87. The C.G.S. System.—A committee of the British Association, specially appointed to recommend a system of units for general adoption in scientific calculation, have recommended that the *centimetre* be adopted as the unit of length, the *gramme* as the unit of mass, and the *second* as the unit of time. We shall first give the rough and afterwards the more exact definitions of these quantities.

The centimetre is approximately $\frac{1}{10^9}$ of the distance of either pole of the earth from the equator; that is to say 1 followed by 9 zeros expresses this distance in centimetres.

The gramme is approximately the mass of a cubic centimetre of cold water. Hence the same number which expresses the specific gravity of a substance referred to water, expresses also the mass of a cubic centimetre of the substance, in grammes.

The second is $\frac{1}{24 \times 60 \times 60}$ of a mean solar day.

More accurately, the centimetre is defined as one hundredth part of the length, at the temperature 0° Centigrade, of a certain standard bar, preserved in Paris, carefully executed copies of which are preserved in several other places; and the gramme is defined as one thousandth part of the mass of a certain standard which is preserved at Paris, and of which also there are numerous copies preserved elsewhere.

For brevity of reference, the committee have recommended that the system of units based on the Centimetre, Gramme, and Second, be called the C.G.S. system.

The unit of area in this system is the square centimetre.

The unit of volume is the cubic centimetre.

The unit of velocity is a velocity of a centimetre per second.

The unit of momentum is the momentum of a gramme moving with a velocity of a centimetre per second.

The unit force is that force which generates this momentum in one second. It is therefore that force which, acting on a gramme for one second, generates a velocity of a centimetre per second. This force is called the *dyne*, an abbreviated derivative from the Greek δύναμις (force).

The unit of work is the work done by a force of a dyne working through a distance of a centimetre. It might be called the dyne-centimetre, but a shorter name has been provided and it is called the *erg*, from the Greek ἔργον (work).

CHAPTER VII.

LAWS OF FALLING BODIES.

88. Effect of the Resistance of the Air.—In air, bodies fall with unequal velocities; a sovereign or a ball of lead falls rapidly, a piece of down or thin paper slowly. It was formerly thought that this difference was inherent in the nature of the materials; but it is easy to show that this is not the case, for if we compress a mass of down or a piece of paper by rolling it into a ball, and compare it with a piece of gold-leaf, we shall find that the latter body falls more slowly than the former. The inequality of the velocities which we observe is due to the resistance of the air, which increases with the extent of surface exposed by the body.

It was Galileo who first discovered the cause of the unequal rapidity of fall of different bodies. To put the matter to the test, he prepared small balls of different substances, and let them fall at the same time from the top of the tower of Pisa; they struck the ground almost at the same instant. On changing their forms, so as to give them very different extents of surface, he observed that they fell with very unequal velocities. He was thus led to the conclusion that gravity acts on all substances with the same intensity, and that in a vacuum all bodies would fall with the same velocity.

This last proposition could not be put to the test of experiment in the time of Galileo, the air-pump not having yet been invented. The experiment was performed by Newton, and is now well known as the "guinea and feather" experiment. For this purpose a tube from a yard and a half to two yards long is used, which can be exhausted of air, and which contains bodies of various densities, such as a coin, pieces of paper, and feathers. When the tube is full of air and is inverted, these different bodies are seen to fall with very unequal velocities; but if the experiment is repeated after the tube has been exhausted of air, no difference can be perceived between the times of their descent.

89. Mass and Gravitation Proportional.—This experiment proves that bodies which have equal weights are equal in mass. For equal masses are defined to be those which, when acted on by equal forces, receive equal accelerations; and the forces, in this experiment, are the weights of the falling bodies.

Newton tested this point still more severely by experiments with pendulums (*Principia*, book III. prop. vi.). He procured two round wooden boxes of the same size and weight, and suspended them by threads eleven feet long. One of them he filled with wood, and he placed very accurately in the centre of oscillation of the other the same weight of gold. The boxes hung side by side, and, when set swinging in equal oscillations, went and returned together for a very long time. Here the forces concerned in producing and checking the motion, namely, the force of gravity and the resistance of the air, were the same for the two pendulums, and as the movements produced were the same, it follows that the masses were equal. Newton remarks that a difference of mass amounting to a thousandth part of the whole could not have escaped detection. He experimented in the same way with silver, lead, glass, sand, salt, water, and wheat, and with the same result. He therefore infers that universally bodies of equal mass gravitate equally towards the earth at the same place. He further extends the same law to gravitation generally, and establishes the conclusion that the mutual gravitating force between any two bodies depends only on their masses and distances, and is independent of their materials.

