normal at n. About n describe a circle $r_1r_2a'b'$; then $oa'=a$, $ob'=b$, $a'r_1$, $b'r_1$, are the directions of the principal stresses with respect to r_1 and $b'r_2$, $a'r_2$ with respect to r_2, *i.e.*, $ob'r_1=xn_1$ and $ob'r_2=xn_2$

$$\therefore\ n_1n_2=ob'r_2-ob'r_1=r_2b'r_1=r_2a'r_1$$

In case the given obliquities are of opposite sign, as they must be in conjugate stresses, for example, it is of no consequence, in so far as obtaining principal stresses a and b is concerned, whether these given obliquities are constructed on the same side of on, or on opposite sides of it; for a point on the opposite side of on, as r_2', and symmetrically situated with respect to r_2, must lie on the same circle about n. But in case opposite obliquities are on the same side of on we have $n_1n_2=ob'r_1+ob'r_2=r_2b'r_2'$.

It is unnecessary to enter into the proof of the preceding construction as its correctness is sufficiently evident from preceding problems.

The algebraic relationships may be written as follows:

$$\tfrac{1}{4}(a-b)^2=\tfrac{1}{4}(a+b)^2+r_1^2-r_1(a+b)\cos n_1r_1$$
$$\tfrac{1}{4}(a-b)^2=\tfrac{1}{4}(a+b)+r_2^2-r^2(a+b)\cos n_2r_2$$
$$\therefore\ (a+b)(r_1\cos n_1r_1-r_2\cos n_2r_2)=r_1^2-r_2^2$$

Also
$$(a-b)\cos 2xn_1+a+b=2r_1\cos n_1r_1$$
$$(a-b)\cos 2xn_2+a+b=2r_2\cos n_2r_2$$

which last equations express twice the respective normal components, and from them the values of xn_1 and xn_2 can be computed.

PROBLEM 5.—If the state of stress be defined by giving the intensity and obliquity of the stress on one plane, and its inclination to the principal stresses, and also the intensity of the stress on a second plane and its inclination to the principal stresses, to find the obliquity of

the stress on the second plane, and the magnitude of the principal stresses.

Let the construction in Fig. 9 be effected thus: from the common normal on lay off or_1 to represent the obliquity and intensity of the stress on the first plane; draw od so that $nod=xn_2-xn_1$ the difference of the given inclinations of the normals of the two planes; through r_1 draw r_1r_2 perpendicular to od; about o as a center describe a circle with radius r_2 the given intensity on the second plane, and let it intersect r_1r_2 at r_2 or r_2', then is nr_2 the required obliquity. This is evident, because

$$xn_1=nb'r_1=\tfrac{1}{2}a'nr_1,\ xn_2=nb'r_2=\tfrac{1}{2}a'nr_2,$$
$$\therefore\ nod=one=\tfrac{1}{2}(onr_1+onr_2)$$
$$=180^\circ-(xn_2-xn_1)$$

If xn_1 and xn_2 are of different sign care must be taken to take their algebraic sum.

The construction is completed as in Problem 4.

PROBLEM 6.—In a state of stress defined by two given obliquities and either both of the normal components or both of the tangential components of the intensities, to find the principal stresses and the relative position of the two planes of action.

If in Fig. 9 the obliquities nr_1, nr_2, and the normal components $ot_1=n_1$, $ot_2=n_2$ are given, draw perpendiculars at t_1 and t_2 intersecting or_1 and or_2 at r_1 and r_2 respectively.

If the tangential components $t_1r_1=t_1$ and $t_2r_2=t_2$ are given instead of the normal components, draw at these distances parallels to on which intersect or_1 or_2 at r_1r_2 respectively. Complete the construction in the same manner as before.

PROBLEM 7.—In a state of stress defined by its principal stresses a and b, to find the positions and obliquities of the stresses on two planes at right angles to each other whose stresses have a given tangential component t.

Fig. 9, slightly changed, will admit of the required construction as follows: lay off on the same normal on, $oa'=a$, $ob'=b$; bisect $a'b'$ at n; erect a perpendicular $ne=t$ to $a'b'$ at n; draw through e a parallel r_1r_2 to on intersecting or_1 and or_2 at r_1 and r_2 respectively. Then the stresses $or_1=r_1$, $or_2=r_2$ have equal tangential components, and as previously shown these belong to planes at right angles to each other provided these tangential components are of opposite sign. So that when we find the position of the planes of action, one obliquity, as nr_2, must be taken on the other side of on, as nr_2'. The rest of the construction is the same as that already given.

PROBLEM 8.—In a state of stress defined by its principal stresses, to find the intensities, obliquities and planes of action of the stresses which have maximum tangential components.

In Fig. 9 make $oa'=a$, $ob'=b$ and describe a circle on $a'b'$ as a diameter; then the maximum tangential component is evidently found by drawing a tangent at r parallel to on, in which case $t=a-b$, and rb', ra' the directions of the principal stresses make angles of 45° with on, which may be otherwise stated by saying that the planes of maximum tangential stress bisect the angles between the principal stresses; or conversely the principal stresses bisect the angles between the pair of planes at right angles to each other on which the tangential stress is a maximum.

It is unnecessary to extend further the list of problems involving the relations just employed as they will be readily solved by the reader.

In particular, a given tangential and normal component may replace a given intensity and obliquity on any plane.

