joining them, or this line produced, in a definite point, whose position depends only on the magnitudes of the given forces, and not at all on the angle which their direction makes with the joining line. This result is important in connection with centres of gravity. The point so determined is called the centre of the two parallel forces. If these two forces are the weights of two particles, the "centre" thus found is their centre of gravity, and the resultant force is the same as if the two particles were collected at this point.

22. Moments of Resultant and of Components Equal.—The following proposition is often useful. Let any straight line be drawn across the lines of action of two parallel forces P_1, P_2 (Fig. 6). Let O be any point on this line, and x_1, x_2 the distances measured from O to the points of section, distances measured in opposite directions being distinguished by opposite signs, and forces in opposite directions being also distinguished by opposite signs. Also let R denote the resultant of P_1 and P_2, and $\bar{x}$ the distance from O to its intersection with the line; then we shall have

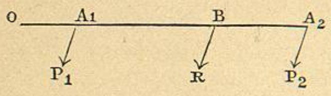

Fig. 6.

$$P_1 x_1 + P_2 x_2 = R \bar{x}.$$

For, taking the standard case, as represented in Fig. 6, in which all the quantities are positive, we have $OA_1 = x_1$, $OA_2 = x_2$, $OB = \bar{x}$, and by § 19 or § 20 we have

$$P_1 . A_1B = P_2 . BA_2,$$

that is,

$$P_1 (\bar{x} - x_1) = P_2 (x_2 - \bar{x}),$$

whence

$$(P_1 + P_2) \bar{x} = P_1 x_1 + P_2 x_2,$$

that is,

$$R \bar{x} = P_1 x_1 + P_2 x_2. \qquad (2)$$

23. Any Number of Parallel Forces in One Plane.—Equation (2) affords the readiest means of determining the line of action of the resultant of several parallel forces lying in one plane. For let P_1, P_2, P_3, &c., be the forces, R_1 the resultant of the first two forces P_1, P_2, and R_2 the resultant of the first three forces P_1, P_2, P_3. Let a line be drawn across the lines of action, and let the distances of the points of section from an arbitrary point O on this line be expressed according to the following scheme:—

Force	P_1	P_2	P_3	R_1	R_2
Distance	x_1	x_2	x_3	$\bar{x}_1$	$\bar{x}_2$

Then, by equation (2) we have

$$R_1 \bar{x}_1 = P_1 x_1 + P_2 x_2.$$

Also since R_2 is the resultant of R_1 and P_3, we have

$$R_2 \bar{x}_2 = R_1 \bar{x}_1 + P_3 x_3,$$

and substituting for the term $R_1 \bar{x}_1$, we have

$$R_2 \bar{x}_2 = P_1 x_1 + P_2 x_2 + P_3 x_3.$$

This reasoning can evidently be extended to any number of forces, so that we shall have finally

$$R\bar{x} = \text{sum of such terms as } Px,$$

where R denotes the resultant of all the forces, and is equal to their algebraic sum; while $\bar{x}$ denotes the value of x for the point where the line of action of R cuts the fixed line. It is usual to employ the Greek letter Σ to denote "the sum of such terms as." We may therefore write

$$R = \Sigma (P)$$
$$R\bar{x} = \Sigma (Px)$$

whence

$$\bar{x} = \frac{\Sigma (Px)}{\Sigma (P)} \qquad (3)$$

24. Moment of a Force about a Point.—When the fixed line is at right angles to the parallel forces, the product Px is called the moment of the force P about the point O. More generally, the *moment of a force about a point* is the *product of the force by the length of the perpendicular dropped upon it from the point.* The above equations show that for parallel forces in one plane, the *moment of the resultant* about any point in the plane *is the sum of the moments of the forces* about the same point.

If the resultant passes through O, the distance $\bar{x}$ is zero; whence it follows from the equations that the algebraical sum of the moments vanishes.

The moment of a force about a point measures the tendency of the force to produce rotation about the point. If one point of a body be fixed, the body will turn in one direction or the other according as the resultant passes on one side or the other of this point (the direction of the resultant being supposed given). If the resultant passes through the fixed point, the body will be in equilibrium.

The moment Px of any force about a point, changes sign with P and also with x; thereby expressing (what is obvious in itself) that the direction in which the force tends to turn the body about the point will be reversed if the direction of P is reversed while its line of action remains unchanged, and will also be reversed if the line of action be shifted to the other side of the point while the direction of the force remains unchanged.

