perpendicular BC on this line; AC will represent the required component. CB represents the other component, which, along with AC, is equivalent to the given force. If the total number of rectangular components, of which AC represents one, is to be three, then the other two will lie in a plane perpendicular to AC, and they can be found by again resolving CB. The magnitude of AC will be the same whether the number of components be two or three, and the component along AC will be F $\frac{AC}{AB}$, or in trigonometrical language,

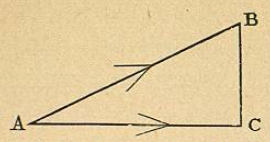

Fig. 9.—Component along a given Line.

$$F \cos . BAC.$$

We have thus the following rule:—*The component of a given force along a given line is found by multiplying the force by the cosine of the angle between its own direction and that of the required component.*

CHAPTER III.

CENTRE OF GRAVITY.

33. Gravity is the force to which we owe the names "up" and "down." The direction in which gravity acts at any place is called the downward direction, and a line drawn accurately in this direction is called *vertical;* it is the direction assumed by a plumb-line. A plane perpendicular to this direction is called *horizontal,* and is parallel to the surface of a liquid at rest. The verticals at different places are not parallel, but are inclined at an angle which is approximately proportional to the distance between the places. It amounts to 180° when the places are antipodal, and to about 1′ when their distance is one geographical mile, or to about 1″ for every hundred feet. Hence, when we are dealing with the action of gravity on a body a few feet or a few hundred feet in length, we may practically regard the action as consisting of parallel forces.

34. Centre of Gravity.—Let A and B be any two particles of a rigid body, let w_1 be the weight of the particle A, and w_2 the weight of B. These weights are parallel forces, and their resultant divides the line AB in the inverse ratio of the forces. As the body is turned about into different positions, the forces w_1 and w_2 remain unchanged in magnitude, and hence the resultant cuts AB always in the same point. This point is called the centre of the parallel forces w_1 and w_2, or the centre of gravity of the two particles A and B. The magnitude of the resultant will be w_1+w_2, and we may substitute it for the two forces themselves; in other words, we may suppose the two particles A and B to be collected at their centre of gravity. We can now combine this resultant with the weight of a third particle of the body, and shall thus obtain a resultant $w_1+w_2+w_3$, passing through a definite point in the line which joins the third particle to the centre of gravity of the first two. The first three particles may now be supposed to be collected at this point, and the same reasoning may be extended until all the particles have been collected at one point. This point will be the *centre of gravity* of the whole body. From the manner in which it has been obtained, it possesses the property that *the resultant of all the forces of gravity on the body passes through it, in every position in which the body can be placed.* The resultant force of gravity upon a rigid body is therefore a single force passing through its centre of gravity.

35. Centres of Gravity of Volumes, Areas, and Lines.—If the body is homogeneous (that is composed of uniform substance throughout), the position of the centre of gravity depends only on the figure, and in this sense it is usual to speak of the centre of gravity of a figure. In like manner it is customary to speak of the centres of gravity of areas and lines, an area being identified in thought with a thin uniform plate, and a line with a thin uniform wire.

It is not necessary that a body should be rigid in order that it may have a centre of gravity. We may speak of the centre of gravity of a mass of fluid, or of the centre of gravity of a system of bodies not connected in any way. The same point which would be the centre of gravity if all the parts were rigidly connected, is still called by this name whether they are connected or not.

36. Methods of Finding Centres of Gravity.—Whenever a homogeneous body contains a point which bisects all lines in the body that can be drawn through it, this point must be the centre of gravity. The centres of a sphere, a circle, a cube, a square, an ellipse, an ellipsoid, a parallelogram, and a parallelepiped, are examples.

Again, when a body consists of a finite number of parts whose weights and centres of gravity are known, we may regard each part as collected at its own centre of gravity.