We shall now give a few problems which exhibit specially the distinction between states of stress defined by principal stresses of like sign and by principal stresses of unlike sign, (*i.e.* the distinction between oblique stress and oblique shearing stress).

PROBLEM 9.—In a state of stress defined by like principal stresses, to find the inclination of the planes on which the obliquity of the stress is a maximum, to find this maximum obliquity and the intensity.

In Fig. 10 let $oa'=a$, $ob'=b$, the principal stresses; on $a'b'$ as a diameter describe a circle; to it draw the tangent or_0; then nr_0 is the required maximum obliquity and or_0 the required intensity. It is evident from inspection that in the given state of stress there can be no greater obliquity than nr_0. The directions of the principal axes are $b'r_0$, $a'r_0$ as has been before shown.

There are two planes of maximum obliquity, and or_0' represents the second; they are situated symmetrically about the principal axes.

Bisect nr_0 by the line od, then

$$oa'r_0=yn \therefore onr_0=2yn, \quad \text{but}$$

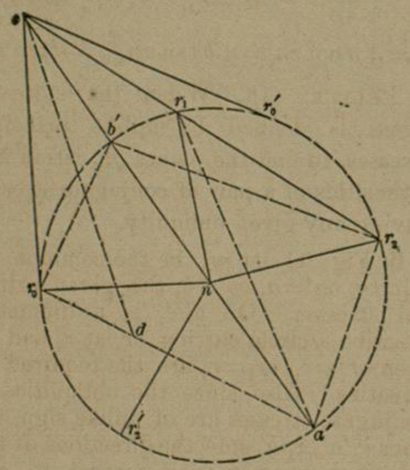
$$onr_0+nor_0=90° \text{ or, } 2yn+nr_0=90°$$

$$\therefore \tfrac{1}{2}nr_0+yn=45°, \quad \text{but}$$

$$odr_0=doa'+oa'd \therefore odr_0=45°,$$

hence the line bisecting the angle of maximum obliquity bisects also the angle between the principal axes. This is the best test for the correctness of the final position of the planes of maximum obliquity with reference to the principal axes.

Fig. 10.



Problem 10.—In a state of stress defined by its maximum obliquity and the intensity at that obliquity, to find the principal stresses.

In Fig. 10 measure the obliquity nr_0 from the normal on and at the extremity of $or_0=r_0$ erect a perpendicular intersecting the normal at n. Then complete the figure as before. The principal axes make angles of 45° at o with od which bisects the obliquity nr_0.

The algebraic statement of Problems 9 and 10 is:

$$\sin nr_0=\frac{a-b}{a+b}=-\cos 2xn,\ r_0^2=ab.$$

$$r_0=a \cot xn=b \tan xn, \quad \therefore \quad a=b \tan^2 xn$$

The normal and tangential components are:

$$n_0=\frac{2r_0^2}{a+b}, \qquad t_0=\frac{r_0(a-b)}{a+b}.$$

Problem 11.—When the state of stress is defined by like principal stresses, to find the planes of action and intensities of a pair of conjugate stresses having a given common obliquity less than the maximum.

In Fig. 10 let $nr_1=nr_2$ be the given obliquity; describe a circle on $a'b'$ as a diameter; then $or_1=r_1$, $or_2=r_2$ are the required intensities. The lines $a'r_1$, $b'r_1$ show the directions of the principal axes with respect to or_1, and $a'r'_2$, $b'r'_2$ with respect to $or_2'=or_2$. The obliquities of conjugate stresses are of opposite sign, and for that reason r_2' is employed for finding the position of the principal stresses. The algebraic expression of these results can be obtained at once from those in Problem 4.

Problem 12.—When the state of stress is defined by the intensities and common obliquity of a pair of like conjugate stresses, to find the principal stresses and maximum obliquity.

This is the case of Problem 4, so far as finding the principal stresses is concerned, and the maximum obliquity is then found by Problem 9. The construction is given in Fig. 10.

Problem 13.—Let the maximum obliquity of a state of oblique stress be given, to find the ratio of the intensities of the pair of conjugate stresses having a given obliquity less than the maximum.

In Fig. 10 let nr_0 be the given maximum obliquity, and $n r_1$ the given obliquity of the conjugate stresses. At any convenient point on or_0, as r_0 erect the perpendicular r_0n, and about n (its point of intersection with on) as a center describe a circle with a radius nr_0 which

cuts nr_1 at r_1 and r_2; then $or_1 \div or_2=r_1 \div r_2$ is the required ratio.

It must be noticed that the scale on which or_1 and or_2 are measured is unknown, for the magnitude of the principal stresses is unknown although their ratio is $ob' \div oa'$. In order to express these results in formulæ, let r represent either of the conjugate stresses, then as previously seen

$$\tfrac{1}{4}(a-b)^2=\tfrac{1}{4}(a+b)^2+r^2-r(a+b) \cos nr$$

$$\therefore\ 2r=(a+b)\cos nr \pm [(a+b)^2\cos^2 nr-4ab]^{1/2}$$

Call the two values of r, r_1 and r_2; and as previously shown $r_0^2=r_1r_2$; also

$$\cos.\ nr_0=r_0 \div \tfrac{1}{2}(a+b)$$

$$\therefore \frac{r_1}{r_2}=\frac{\cos nr - (\cos^2 nr-\cos^2 nr_0)^{1/2}}{\cos nr + (\cos^2 nr-\cos^2 nr_0)^{1/2}}$$

When $nr=0$ the ratio becomes

$$\frac{b}{a}=\frac{1-\sin nr_0}{1+\sin nr_0}$$

Problem 14.—In a state of stress defined by unlike principal stresses, to find the inclination of the planes on which the stress is a shear only, and to find its intensity.

