CHAPTER VIII.

THE PENDULUM.

106. The Pendulum.—When a body is suspended so that it can turn about a horizontal axis which does not pass through its centre of gravity, its only position of stable equilibrium is that in which its centre of gravity is in the same vertical plane with the axis and below it (§ 42). If the body be turned into any other position, and left to itself, it will oscillate from one side to the other of the position of equilibrium, until the resistance of the air and the friction of the axis gradually bring it to rest. A body thus suspended, whatever be its form, is called a pendulum. It frequently consists of a rod which can turn about an axis O (Fig. 44) at its upper end, and which carries at its lower end a heavy lens-shaped piece of metal M called the bob; this latter can be raised or lowered by means of the screw V. The applications of the pendulum are very important: it regulates our clocks, and it has enabled us to measure the intensity of gravity in different parts of the world; it is important then to know at least the fundamental points in its theory. For explaining these, we shall begin with the consideration of an ideal body called the *simple pendulum.*

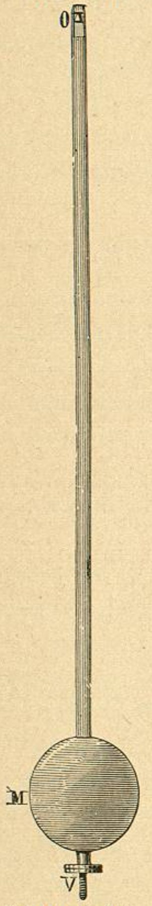

Fig. 44.—Pendulum.

107. Simple Pendulum.—This is the name given to a pendulum consisting of a heavy particle M (Fig. 45) attached to one end of an inextensible thread without weight, the other end of the thread being fixed at A. When the thread is vertical, the weight of the particle acts in the direction of its length, and there is equilib-

rium. But suppose it is drawn aside into another position, as AM. In this case, the weight MG of the particle can be resolved into two forces MC and MH. The former, acting along the prolongation of the thread, is destroyed by the resistance of the thread; the other, acting along the tangent MH, produces the motion of the particle. This effective component is evidently so much the greater as the angle of displacement from the vertical position is greater. The particle will therefore move along an arc of a circle described from A as centre, and the force which urges it forward will continually diminish till it arrives at the lowest point M′. At M′ this force is zero, but, in virtue of the velocity acquired, the particle will ascend on the opposite side, the effective component of gravity being now opposed to the direction of its motion; and, inasmuch as the magnitude of this component goes through the same series of values in this part of the motion as in the former part, but in reversed order, the velocity will, in like manner, retrace its former values, and will become zero when the particle has risen to a point M″ at the same height as M. It then descends again and performs an oscillation from M″ to M precisely similar to the first, but in the reverse direction. It will thus continue to vibrate between the two points M, M″ (friction being supposed excluded), for an indefinite number of times, all the vibrations being of equal extent and performed in equal periods.

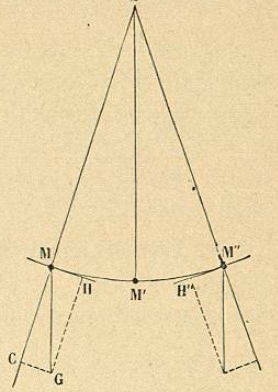

Fig. 45.—Motion of Simple Pendulum.

The distance through which a simple pendulum travels in moving from its lowest position to its furthest position on either side, is called its *amplitude.* It is evidently equal to half the complete arc of vibration, and is commonly expressed, not in linear measure, but in degrees of arc. Its numerical value is of course equal to that of the angle MAM′, which it subtends at the centre of the circle.

The *complete period* of the pendulum's motion is the time which it occupies in moving from M to M″ and back to M, or more generally, is the time from its passing through any given position to its next passing through the same position *in the same direction.*

What is commonly called the time of vibration, or the time of a single vibration, is the half of a complete period, being the time of passing from one of the two extreme positions to the other. Hence what we have above defined as a complete period is often called a double vibration.