When the parts are at all numerous, the final result will most readily be obtained by the use of the formula

$$\bar{x} = \frac{\Sigma\,(Px)}{\Sigma\,(P)}, \qquad (3)$$

where P denotes the weight of any part, x the distance of its centre of gravity from any plane, and $\bar{x}$ the distance of the centre of gravity of the whole from that plane. We have already in § 23

proved this formula for the case in which the centres of gravity lie in one straight line and x denotes distance from a point in this line; and it is not difficult, by the help of the properties of similar triangles, to make the proof general.

37. Centre of Gravity of a Triangle.—To find the centre of gravity of a triangle ABC (Fig. 10), we may begin by supposing it divided into narrow strips by lines (such as bc) parallel to BC. It can be shown, by similar triangles, that each of these strips is bisected by the line AD drawn from A to D the middle point of BC. But each strip may be collected at its own centre of gravity, that is at its own middle point; hence the whole triangle may be collected on the line AD; its centre of gravity must therefore be situated upon this line. Similar reasoning shows that it must lie upon the line BE drawn from B to the middle point of AC. It is therefore the intersection of these two lines. If we join DE we can show that the triangles AGB, DGE, are similar, and that

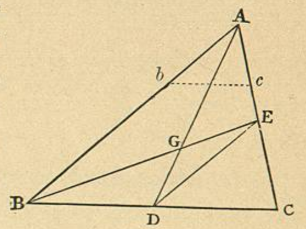

Fig. 10.

$$\frac{AG}{GD} = \frac{AB}{DE} = 2.$$

DG is therefore one third of DA. The centre of gravity of a triangle therefore lies upon the line joining any corner to the middle point of the opposite side, and is at one-third of the length of this line from the end where it meets that side.

It is worthy of remark that if three equal particles are placed at the corners of any triangle, they have the same centre of gravity as the triangle. For the two particles at B and C may be collected at the middle point D, and this double particle at D, together with the single particle at A, will have their centre of gravity at G, since G divides DA in the ratio of 1 to 2.

38. Centre of Gravity of a Pyramid.—If a pyramid or a cone be divided into thin slices by planes parallel to its base, and a straight line be drawn from the vertex to the centre of gravity of the base, this line will pass through the centres of gravity of all the slices, since all the slices are similar to the base, and are similarly cut by this line.

In a tetrahedron or triangular pyramid, if D (Fig. 11) be the centre of gravity of one face, and A be the corner opposite to this face, the centre of gravity of the pyramid must lie upon the line AD. In like manner, if E be the centre of gravity of one face, the centre of gravity of the pyramid must lie upon the line joining E with the opposite corner B. It must therefore be the intersection G of these two lines. That they do intersect is otherwise obvious, for the lines AE, BD meet in C, the middle point of one edge of the pyramid, E being found by taking CE one third of CA, and D by taking CD one third of CB.

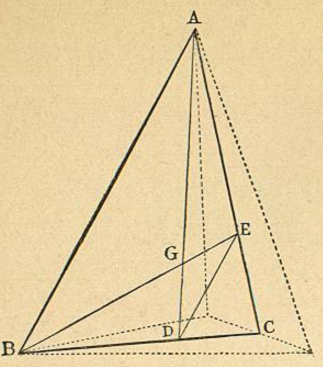

Fig. 11.—Centre of Gravity of Tetrahedron.

If D, E be joined, we can show that the joining line is parallel to BA, and that the triangles AGB, DGE are similar. Hence

$$\frac{AG}{GD} = \frac{AB}{DE} = \frac{BC}{DC} = 3.$$

That is, the line AD joining any corner to the centre of gravity of the opposite face, is cut in the ratio of 3 to 1 by the centre of gravity G of the triangle. DG is therefore one-fourth of DA, and the distance of the centre of gravity from any face is one-fourth of the distance of the opposite corner.