The time of revolution of the moon round the earth, considered in conjunction with her distance from the earth, shows that the relation between mass and gravitation is the same for the material of which the moon is composed as for terrestrial matter; and the same conclusion is proved for the planets by the relation which exists between their distances from the sun and their times of revolution in their orbits.

90. Uniform Acceleration.—The fall of a heavy body furnishes an illustration of the second law of motion, which asserts that the change of momentum in a body in a given time is a measure of the force which acts on the body. It follows from this law that if the same force continues to act upon a body the changes of momentum in successive equal intervals of time will be equal. When a heavy

body originally at rest is allowed to fall, it is acted on during the time of its descent by its own weight and by no other force, if we neglect the resistance of the air. As its own weight is a constant force, the body receives equal changes of momentum, and therefore of velocity, in equal intervals of time. Let g denote its velocity in centimetres per second, at the end of the first second. Then at the end of the next second its velocity will be $g+g$, that is $2g$; at the end of the next it will be $2g+g$, that is $3g$, and so on, the gain of velocity in each second being equal to the velocity generated in the first second. At the end of t seconds the velocity will therefore be tg. Such motion as this is said to be *uniformly accelerated*, and the constant quantity g is the measure of the acceleration. Acceleration is defined as the gain of velocity per unit of time.

91. Weight of a Gramme in Dynes. Value of g.—Let m denote the mass of the falling body in grammes. Then the change of momentum in each second is mg, which is therefore the measure of the force acting on the body. The weight of a body of m grammes is therefore mg dynes, and the weight of 1 gramme is g dynes. The value of g varies from 978·1 at the equator to 983·1 at the poles; and 981 may be adopted as its average value in temperate latitudes. Its value at any part of the earth's surface is approximately given by the formula

$$g = 980{\cdot}6056 - 2{\cdot}5028 \cos 2\lambda - {\cdot}000{,}003h,$$

in which λ denotes the latitude, and h the height (in centimetres) above sea-level.[1]

In § 79 we distinguished between the intensity and the amount of a force. The amount of the force of gravity upon a mass of m grammes is mg dynes. The intensity of this force is g dynes per gramme. The intensity of a force, in dynes per gramme of the body acted on, is always equal to the change of velocity which the force produces per second, this change being expressed in centimetres per second. In other words the intensity of a force is equal to the acceleration which it produces. The best designation for g is *the intensity of gravity*.

92. Distance fallen in a Given Time.—The distance described in a given time by a body moving with uniform velocity is calculated by multiplying the velocity by the time; just as the area of a rectangle is calculated by multiplying its length by its breadth. Hence if we draw a line such that its ordinates AA′, BB′, &c., represent the

[1] For the method of determination see § 120.

velocities with which a body is moving at the times represented by OA, OB (time being reckoned from the beginning of the motion), it can be shown that the whole distance described is represented by the area OB′B bounded by the curve, the last ordinate, and the base line. In fact this area can be divided into narrow strips (one of which is shown at AA′, Fig. 37) each of which may practically be regarded as a rectangle, whose height represents the velocity with which the body is moving during the very small interval of time represented by its base, and whose area therefore represents the distance described in this time.

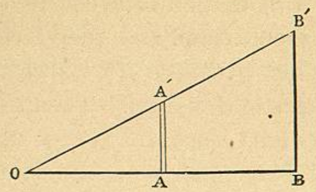

Fig. 37.