In Fig. 11 let $oa'=a$, $ob'=b$, the given principal stresses of unlike sign; on $a'b'$ as a diameter describe a circle; at o erect the perpendicular or_0 cutting the circle at r_0; then is $or_0=r_0$ the required intensity, and $b'r_0$, $a'r_0$ are the directions of the principal stresses.

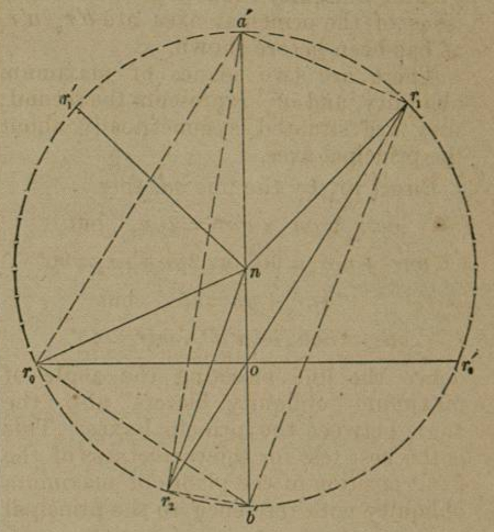
It is evident from inspection that there is no other position of r_0 except r_0' which will cause the stress to reduce to a shear alone. Hence as previously stated the principal stresses bisect the angles between the planes of shear.

Problem 15.—In a state of stress defined by the position of its planes of shear and the common intensity of the stress on these planes, to find the principal stresses.

In Fig. 11 let $or_0=r_0$, the common intensity of the shear, and $or_0b'=xn$, $or_0a'=yn$ the given inclinations of a plane of shear; then $oa'=a$ and $ob'=b$, the principal stresses.

The algebraic statement of Problems 14 and 15, when n_0 denotes the normal to a plane of shear, is:

Fig. 11.



$$\frac{a+b}{a-b}=-\cos 2xn_0, \qquad r_0^2=-ab=t_0^2$$

$$r_0=\pm a \cot xn_0=\pm b \tan xn_0, a=-b\tan^2 xn_0$$

Problem 16.—When the state of stress is defined by unlike principal stresses, to find the planes of action and intensities of a pair of conjugate stresses having any given obliquity.

In Fig. 11 let nr_1 be the common obliquity, $oa'=a$, $ob'=b$, the given principal stresses. On $a'b'$, as a diameter, describe a circle cutting or_1 at r_1 and r_2; then $or_1=r_1$, $or_2=r_2$ are the required intensities. Also, since the obliquities of conjugate stresses are of unlike sign, the lines $r_1'a'$, $r_1'b'$ show the directions of the principal stresses with respect to on_1, and r_2a', r_2b' with respect to on_2.

Problem 17.—When the state of stress is defined by the intensities and common obliquities of unlike conjugate stresses, to find the principal stresses and planes of shear.

In finding the principal stresses this problem is constructed as a case of Problem 4, and then the planes of shear are found by Problem 14. The construction is given in Fig. 11.

Problem 18.—Let the position of the

planes of shear be given in a state of oblique shearing stress, to find the ratio of the intensities of a pair of conjugate stresses having any given obliquity.

In Fig. 11 at any convenient point r_0 make $or_0b'=xn$, $or_0a'=yn$, the given angles which fix the position of the planes of shear. On $a'b'$ as a diameter describe a circle; make nr_1 equal to the common obliquity of the conjugate stresses; then is $or_1 \div or_2 = r_1 \div r_2$ the ratio required.

The ratio may be expressed as in Problem 13, and after reducing by the relations

$$r_0^2=-ab, \quad r_0 \div \tfrac{1}{2}(a+b)=-\tan 2xn,$$

we have,

$$\frac{r_1}{r_2}=\frac{\cos nr + (\cos^2 nr+\tan^2 2xn_0)^{1/2}}{\cos nr - (\cos^2 nr+\tan^2 2xn_0)^{1/2}}$$

When $nr=0$ the ratio becomes

$$-\frac{a}{b}=\frac{1+\cos 2xn_0}{1-\cos 2xn_0}$$

COMBINATION AND SEPARATION OF STATES OF STRESS.

Problem 19.—When two given states of right shearing stress act at the same point, and their principal stresses have a given inclination to each other, to combine these states of stress and find the resultant state.

In Fig. 12 let ox_1, ox_2 denote the directions of the two given principal + stresses, and let $a_1=on_1$, $a_2=on_2$ represent the position and magnitude of these principal stresses. Since the given stresses are right shearing stresses $a_1=-b_1$, $a_2=-b_2$ and the respective planes of shear bisect the angles between the principal stresses. Now it has been previously shown that the intensity of the stress caused by the principal stresses $a_1=-b_1$ is the same on every plane traversing o: the same is true of the principal stresses $a_2=-b_2$: hence, when combined, they together produce a stress of the same intensity on every plane traversing o. This resultant state of stress evidently does not cause a normal stress on every plane, hence the resultant state must be a right shearing stress.