A pyramid standing on a polygonal base can be cut up into triangular pyramids standing on the triangular bases into which the polygon can be divided, and having the same vertex as the whole pyramid. The centres of gravity of these triangular pyramids are all at the same perpendicular distance from the base, namely at one-fourth of the distance of the vertex, which is therefore the distance of the centre of gravity of the whole from the base. The centre of gravity of any pyramid is therefore found by joining the vertex to the centre of gravity of the base, and cutting off one-fourth of the joining line from the end where it meets the base. The same rule applies to a cone, since a cone may be regarded as a polygonal pyramid with a very large number of sides.

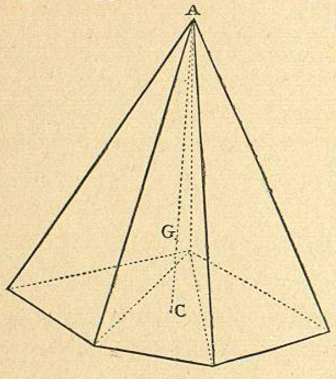

Fig. 12.—Centre of Gravity of Pyramid.

39. If four equal particles are placed at the corners of a triangular pyramid, they will have the same centre of gravity as the pyramid. For three of them may, as we have seen (§ 37) be collected at the centre of gravity of one face; and the centre of gravity of the four particles will divide the line which joins this point to the fourth, in the ratio of 1 to 3.

40. Condition of Standing or Falling.—When a heavy body stands on a base of finite area, and remains in equilibrium under the action of its own weight and the reaction of this base, the vertical through its centre of gravity must fall within the base. If the body is supported on three or more points, as in Fig. 13, we are to understand by the base the convex[1] polygon whose corners are the points of support; for if a body so supported turns over, it must turn about the line joining two of these points.

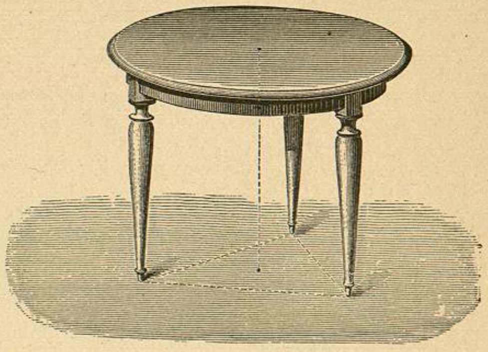

Fig. 13.—Equilibrium of a Body supported on a Horizontal Plane at three or more Points.

41. Body supported at one Point.—When a heavy body supported at one point remains at rest, the reaction of the point of support equilibrates the force of gravity. But two forces cannot be in equilibrium unless they have the same line of action; hence the vertical through the centre of gravity of the body must pass through the point of support. If instead of being supported at a point, the heavy body is supported by an axis about which it is free to turn, the vertical through the centre of gravity must pass through this axis.

42. Stability and Instability.—When the point of support, or axis of support, is vertically *below* the centre of gravity, it is easily seen that, if the body were displaced a little to either side, the forces acting upon it would turn it still further away from the position of equilibrium. On the other hand, when the point or axis of support is vertically *above* the centre of gravity, the forces which would

[1] The word *convex* is inserted to indicate that there must be no re-entrant angles. Any points of support which lie within the polygon formed by joining the rest, must be left out of account.

act upon it if it were slightly displaced would tend to restore it. In the latter case the equilibrium is said to be *stable*, in the former *unstable*.

When the centre of gravity coincides with the point of support, or lies upon the axis of support, the body will still be in equilibrium when turned about this point or axis into any other position. In this case the equilibrium is neither stable nor unstable but is called *neutral*.

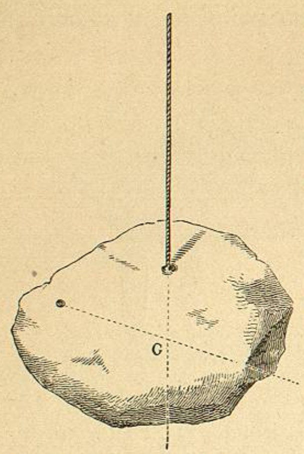

Fig. 14.—Experimental Determination of Centre of Gravity.