This would be true for the distance described by a body moving from rest with any law of velocity. In the case of falling bodies the law is that the velocity is simply proportional to the time; hence the ordinates AA′, BB′, &c., must be directly as the abscissæ OA, OB; this proves that the line OA′ B′ must be straight; and the figure OB′ B is therefore a triangle. Its area will be half the product of OB and BB′. But OB represents the time t occupied by the motion, and BB′ the velocity gt at the end of this time. The area of the triangle therefore represents half the product of t and gt, that is, represents $\frac{1}{2}gt^2$, which is accordingly the distance described in the time t. Denoting this distance by s, and the velocity at the end of time t by v, we have thus the two formulæ

$$v = gt, \qquad (1)$$

$$s = \tfrac{1}{2}gt^2, \qquad (2)$$

from which we easily deduce

$$gs = \tfrac{1}{2}v^2. \qquad (3)$$

93. Work spent in Producing Motion.—We may remark, in passing, that the third of these formulæ enables us to calculate the work required to produce a given motion in a given mass. When a body whose mass is 1 gramme falls through a distance s, the force which acts upon it is its own weight, which is g dynes, and the work done upon it is gs ergs. Formula (3) shows that this is the same as $\frac{1}{2}v^2$ ergs. For a mass of m grammes falling through a distance s, the work is $\frac{1}{2}mv^2$ ergs. *The work required to produce a velocity* v (centimetres per second) *in a body of mass* m (grammes) originally at rest *is* $\frac{1}{2}mv^2$ (ergs).

94. Body thrown Upwards.—When a heavy body is projected vertically upwards, the formulæ (1) (2) (3) of § 92 will still apply to its motion, with the following interpretations:—

v denotes the velocity of projection.

t denotes the whole time occupied in the ascent.

s denotes the height to which the body will ascend.

When the body has reached the highest point, it will fall back, and its velocity at any point through which it passes twice will be the same in going up as in coming down.

95. Resistance of the Air.—The foregoing results are rigorously applicable to motion in vacuo, and are sensibly correct for motion in air as long as the resistance of the air is insignificant in comparison with the force of gravity. The force of gravity upon a body is the same at all velocities; but the resistance of the air increases with the velocity, and increases more and more rapidly as the velocity becomes greater; so that while at very slow velocities an increase of 1 per cent. in velocity would give an increase of 1 per cent. in the resistance, at a higher velocity it would give an increase of 2 per cent., and at the velocity of a cannon-ball an increase of 3 per cent.[1] The formulæ are therefore sensibly in error for high velocities. They are also in error for bodies which, like feathers or gold-leaf, have a large surface in proportion to their weight.

96. Projectiles.—If, instead of being simply let fall, a body is projected in any direction, its motion will be compounded of the motion of a falling body and a uniform motion in the direction of projection. Thus if OP (Fig. 38) is the direction of projection, and OQ the vertical through the point of projection, the body would move along OP keeping its original velocity unchanged, if it were not disturbed by gravity. To find where the body will be at any time t, we must lay off a length OP equal to Vt, V denoting the velocity of projection, and must then draw from P the vertical line PR downwards equal to $\frac{1}{2}gt^2$, which is the distance that the body would have fallen in the time if simply dropped. The point R thus determined, will be the actual position of the body. The velocity of the body at any time will in like manner be found by compounding the initial

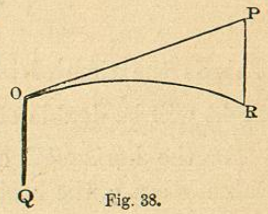

Fig. 38.

[1] This is only another way of saying that the resistance varies approximately as the velocity when very small, and approximately as the cube of the velocity for velocities like that of a cannon-ball.

velocity with the velocity which a falling body would have acquired in the time.

The path of the body will be a curve, as represented in the figure, OP being a tangent to it at O, and its concavity being downwards. The equations above given, namely

$$OP = Vt, \; PR = \tfrac{1}{2}gt^2,$$

show that PR varies as the square of OP, and hence that the path (or *trajectory* as it is technically called) is a parabola, whose axis is vertical.

97. Time of Flight, and Range.—If the body is projected from a point at the surface of the ground (supposed level) we can calculate the time of flight and the range in the following way.