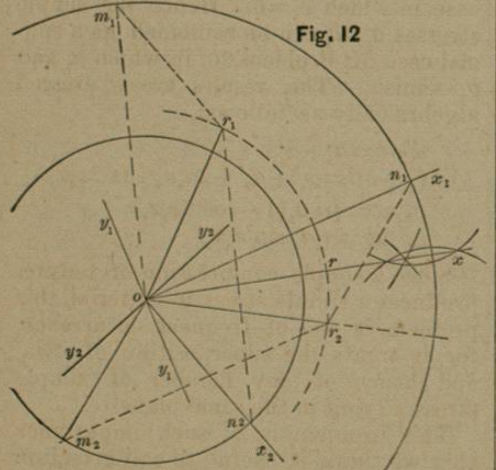


Fig. 12

Let us find its intensity as follows: The principal stresses $a_1=-b_1$ cause a stress on_1 on the plane y_2y_2, and the principal stresses $a_2=-b_2$ cause a stress om_2 on the same plane in such a direction that $x_2om_2=x_1ox_2$, as has been before shown. Complete the parallelogram $n_1om_2r_2$; then or_2 represents the intensity and direction of the stress on y_2y_2. But the principal stresses bisect the angles between the normal and the resultant intensity, therefore, ox, which bisects x_2or_2, is the direction of a principal stress of the resultant state, and $or=or_2=a$ is the intensity of the resultant stress on any plane through o.

The same result is obtained by finding the stress the plane y_1y_1, in which case we have $on_2=a_2$ acting normal to the plane, and $om_1=a_1$ in such a direction that $x_1om_1=x_2ox_1$. The sides and angles of $n_2om_1r_1$ and $n_1om_2r_2$ are evidently equal, hence the resultants are the same, $or_1=or_2=a$, and ox bisects x_2or_1.

The algebraic solution of the problem is expressed by the equation,

$$a^2=a_1^2+a_2^2+2a_1a_2 \cos 2\, x_1x_2,$$

from which a may be found, and, finally, the position of or is found from the proportion,

$$\sin 2xx_1 : a_2 :: \sin 2xx_2 : a_1 :: \sin 2x_1x_2 : a.$$

Problem 20.—When any two states of stress, defined by their principal stresses, act at the same point, and their principal stresses have a given inclination to each other, to combine these states and find the resultant state.

Let a_1, b_1, and a_2, b_2 be the given principal stresses, of which a_1 and a_2 have the same sign and are inclined at a known angle x_1x_2, but in so taking a_1 and a_2 they may not both be numerically greater than b_1 and b_2 respectively.

Separate the pair of principal stresses a_1b_1 into the fluid stress $+\frac{1}{2}(a_1+b_1)$, and the right shearing stress $\pm\frac{1}{2}(a_1-b_1)$ as has been previously done; and in a similar manner the principal stresses $a_2\, b_2$ into $+\frac{1}{2}(a_2+b_2)$ and $\pm\frac{1}{2}(a_2-b_2)$. Then the combined fluid stresses produce a fluid stress of $+\frac{1}{2}(a_1+b_1+a_2+b_2)$ on every plane through o; and the combined right shearing stresses cause a stress whose intensity and position can be found by Problem 19.

The total stress is obtained by combining the total fluid stress with the resultant right shearing stress.

Of course, any greater number of states of stress than two, can be combined by this problem by combining the resultant of two states with a third state and so on.

The algebraic expression of the combination of any two states of stress is as follows:

$$(a+b)=(a_1+b_1+a_2+b_2),$$

$$(a-b)^2=(a_1-b_1)^2+(a_2-b_2)^2 + 2(a_1-b_1)(a_2-b_2)\cos 2x_1x_2,$$

$$\therefore a=\tfrac{1}{2}(a_1+b_1+a_2+b_2+[(a_1-b_1)^2 + (a_2-b_2)^2+2(a_1-b_1)(a_2-b_2)\cos 2x_1x_2]^{1/2}),$$

$$b=\tfrac{1}{2}(a_1+b_1+a_2+b_2-[(a_1-b_1)^2+(a_2-b_2)^2 + 2(a_1-b_1)(a_2-b_2)\cos 2x_1x_2]^{1/2}),$$

in which a and b are the resultant principal stresses. Also, $\sin 2xx_1 : a_2-b_2$

$$:: \sin 2xx_2 : a_1-b_1 :: \sin 2x_1x_2 : a-b.$$

Problem 21.—In a state of stress defined by the stresses upon two planes at right angles to each other, to find the principal stresses.

Let the given stresses be resolved into tangential and normal components; it has been shown that the tangential components upon these planes are of equal intensity and unlike sign. Let the intensity of the tangential component be a_t, and that of the normal components a_n and b_n respectively. The tangential components together constitute a state of right shearing stress of which the given planes are the planes of shear, and the principal stresses bisect the angles between the given planes.

Separate the remaining state of stress into the fluid stress $+\frac{1}{2}(a_n+b_n)$ and the right shearing stress $\pm\frac{1}{2}(a_n-b_n)$, and combine this last right shearing stress with that due to the tangential components. The final result is found, just as in Problem 20, by combining the fluid stress $\frac{1}{2}(a_n+b_n)$ with the resulting right shearing stress.

This problem can also be solved in a manner similar to that employed in Problem 6.