43. Experimental determination of Centre of Gravity.—In general, if we suspend a body by any point, in such a manner that it is free to turn about this point, it will come to rest in a position of stable equilibrium. The centre of gravity will then be vertically beneath the point of support. If we now suspend the body from another point, the centre of gravity will come vertically beneath this. The intersection of these two verticals will therefore be the centre of gravity (Fig. 14).

44. To find the centre of gravity of a flat plate or board (Fig. 15), we may suspend it from a point near its circumference, in such a manner that it sets itself in a vertical plane. Let a plumb-line be at the same time suspended from the same point, and made to leave its trace upon the board by chalking and "snapping" it. Let the board now be suspended from another point, and the operation be repeated. The two chalk lines will intersect each other at that point of the face which is opposite to the centre of gravity; the centre of gravity itself being of course in the substance of the board.

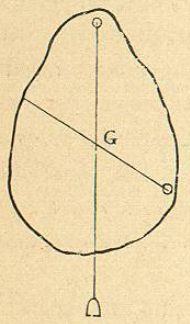

Fig. 15.—Centre of Gravity of Board.

45. Work done against Gravity.—When a heavy body is raised, work is said to be done against gravity, and the amount of this work is reckoned by multiplying together the weight of the body and the height through which it is raised. Horizontal movement does not count, and when a body is raised obliquely, only the vertical component of the motion is to be reckoned.

Suppose, now, that we have a number of particles whose weights

are w_1, w_2, w_3 &c., and let their heights above a given horizontal plane be respectively h_1, h_2, h_3 &c. We know by equation (3), § 23, that if $\bar{h}$ denote the height of their centre of gravity we have

$$(w_1+w_2+\&c.)\,\bar{h}=w_1 h_1+w_2 h_2+\&c. \qquad (4)$$

Let the particles now be raised into new positions in which their heights above the same plane of reference are respectively H_1, H_2, H_3 &c. The height $\bar{H}$ of their centre of gravity will now be such that

$$(w_1+w_2+\&c.)\,\bar{H}=w_1 H_1+w_2 H_2+\&c. \qquad (5)$$

From these two equations, we find, by subtraction

$$(w_1+w_2+\&c.)\,(\bar{H}-\bar{h})=w_1\,(H_1-h_1)+w_2\,(H_2-h_2)+\&c. \qquad (6)$$

Now H_1-h_1 is the height through which the particle of weight w_1 has been raised; hence the work done against gravity in raising it is $w_1\,(H_1-h_1)$ and the second member of equation (6) therefore expresses the whole amount of work done against gravity. But the first member expresses the work which would be done in raising all the particles through a uniform height $\bar{H}-\bar{h}$, which is the height of the new position of the centre of gravity above the old. The work done against gravity in raising any system of bodies will therefore be correctly computed by supposing all the system to be collected at its centre of gravity. For example, the work done in raising bricks and mortar from the ground to build a chimney, is equal to the total weight of the chimney multiplied by the height of its centre of gravity above the ground.

46. The Centre of Gravity tends to Descend.—When the forces which tend to move a system are simply the weights of its parts, we can determine whether it is in equilibrium by observing the path in which its centre of gravity would travel if movement took place. If we suppose this path to represent a hard frictionless surface, and the centre of gravity to represent a heavy particle placed upon it, the conditions of equilibrium will be the same as in the actual case. The centre of gravity tends to run down hill, just as a heavy particle does. There will be stable equilibrium if the centre of gravity is at the bottom of a valley in its path, and unstable equilibrium if it is at the top of a hill. When a rigid body turns about a horizontal axis, the path of its centre of gravity is a circle in a vertical plane. The highest and lowest points of this circle are the positions of the centre of gravity in unstable and stable equilibrium respectively;

except when the axis traverses the centre of gravity itself, in which case the centre of gravity can neither rise nor fall, and the equilibrium is neutral.