Let α be the angle which the direction of projection makes with the horizontal. Then the velocity of projection can be resolved into two components, $V \cos \alpha$ and $V \sin \alpha$, the former being horizontal, and the latter vertically upward. The horizontal component of the velocity of the body is unaffected by gravity and remains constant. The vertical velocity after time t will be compounded of $V \sin \alpha$ upwards and gt downwards. It will therefore be an upward velocity $V \sin \alpha - gt$, or a downward velocity $gt - V \sin \alpha$. At the highest point of its path, the body will be moving horizontally and the vertical component of its velocity will be zero; that is, we shall have

$$V \sin \alpha - gt = 0; \text{ whence } t = \frac{V \sin \alpha}{g}.$$

This is the time of attaining the highest point; and the time of flight will be double of this, that is, will be $\frac{2V \sin \alpha}{g}$.

As the horizontal component of the velocity has the constant value $V \cos \alpha$, the horizontal displacement in any time t is $V \cos \alpha$ multiplied by t. The range is therefore

$$\frac{2V^2 \sin \alpha \cos \alpha}{g} \text{ or } \frac{V^2 \sin 2\alpha}{g}.$$

The range (for a given velocity of projection) will therefore be greatest when $\sin 2\alpha$ is greatest, that is when $2\alpha = 90°$ and $\alpha = 45°$.

We shall now describe two forms of apparatus for illustrating the laws of falling bodies.

98. Morin's Apparatus.—Morin's apparatus consists of a wooden cylinder covered with paper, which can be set in uniform rotation about its axis by the fall of a heavy weight. The cord which sup-

ports the weight is wound upon a drum, furnished with a toothed wheel which works on one side with an endless screw on the axis of the cylinder, and on the other drives an axis carrying fans which serve to regulate the motion.

In front of the turning cylinder is a cylindro-conical weight of cast-iron carrying a pencil whose point presses against the paper, and having ears which slide on vertical threads, serving to guide it in its fall. By pressing a lever, the weight can be made to fall at a chosen moment. The proper time for this is when the motion of the cylinder has become sensibly uniform. It follows from this arrangement that during its vertical motion the pencil will meet in succession the different generating lines[1] of the revolving cylinder, and will consequently describe on its surface a certain curve, from the study of which we shall be able to gather the law of the fall of the body which has traced it. With this view, we describe (by turning the cylinder while the pencil is stationary) a circle passing through the commencement of the curve, and also draw a vertical line through this point. We cut the paper along this latter line and develop it (that is, flatten

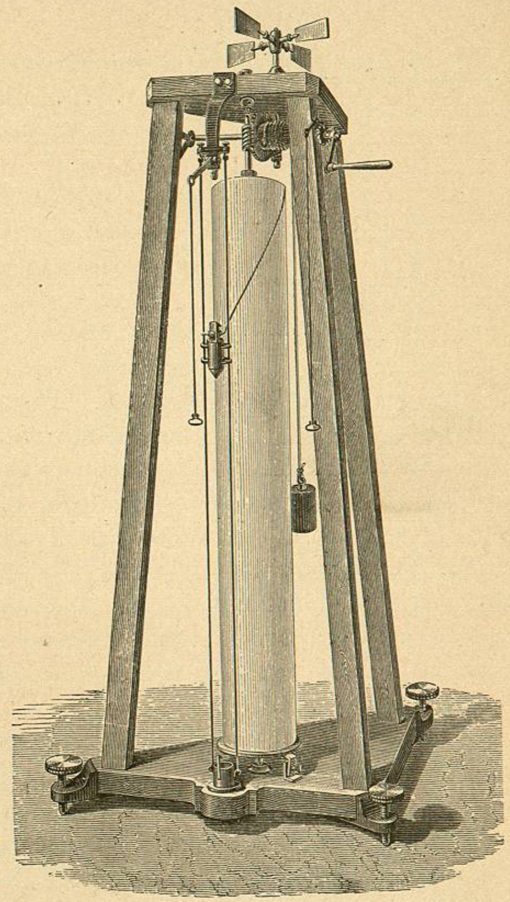

Fig. 39.—Morin's Apparatus.

[1] A cylindric surface could be swept out or "generated" by a straight line moving round the axis and remaining always parallel to it. The successive positions of this generating line are called the "generating lines of the cylinder."