25. Experimental Illustration.—Fig. 7 represents a simple apparatus (called the *arithmetical lever*) for illustrating the laws of par-

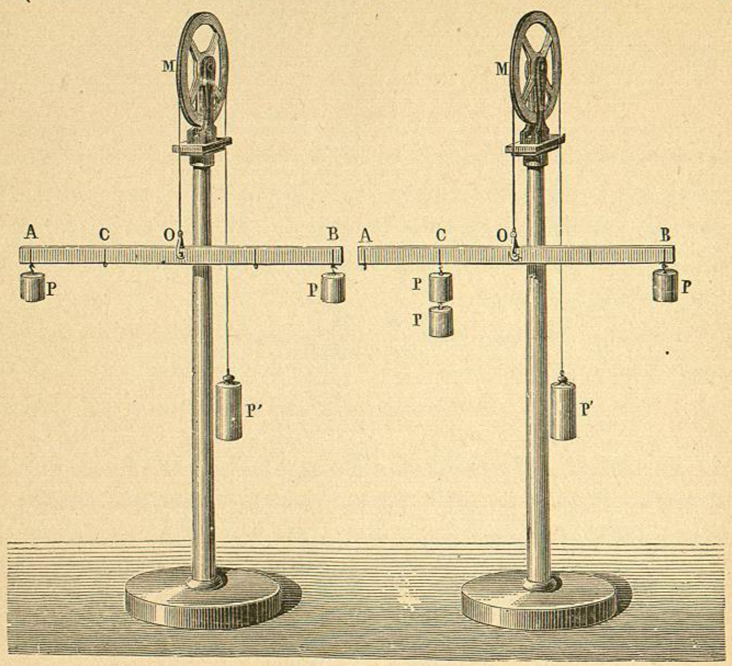

Fig. 7.—Composition of Parallel Forces.

allel forces. The lever AB is suspended at its middle point by a cord, so that when no weights are attached it is horizontal. Equal weights P, P are hung at points A and B equidistant from the centre, and the suspending cord after being passed over a very freely moving pulley M, has a weight P′ hung at its other end sufficient to produce equilibrium. It will be found that P′ is equal to the sum of the two weights P together with the weight required to counterpoise the lever itself.

In the second figure, the two weights hung from the lever are not equal, but one of them is double of the other, P being hung at B, and 2 P at C; and it is necessary for equilibrium that the distance OB be double of the distance OC. The weight P′ required

to balance the system will now be 3 P together with the weight of the lever.

26. Couple.—There is one case of two parallel forces in opposite directions which requires special attention; that in which the two forces are equal.

To obtain some idea of the effect of two such forces, let us first suppose them not exactly equal, but let their difference be very small compared with either of the forces. In this case, the resultant will be equal to this small difference, and its line of action will be at a great distance from those of the given forces. For in § 19 if Q is very little greater than P, so that Q-P, or R is only a small fraction of P, the equation $\frac{P}{BC}=\frac{R}{AB}$ shows that AB is only a small fraction of BC, or in other words that BC is very large compared with AB.

If Q gradually diminishes until it becomes equal to P, R will gradually diminish to zero; but while it diminishes, the product R . BC will remain constant, being always equal to P . AB.

A very small force R at a very great distance would have sensibly the same moment round all points between A and B or anywhere in their neighbourhood, and the moment of R is always equal to the algebraic sum of the moments of P and Q.

When Q is equal to P, they compose what is called *a couple*, and the algebraic sum of their moments about any point in their plane is constant, being always equal to P . AB, which is therefore called the moment of the couple.

A couple consists of two equal and parallel forces in opposite directions applied to the same body. The distance between their lines of action is called the arm of the couple, and the product of one of the two equal forces by this arm is called the moment of the couple.

27. Composition of Couples. Axis of Couple.—A couple cannot be balanced by a single force; but it can be balanced by any couple of equal moment, opposite in sign, if the plane of the second couple be either the same as that of the first or parallel to it.

Any number of couples in the same or parallel planes are equivalent to a single couple whose moment is the algebraic sum of theirs.

The laws of the composition of couples (like those of forces) can be illustrated by geometry.

Let a couple be represented by a line perpendicular to its plane, marked with an arrow according to the convention that if an ordinary screw were made to turn in the direction in which the couple tends to turn, it would advance in the direction in which the arrow points. Also let the length of the line represent the moment of the couple. Then the same laws of composition and resolution which hold for forces acting at a point will also hold for couples. A line thus drawn to represent a couple is called the *axis* of the couple.

Just as any number of forces acting at a point are either in equilibrium or equivalent to a single force, so any number of couples applied to the same rigid body (no matter to what parts of it) are either in equilibrium or equivalent to a single couple.

28. Resultant of Force and Couple in Same Plane.—The resultant of a force and a couple in the same plane is a single force. For the couple may be replaced by another of equal moment having its equal forces P, Q, each equal to the given force F, and the latter couple may then be turned about in its own plane and carried into such a position that one of its two forces destroys the force F, as represented in Fig. 8. There will then only remain the force P, which is equal and parallel to F.

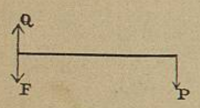

Fig. 8.