The result is expressed by the equations,

$$a+b=a_n+b_n,$$

$$(a-b)^2=(a_n-b_n)^2+4a_t{}^2$$

for the angle which has been heretofore denoted by x_1x_2 is in this case 45° $\therefore \cos 2x_1x_2=0$

$$\therefore\ a=\tfrac{1}{2}(a_n+b_n+[(a_n-b_n)^2+4a_t^2]^{1/2})$$

$$b=\tfrac{1}{2}(a_n+b_n-[(a_n-b_n)^2+4a_t^2]^{1/2})$$

$$\sin.\ 2xx_1 : 2a_t :: \sin.\ 2xx_2 : a_n-b_n :: 1 : a-b,$$

but $2xx_1=90°-2xx_2$,

$$\therefore \tan 2xx_1=2a_t \div (a_n-b_n).$$

Problem 22.—In a state of stress defined by two simple stresses which act at the same point and have a given inclination to each other, to combine them and find the resultant state.

It has been previously mentioned that any simple stress as a_1 can be separated into the fluid stress $+\frac{1}{2}a_1$ and the right shearing stress $\pm\frac{1}{2}a_1$, as it is simply a case in which $b_1=0$. Hence the simple stresses a_1, a_2 can be combined as a special case of Problem 20, in which b_1 and b_2 vanish. The results are expressed algebraically as follows:

$$a+b=a_1+a_2,$$

$$(a-b)^2=a_1^2+a_2^2+2a_1a_2\cos 2x_1x_2$$

$$\therefore\ ab=\tfrac{1}{2}a_1a_2(1-\cos 2x_1x_2)$$

$$\therefore\ ab=a_1a_2\sin^2 x_1x_2.$$

Since a simple compression or tension produces a simple stress in material, this problem is one of frequent occurrence, for it treats the superposition of two, and hence of any number of simple stresses lying in the same plane.

This problem is of such importance that we think it useful to call attention to another solution of it, suggested by the algebraic expressions just found.

In Fig. 13 let

$$o'a'=a_1,\ o'b'=a_2\ \therefore\ o'r'=\sqrt{a_1a_2}=oi.$$

Now, if $oir=x_1x_2$, then $or=o'r'\sin x_1x_2$

$$\therefore\ \overline{or}^2=oa'.ob'=o'a'.o'b'\sin^2 x_1x_2$$

$$\therefore\ oa'=a \text{ and } ob'=b.$$

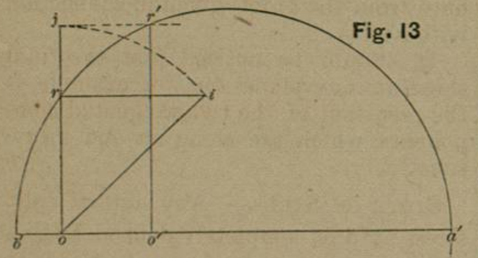


Fig. 13

This solution is treated more fully in Problem 23.

Problem 23.—When a state of stress is defined by its principal stresses, it is required to separate it into two simple stresses having a given inclination to each other.

It was shown in Problem 22 that

$$a+b=a_1+a_2, \text{ and } ab=a_1a_2\sin x_1x_2.$$

Let us apply these equations in Fig. 13 to effect the required construction. Make $oa'=a$, $ob'=b$; then $a'b'=a_1+a_2$. At o erect a perpendicular to $a'b'$ cutting the circle of which $a'b'$ is the diameter at r; then $\overline{or}^2=ab$, the product of the principal stresses. Also make $a'oi=x_1x_2$ the given inclination of the simple stresses, and let $ri \parallel a'b'$ intersect oi at i; then $or=oi\sin x_1x_2\ \therefore\ \overline{oi}^2=a_1a_2$.

Make $oj=oi$ and draw $jr' \parallel a'b'$, then

$$o'r'=oi, \text{ and } o'a'.o'b'=\overline{o'r'}^2,$$

$$\because\ o'a'=a_1 \text{ and } o'b'=a_2,$$

the required simple stresses. This construction applies equally whether the given principal stresses are of like or unlike sign, and also equally whether the two simple stresses are required to have like or unlike signs.

Problem 24.—When a state of stress is defined by its principal stresses, to find the inclination of two given simple stresses into which it can be separated.

In Fig. 13 let $oa'=a$, $ob'=b$ be the intensities of the principal stresses, and $o'a'=a_1$, $o'b'=a_2$ be the intensities of the given simple stresses. It has been already shown that $a+b=a_1+a_2$. Draw the two perpendiculars or and $o'r'$; through r draw $ri \parallel a'b'$; make $oi=oj=o'r'$; then is $oir=ioa'$ the required inclination, for it is such that

$$ab=a_1a_2\sin^2 x_1x_2$$

Problem 25.—To separate a state of right shearing stress of given intensity into two component states of right shearing stress whose intensities are given, and to find the mutual inclination of the principal stresses of the component states.

In Fig. 12, about the center o, describe circles with radii $on_1=a_1$, $on_2=a_2$, the given component intensities; and also about o at a distance $or_1=a$, the given intensity. Also describe circles with radii $r_1m_1=on_2$, $r_1n_2=on_1$ cutting the first mentioned circles at m_1 and n_2: then is $\frac{1}{2}n_2om_1=x_1x_2$ the required mutual inclination of the principal stresses of the component states. This is evident from considerations previously adduced in connection with this figure. The relative position of the principal stresses and principal component stresses is also readily found from the figure.