A uniform sphere or cylinder lying on a horizontal plane is in neutral equilibrium, because its centre of gravity will neither be raised nor lowered by rolling. An egg balanced on its end as in Fig. 16, is in unstable equilibrium, because its centre of gravity is at the top of a hill which it will descend when the egg rolls to one side. The position of equilibrium shown in Fig. 17 is stable as regards rolling to left or right, because the path of its centre of gravity in

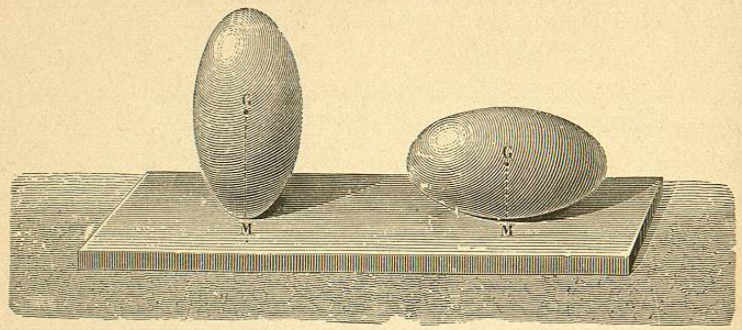

Fig. 16.—Unstable Equilibrium. Fig. 17.—Stable Equilibrium.

such rolling would be a curve whose lowest point is that now occupied by the centre of gravity. As regards rolling in the direction at right angles to this, if the egg is a true solid of resolution, the equilibrium is neutral.

47. Work done by Gravity.—When a heavy body is lifted, the lifting force does work against gravity. When it descends gravity does work upon it; and if it descends to the same position from which it was lifted, the work done by gravity in the descent is equal to the work done against gravity in the lifting; each being equal to the weight of the body multiplied by the vertical displacement of its centre of gravity. The tendency of the centre of gravity to descend is a manifestation of the tendency of gravity to do work; and this tendency is not peculiar to gravity.

48. Work done by any Force.—A force is said to do work when its point of application moves in the direction of the force, or in any direction making an acute angle with this, so as to give a component displacement in the direction of the force; and the amount of work done is the product of the force by this component. If F denote

the force, a the displacement, and θ the angle between the two, the work done by F is

$$F\,a\cos\theta.$$

which is what we obtain either by the above rule or by multiplying the whole displacement by the effective component of F, that is the component of F in the direction of the displacement. If the angle θ is obtuse, $\cos\theta$ is negative and the force F does negative work. If θ is a right angle F does no work. In this case F neither assists nor resists the displacement. When θ is acute, F assists the displacement, and would produce it if the body were constrained by guides which left it free to take this displacement and the directly opposite one, while preventing all others.

If θ is obtuse, F resists the displacement, and would produce the opposite displacement if the body were constrained in the manner just supposed.

49. Principle of Work.—If any number of forces act upon a body which is only free to move in a particular direction and its opposite, we can tell in which of these two directions it will move by calculating the work which each force would do. Each force would do positive work when the displacement is in one direction, and negative work when it is in the opposite direction, the absolute amounts of work being the same in both cases if the displacements are equal. The body will upon the whole be urged in that direction which gives an excess of positive work over negative. If no such excess exists, but the amounts of positive and negative work are exactly equal, the body is in equilibrium. This principle (which has been called the principle of *virtual velocities*, but is better called the *principle of work*) is often of great use in enabling us to calculate the ratio which two forces applied in given ways to the same body must have in order to equilibrate each other. It applies not only to the "mechanical powers" and all combinations of solid machinery, but also to hydrostatic arrangements; for example to the hydraulic press. The condition of equilibrium between two forces applied to any frictionless machine and tending to drive it opposite ways, is that in a small movement of the machine they would do equal and opposite amounts of work. Thus in the screw-press (Fig. 30) the force applied to one of the handles, multiplied by the distance through which this handle moves, will be equal to the pressure which this force produces at the foot of the screw, multiplied by the distance that the screw travels.