By reversing this procedure, we can show that a force P which does not pass through a given point A is equivalent to an equal and parallel force F which does pass through it, together with a couple; the moment of the couple being the same as the moment of the force P about A.

29. General Resultant of any Number of Forces applied to a Rigid Body.—Forces applied to a rigid body in lines which do not meet in one point are not in general equivalent to a single force. By the process indicated in the concluding sentence of the preceding section, we can replace the forces by forces equal and parallel to them, acting at any assumed point, together with a number of couples. These couples can then be reduced (by the principles of § 27) to a single couple, and the forces at the point can be replaced by a single force; so that we shall obtain, as the complete resultant, a single force applied at any point we choose to select, and a couple.

We can in general make the couple smaller by resolving it into two components whose planes are respectively perpendicular and parallel to the force, and then compounding one of these components (the latter) with the force as explained in § 28, thus moving the force parallel to itself without altering its magnitude. This is the greatest simplification that is possible. The result is that we have a single force and a couple whose plane is perpendicular to the force. Any combination of forces that can be applied to a rigid body is reducible to a force acting along one definite line and a couple in a plane perpendicular to this line. Such a combination of a force and couple is called a *wrench*, and the "one definite line" is called the *axis* of the wrench. The point of application of the force is not definite, but is any point of the axis.

30. Application to Action and Reaction.—Every action of force that one body can exert upon another is reducible to a wrench, and the law of reaction is that the second body will, in every case, exert upon the first an equal and opposite wrench. The two wrenches will have the same axis, equal and opposite forces along this axis, and equal and opposite couples in planes perpendicular to it.

31. Resolution the Inverse of Composition.—The process of finding the resultant of two or more forces is called *composition*. The inverse process of finding two or more forces which shall together be equivalent to a given force, is called *resolution*, and the two or more forces thus found are called *components*.

The problem to resolve a force into two components along two given lines which meet it in one point and are in the same plane with it, has a definite solution, which is obtained by drawing a triangle whose sides are parallel respectively to the given force and the required components. The given force and the required components will be proportional to the sides of this triangle, each being represented by the side parallel to it.

The problem to resolve a force into three components along three given lines which meet it in one point and are not in one plane, also admits of a definite solution.

32. Rectangular Resolution.—In the majority of cases which occur in practice the required components are at right angles to each other, and the resolution is then said to be rectangular. When "the component of a force along a given line" is mentioned, without anything in the context to indicate the direction of the other component or components, it is always to be understood that the resolution is rectangular. The process of finding the required component in this case is illustrated by Fig. 9. Let AB represent the given force F, and let AC be the line along which the component of F is required. It is only necessary to drop from B a perpendicular BC on this line; AC will represent the required component. CB represents the other component, which, along with AC, is equivalent to the given force. If the total number of rectangular components, of which AC represents one, is to be three, then the other two will lie in a plane perpendicular to AC, and they can be found by again resolving CB. The magnitude of AC will be the same whether the number of components be two or three, and the component along AC will be F $\frac{AC}{AB}$, or in trigonometrical language,

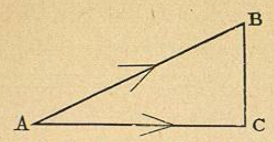

Fig. 9.—Component along a given Line.

$$F \cos . BAC.$$

We have thus the following rule:—*The component of a given force along a given line is found by multiplying the force by the cosine of the angle between its own direction and that of the required component.*

CHAPTER III.

CENTRE OF GRAVITY.

33. Gravity is the force to which we owe the names "up" and "down." The direction in which gravity acts at any place is called the downward direction, and a line drawn accurately in this direction is called *vertical;* it is the direction assumed by a plumb-line. A plane perpendicular to this direction is called *horizontal,* and is parallel to the surface of a liquid at rest. The verticals at different places are not parallel, but are inclined at an angle which is approximately proportional to the distance between the places. It amounts to 180° when the places are antipodal, and to about 1′ when their distance is one geographical mile, or to about 1″ for every hundred feet. Hence, when we are dealing with the action of gravity on a body a few feet or a few hundred feet in length, we may practically regard the action as consisting of parallel forces.

34. Centre of Gravity.—Let A and B be any two particles of a rigid body, let w_1 be the weight of the particle A, and w_2 the weight of B. These weights are parallel forces, and their resultant divides the line AB in the inverse ratio of the forces. As the body is turned about into different positions, the forces w_1 and w_2 remain unchanged in magnitude, and hence the resultant cuts AB always in the same point. This point is called the centre of the parallel forces w_1 and w_2, or the centre of gravity of the two particles A and B. The magnitude of the resultant will be w_1+w_2, and we may substitute it for the two forces themselves; in other words, we may suppose the two particles A and B to be collected at their centre of gravity. We can now combine this resultant with the weight of a third particle of the body, and shall thus obtain a resultant $w_1+w_2+w_3$, passing through a definite point in the line which joins