Problem 26.—In a state of right shearing stress of given intensity to separate it into two component states of right shearing stress, when the intensity of one of these components is given and also the mutual inclination of the principal stresses of the component states.

In Fig. 12, about the center o describe a circle rr with radius $or=a$, the intensity of the given right shearing stress, and at n_1, at a distance $on_1=a_1$ from o which is the intensity of the given component, make $x_1n_1r_2=2x_1x_2$, twice the given mutual inclination; then is n_1r_2 the distance from n_1 to the circle rr the intensity of the required component stress. The figure can be completed as was done previously.

It is evident, when the component a_1 exceed a, that there is a certain maximum value of the double inclination, which can be obtained by drawing n_1r_2 tangent to the circle rr, and the given inclination is subject to this restriction.

Other problems concerning the combination and separation of states of stress can be readily solved by methods like those already employed, for such problems can be made to depend on the combination and separation of the fluid stresses and right shearing stresses into which every state of stress can be separated.

PROPERTIES OF SOLID STRESS.

We shall call that state of stress at a point *a solid stress* which causes a stress on every plane traversing the point. In the foregoing discussion of plane stress no mention was made of a stress on the plane of the paper, to which the plane stress was assumed to be parallel. It is, evidently, possible to combine a simple stress perpendicular to the plane of the paper with any of the states of stress heretofore treated without changing the stress on any plane perpendicular to the paper.

Hence in treating plane stress we have already treated those cases of solid stress which are produced by a plane stress combined with any stress perpendicular to its plane, acting on planes also perpendicular to the plane of the paper.

We now wish to treat solid stress in a somewhat more general manner, but as most practical cases are included in plane stress, and the difficulties in the treatment of solid stress are much greater than those of plane stress, we shall make a much less extensive investigation of its properties.

Conjugate Stresses.—Let xx, yy, zz be any three lines through o; now, if any state of stress whatever exists at o, and xx be the direction of the stress on the plane yoz, and yy that on zox, then is zz the direction of the stress on xoy: *i.e.*, each of these three stresses lies in the intersection of the planes of action of the other two.

Reasoning like that employed in connection with Fig. 1, shows that no other direction than that stated could cause internal equilibrium; but a state of stress is a state of equilibrium, hence follows the truth of the above statement.

Tangential Components.—Let xx, yy, zz be rectangular axes through o; then, whatever may be the state of stress at o, the tangential components along xx and yy are equal, as also are those along yy and zz, as well as those along zz and xx.

The truth of this statement flows at once from the proof given in connection with Fig. 3.

It should be noticed that the total shear on any plane xoy, for example, is the resultant of the two tangential components which are along xx and yy respectively.

State of Stress.—Any state of solid stress at o is completely defined, so that the intensity and direction of the stress on any plane traversing o can be completely determined, when the stresses on any three planes traversing o are given in magnitude and direction.

This truth appears by reasoning similar to that employed with Fig. 4, for the three given planes with the fourth enclose a tetrahedron, and the total distributed force acting against the fourth plane is in equilibrium with the resultant of the forces acting on the first three.

Principal Stresses.—In any state of solid stress there is one set of three conjugate stresses at right angles to each other, *i.e.* there are three planes at right angles on which the stresses are normal only.

Since the direction of the stress on any plane traversing a given point o can only change gradually, as the plane through o changes in direction, it is evident from the directions of the stresses on conjugate planes that there must be at least one plane through o on which the stress is normal to the plane. Take that plane as the plane of the paper; then, as proved in plane stresses, there are two more principal stresses lying in the plane of the paper, for the stress normal to the plane of the paper has no component on any plane also perpendicular to the paper.

Fluid Stress.—Let the stresses on three rectangular planes through o be normal stresses of equal intensity and like sign; then the stress on any plane through o is also normal of the same intensity and same sign.

This is seen to be true when we combine with the stresses already acting in Fig. 5, another stress of the same intensity normal to the plane of the paper.

RIGHT SHEARING STRESS.—Let the stresses on three rectangular planes through o be normal stresses of equal intensity, but one of them, say the one along xx, of sign unlike that of the other two; then the stress on any plane through o, whose normal is $x'x'$, is of the same intensity and lies in the plane xox' in such a direction rr that xx and the plane yz bisect the angles in the plane xox' between rr and its plane of action, and rox' respectively.

The stress parallel to yz is a plane fluid stress, and causes therefore a normal stress on the plane xox'. Hence the resultant stress is in the direction stated, as was proved in Fig. 6.

COMPONENT STATES OF STRESS.—Any state of solid stress, defined by its principal stresses abc along the rectanglar axes of xyz respectively, is equivalent to the combination of three fluid stresses, as follows:

$\frac{1}{2}(a+b)$ along x and y, $-\frac{1}{2}(a+b)$ along z;
$\frac{1}{2}(c+a)$ along z and x, $-\frac{1}{2}(c+a)$ along y;
$\frac{1}{2}(b+c)$ along y and z, $-\frac{1}{2}(b+c)$ along y;

For these together give rise to the following combination:

$\frac{1}{2}(a+b)+\frac{1}{2}(c+a)-\frac{1}{2}(b+c)=a$, along x;
$\frac{1}{2}(a+b)-\frac{1}{2}(c+a)+\frac{1}{2}(b+c)=b$, along y;
$\frac{1}{2}(a+b)+\frac{1}{2}(c+a)+\frac{1}{2}(b+c)=c$, along x.