This is on the supposition of no friction. A frictionless machine gives out the same amount of work which is spent in driving it. The effect of friction is to make the work given out less than the work put in. Much fruitless ingenuity has been expended upon contrivances for circumventing this law of nature and producing a machine which shall give out more work than is put into it. Such contrivances are called "perpetual motions."

50. General Criterion of Stability.—If the forces which act upon a body and produce equilibrium remain unchanged in magnitude and direction when the body moves away from its position, and if the velocities of their points of application also remain unchanged in direction and in their ratio to each other, it is obvious that the equality of positive and negative work which subsists at the beginning of the motion will continue to subsist throughout the entire motion. The body will therefore remain in equilibrium when displaced. Its equilibrium is in this case said to be neutral.

If the forces which are in equilibrium in a given position of the body, gradually change in direction or magnitude as the body moves away from this position, the equality of positive and negative work will not in general continue to subsist, and the inequality will increase with the displacement. If the body be displaced with a constant velocity and in a uniform manner, the rate of doing work, which is zero at first, will not continue to be zero, but will have a value, whether positive or negative, increasing in simple proportion to the displacement. Hence it can be shown that the whole work done in a small movement is proportional to the square of the displacement, for when we double the displacement we, at the same time, double the mean working force.

If this work is positive, the forces assist the displacement and tend to increase it; the equilibrium must therefore have been unstable.

On the other hand, if the work is negative in all possible displacements from the position of equilibrium, the forces oppose the displacements and the equilibrium is stable.

51. Illustration of Stability.—A good example of stable equilibrium of this kind is furnished by Gravesande's apparatus (Fig. 3) simplified by removing the parallelogram and employing a string to support the three weights, one of them P″ being fastened to it at a point A near its middle, and the others P, P′ to its ends. The point A will take the same position as in the figure, and will return to it again when displaced. If we take hold of the point A and

move it in any direction whether in the plane of the string or out of it, we feel that at first there is hardly any resistance and the smallest force we can apply produces a sensible disturbance; but that as the displacement increases the resistance becomes greater. If we release the point A when displaced, it will execute oscillations, which will become gradually smaller, owing to friction, and it will finally come to rest in its original position of equilibrium.

The centre of gravity of the three weights is in its lowest position when the system is in equilibrium, and when a small displacement is produced the centre of gravity rises by an amount proportional to its square, so that a double displacement produces a quadruple rise of the centre of gravity.

In this illustration the three forces remain unchanged, and the directions of two of them change gradually as the point A is moved. Whenever the circumstances of stable equilibrium are such that the forces make no abrupt changes either in direction or magnitude for small displacements, the resistance will, as in this case, be proportional to the displacement (when small), and the work to the square of the displacement, and the system will oscillate if displaced and then left to itself.

52. Stability where Forces vary abruptly with Position.—There are other cases of stable equilibrium which may be illustrated by the example of a book lying on a table. If we displace it by lifting one edge, the force which we must exert does not increase with the displacement, but is sensibly constant when the displacement is small, and as a consequence the work will be simply proportional to the displacement. The reason is, that one of the forces concerned in producing equilibrium, namely, the upward pressure of the table, changes *per saltum* at the moment when the displacement begins. In applying the principle of work to such a case as this, we must employ, instead of the actual work done by the force which changes abruptly, the work which it would do if its magnitude and direction remained unchanged, or only changed gradually.

53. Illustrations from Toys.—The stability of the "balancer" (Fig. 18) depends on the fact that, owing to the weight of the two leaden balls, which are rigidly attached to the figure by stiff wires, the centre of gravity of the whole is below the point of support. If the figure be disturbed it oscillates, and finally comes to rest in a position in which the centre of gravity is vertically under the toe on which the figure stands.

The "tumbler" (Fig. 19) consists of a light figure attached to a hemisphere of lead, the centre of gravity of the whole being between the centre of gravity of the hemisphere and the centre of the sphere to which it belongs. When placed upon a level table, the lowest position of the centre of gravity is that in which the figure is upright, and it accordingly returns to this position when displaced.