In case $b=0$ and $c=0$ this is a simple stress along x.

COMPONENT STRESSES.—Any state of solid stress defined by its principal stresses can also be separated into a fluid stress and three right shearing stresses, as follows:

$\frac{1}{4}(a+b+c)$ along x, y, z;
$\frac{1}{4}(a-b-c)$ along x, and
$-\frac{1}{4}(a-b-c)$ along y and z;
$\frac{1}{4}(b-c-a)$ along y, and
$-\frac{1}{4}(b-c-a)$ along z and x;
$\frac{1}{4}(c-a-b)$ along z, and
$-\frac{1}{4}(c-a-b)$ along x and y;

It will be seen that the total stresses along $x\ y\ z$ are $a\ b\ c$ respectively. This system of component stresses is remarkable because it is strictly analagous in its geometric relationships to the trammel method used in plain stress. We shall simply state this relationship without proof, as we shall not use its properties in our construction.

If the distances $pa_1=a$, $pb_1=b$, $pc_1=c$ be laid off along a straight line from the point p, and then this straight be moved so that the points $a_1\ b_1\ c_1$ move respectively in the planes yz, zx, xy; then p will describe an ellipsoid, as is well known, whose principal semiaxes are along xyz, and are abc respectively. Now the distances pa_1, pb_1, pc_1, may be laid off in the same direction from p or in different directions; so that, in all, four different combinations can be made, either of which will describe the same ellipsoid. But the position of these four generating lines through any assumed point $x_1y_1z_1$ of the ellipsoid is such that their equations are

$$\frac{a}{x_1}(x-x_1)=\pm\frac{b}{y_1}(y-y_1)=\pm\frac{c}{z_1}(z-z_1)$$

Now if the fluid stress $\frac{1}{4}(a+b+c)=or_1$ be laid off along the normal to any plane, *i.e.* parallel to that generating line which in the above equation has all its signs positive, and the other three right shearing stresses r_1r_2, r_2r_3, r_3r_4 be laid off successively parallel to the other generating lines, as was done in plane stresses, the line or_4 will be the resultant stress on the plane.

PROBLEMS IN SOLID STRESS.

PROBLEM 27.—In any state of stress defined by the stresses on three rectangular planes, to find the stress on any given plane.

Let the intensities of the normal components along $x\ y\ z$ be $a_n\ b_n\ c_n$ respectively, and the intensities of the pairs of tangential components which lie in the planes which intersect in $x\ y\ z$ and are

perpendicular to those axes be $a_t\ b_t\ c_t$ respectively, *e.g.*, a_t is the intensity of the tangential component on xoy along y, or its equal on xoz along z.

In Fig. 14 let a plane parallel to the given plane cut the axes at $x_1y_1z_1$; then the total forces on the area $x_1y_1z_1$ along xyz are respectively:

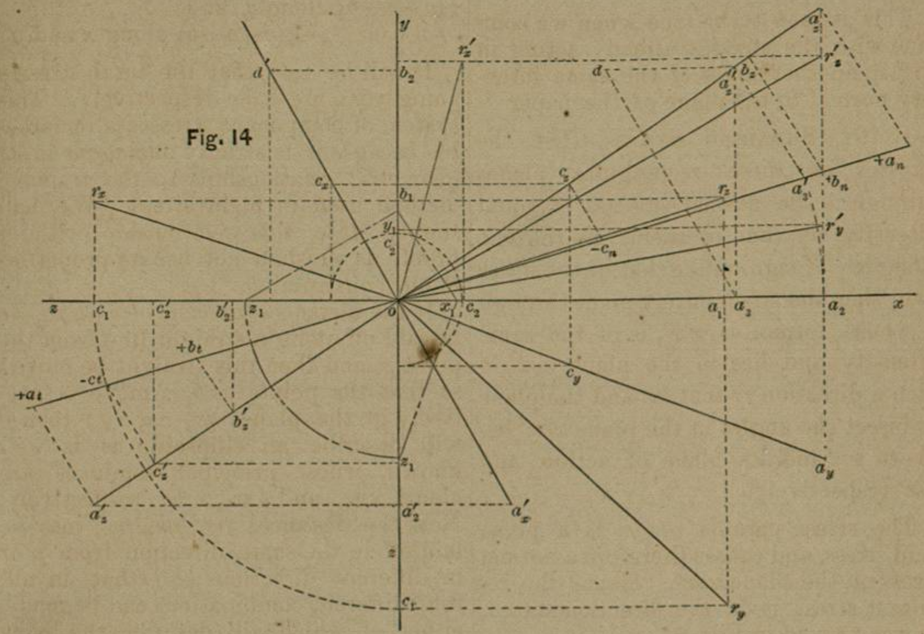


Fig. 14

$$\overline{x_1y_1z_1}.a_1=\overline{y_1oz_1}.a_n+\overline{x_1oy_1}.b_t+\overline{z_1ox_1}.c_t$$
$$\overline{x_1y_1z_1}.b_1=\overline{y_1oz_1}.c_t+\overline{x_1oy_1}.a_t+\overline{z_1ox_1}.b_n$$
$$\overline{x_1y_1z_1}.c_1=\overline{y_1oz_1}.b_t+\overline{x_1oy_1}.c_n+\overline{z_1ox_1}.a_t$$