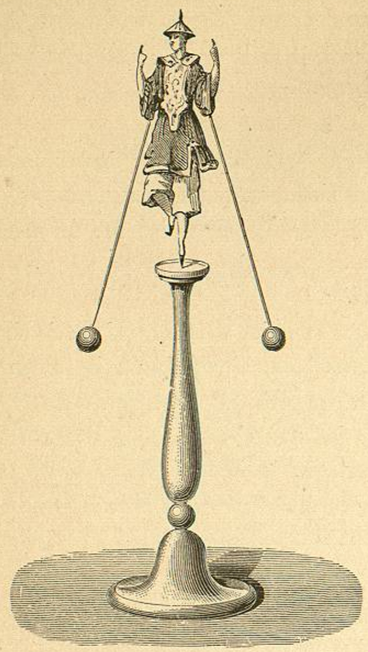

Fig. 18.—Balancer.

54. Limits of Stability.—In the foregoing discussion we have employed the term "stability" in its strict mathematical sense. But there are cases in which, though small displacements would merely produce small oscillations, larger displacements would cause the body, when left to itself, to fall entirely away from the given position of equilibrium. This may be expressed by saying that the equilibrium is stable for displacements lying within certain limits, but unstable for displacements beyond these limits. The equilibrium

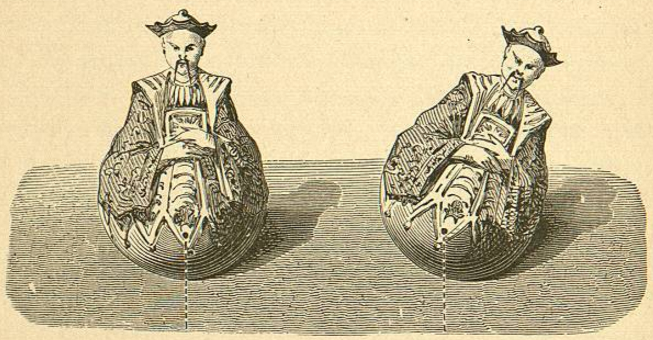

Fig. 19.—Tumblers.

of a system is *practically* unstable when the displacements which it is likely to receive from accidental disturbances lie beyond its limits of stability.

CHAPTER IV.

THE MECHANICAL POWERS.

55. We now proceed to a few practical applications of the foregoing principles; and we shall begin with the so-called "mechanical powers," namely, the *lever*, the *wheel and axle*, the *pulley*, the *inclined plane*, the *wedge*, and the *screw*.

56. Lever.—Problems relating to the lever are usually most conveniently solved by taking moments round the fulcrum. The general condition of equilibrium is, that the moments of the power and the weight about the fulcrum must be in opposite directions, and must be equal. When the power and weight act in parallel directions, the conditions of equilibrium are precisely those of three parallel forces (§ 19), the third force being the reaction of the fulcrum.

It is usual to distinguish three "orders" of lever. In levers of the first order (Fig. 20) the fulcrum is between the power and the

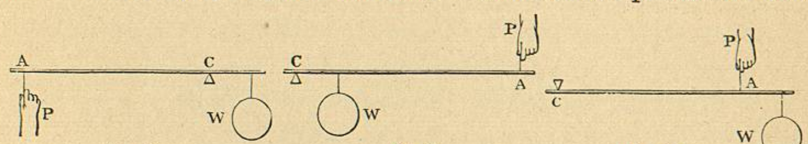

Fig. 20. Fig. 21. Fig. 22.

Three Orders of Lever.

weight. In those of the second order (Fig. 21) the weight is between the power and the fulcrum. In those of the third order (Fig. 22) the power is between the weight and the fulcrum.

In levers of the second order (supposing the forces parallel), the weight is equal to the sum of the power and the pressure on the fulcrum; and in levers of the third order, the power is equal to the sum of the weight and the pressure on the fulcrum; since the middle one of three parallel forces in equilibrium must always be equal to the sum of the other two.