in which $a_1b_1c_1$ are the intensities of the components of the stress on the plane $x_1y_1z_1$ along xyz respectively. Now

$$\overline{y_1oz_1}\div\overline{x_1y_1z_1}=\cos xn$$
$$\overline{z_1ox_1}\div\overline{x_1y_1z_1}=\cos yn$$
$$\overline{x_1oy_1}\div\overline{x_1y_1z_1}=\cos zn.$$

$$\therefore a_1=a_n\cos xn+b_t.\cos zn+c_t\cos yn$$
$$b_1=c_t\cos xn+a_t.\cos zn+b_n\cos yn$$
$$c_1=b_t\cos xn+c_n.\cos zn+a_t\cos yn$$

and $r^2=a_1^2+b_1^2+c_1^2$, therefore the resultant stress r is the diagonal of the right parallelopiped whose edges are $a_1b_1c_1$. In order to construct $a_1b_1c_1$ it is only necessary to lay off $a_n\ b_n\ c_n$, $a_t\ b_t\ c_t$ along the normal, and take the sums of such projections along xyz as are indicated in the above values of $a_1b_1c_1$.

Thus, in Fig. 14, let $x_1y_1z_1$ be the traces of a plane, and it is required to construct the stress upon a plane parallel to it through o.

The ground line between the planes of xoy and xoz is ox. The planes xoz and yoz on being revolved about ox and oy respectively, as in ordinary descriptive geometry, leave oz in two revolved positions at right angles to each other.

The three projections of the normal at o to the given plane are, as is well known, perpendicular to the traces of the given plane, and they are so represented. Let oa_z be the projection of the normal on xoy, and oa_y that on xoz. To find the true length of the normal, revolve it about one projection, say about oa_z, and if $a_z\,a_n=a_2\,a_y$ then is oa_n the revolved position of the normal.

Upon the normal let $oa_n=a_n$, $ob_n=b_n$, $oc_n=c_n$, the given normal components of the stresses upon the rectangular planes, and also let $oa_t=a_t$, $ob_t=b_t$, $oc_t=c_t$, the given tangential components upon the same planes.

Let $a_2b_2c_2$, $a_2'b_2'c_2'$ be the respective projections of the points $a_n\ b_n\ c_n$, $a_t\ b_t\ c_t$ of the normal upon the plane xoy by lines parallel to oz, similarly a_y, etc., are projections by parallels to oy, and a_x', etc., by parallels to ox.

We have taken the stresses c_n and c_t of different sign from the others, and so have called them negative and the others positive.

It is readily seen that the first of the above equations is constructed as follows:

$$a_1 = oa_1 = oa_2 + b_t\, b_z' - c_z' c_2'$$

Similarly, the other two equations become:

$$b_1 = ob_1 = -oc_2' + a_t\, a_2' + ob_2$$

$$c_1 = oc_1 = ab_2' - c_z c_t + oa_2'$$

We have thus found the coördinates of the extremity r of the stress or upon the given plane; hence its projections upon the planes of refererence are respectively or_x, or_y, or_z.

Problem 28.—In any state of stress defined by its three principal stresses, to find the stress on any given plane.

This problem is the special case of Problem 27, in which the tangential components are each zero. Taking the normal components given in Fig. 14 as principal stresses we find $oa_2 = a_n \cos xn$, $ob_2 = b_n \cos yn$, $oc_2 = c_n \cos zn$, as the coordinates which determine the stress or' upon the given plane, and the projections of or' are or_x', or_y', or_z', respectively.

From these results it is easy to show that the sum of the normal components of the stresses on any three planes is constant and equal to the sum of the principal stresses. This is a general property of solid stress in addition to those previously stated.

Problem 29.—Any state of stress being defined by given simple stresses, to find the stresses on three planes at right angles to each other.

In Fig. 14 let a simple stress act along the normal to the plane $x_1y_1z_1$, and cause a stress on that plane whose intensity is $a_n = oa_n$, then is $a_n \cos xn = oa_2$ the intensity of the stress in the same direction acting on the plane yoz. The normal component of this latter intensity is

$$a_n \cos^2 xn = oa_2 . \cos xn = oa_3,$$

and it is obtained by making $oa_3' = oa_2$, $a_3' a_z'' \parallel x_1y_1$, and $a_z'' a_3 \parallel oy$. The tangential component on yoz is od' in magnitude and direction, and it is obtained thus: make $a_z'' d = a_z'' a_3'$, then in the right angled triangle da_3a_2'', da_3 is the magnitude of the tangential component; now make $od' = da_3$. This tangential component can be resolved along the axes of y and z. The stress on the planes zox and xoy can be found in similar manner, since the tangential components which act on two planes at right angles to each other and in a direction perpendicular to their intersection are, as has been shown, equal; the complete construction will itself afford a test of its accuracy.

Other simple stresses may be treated in the same manner, and the resultant stress on either of the three planes, due to these simple stresses, is found by combining together the components which act on that plane due to each of the simple stresses.

It is useless to make the complete combination. It is sufficient to take the algebraic sum of the normal components acting on the plane, and then the algebraic sum of the tangential components along two directions in the plane which are at right angles, as along y and z in yoz.

The treatment of conjugate stresses in general appears to be too complicated to be practically useful, and we shall not at present construct the problems arising in its treatment.