When the amplitude changes, the time of vibration changes also, being greater as the amplitude is greater; but the connection between the two elements is very far from being one of simple proportion. The change of time (as measured by a ratio) is much less than the change of amplitude, especially when the amplitude is small; and when the amplitude is less than about 5°, any further diminution of it has little or no sensible effect in diminishing the time. *For small vibrations,* then, *the time of vibration is independent of the amplitude.* This is called the law of *isochronism.*

108. Law of Acceleration for Small Vibrations.—Denoting the length of a simple pendulum by l, and its inclination to the vertical at any moment by θ, we see from Fig. 45 that the ratio of the effective component of gravity to the whole force of gravity is $\frac{\text{MH}}{\text{MG}}$, that is $\sin\theta$; and when θ is small this is sensibly equal to θ itself as measured by $\frac{\text{arc}}{\text{radius}}$. Let s denote the length of the arc MM′ intervening between the lower end of the pendulum and the lowest point of its swing, at any time; then θ is equal to $\frac{s}{l}$, and the intensity of the effective force of gravity when θ is small is sensibly equal to $g\theta$, that is to $\frac{gs}{l}$. Since g and l are the same in all positions of the pendulum, this effective force varies as s. Its direction is always *towards* the position of equilibrium, so that it accelerates the motion during the approach to this position, and retards it during the recess; the acceleration or retardation being always in direct proportion to the distance from the position of equilibrium. This species of motion is of extremely common occurrence. It is illustrated by the vibration of either prong of a tuning-fork, and in general by the motion of any body vibrating in one plane in such a manner as to yield a simple musical tone.

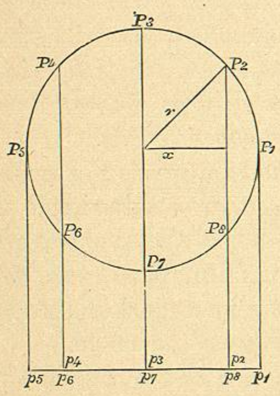

Fig. 46.—Projection of Circular Motion.

109. General Law for Period.—Suppose a point P to travel with uniform velocity round a circle (Fig. 46), and from its successive

positions P_1, P_2, &c., let perpendiculars P_1p_1, P_2p_2, &c., be drawn to a fixed straight line in the plane of the circle. Then while P travels once round the circle, its projection p executes a complete vibration.

The acceleration of P is always directed towards the centre of the circle, and is equal to $\left(\frac{2\pi}{\text{T}}\right)^2 r$ (§ 101). The component of this acceleration parallel to the line of motion of p, is the fraction $\frac{x}{r}$ of the whole acceleration (x denoting the distance of p from the middle point of its path), and is therefore $\left(\frac{2\pi}{\text{T}}\right)^2 x$. This is accordingly the acceleration of p, and as it is simply proportional to x we shall denote it for brevity by μx. To compute the periodic time T of a complete vibration, we have the equation $\mu = \left(\frac{2\pi}{\text{T}}\right)^2$, which gives

$$\text{T} = \frac{2\pi}{\sqrt{\mu}}. \tag{1}$$

110. Application to the Pendulum.—For the motion of a pendulum in a small arc, we have

$$\text{acceleration} = \frac{g}{l}\, s,$$

where s denotes the displacement in linear measure. We must therefore put $\mu = \frac{g}{l}$, and we then have

$$\text{T} = 2\pi\sqrt{\frac{l}{g}} \tag{2}$$

which is the expression for the time of a complete (or double) vibration. It is more usual to understand by the "time of vibration" of a pendulum the half of this, that is the time from one extreme position to the other, and to denote this time by T. In this sense we have

$$\text{T} = \pi\sqrt{\frac{l}{g}}. \tag{3}$$

To find the length of the seconds' pendulum we must put T = 1. This gives

$$\pi^2 \cdot \frac{l}{g} = 1, \quad l = \frac{g}{\pi^2} = \frac{g}{9{\cdot}87} \text{ nearly.}$$

If g were 987 we should have $l = 100$ centimetres or 1 metre. The actual value of g is everywhere a little less than this. The length of the seconds' pendulum is therefore everywhere rather less than a metre.

111. Simple Harmonic Motion.—Rectilinear motion consisting of vibration about a point with acceleration μx, where x denotes distance from this point, is called *Simple Harmonic Motion*, or Simple Harmonic Vibration. The above investigation shows that such vibration is isochronous, its period being $\frac{2\pi}{\sqrt{\mu}}$ whatever the amplitude may be.

To understand the reason of this isochronism we have only to remark that, if the amplitude be changed, the velocity at corresponding points (that is, points whose distances from the middle point are the same fractions of the amplitudes) will be changed in the same ratio. For example, compare two simple vibrations in which the values of μ are the same, but let the amplitude of one be double that of the other. Then if we divide the paths of both into the same number of small equal parts, these parts will be twice as great for the one as for the other; but if we suppose the two points to start simultaneously from their extreme positions, the one will constantly be moving twice as fast as the other. The number of parts described in any given time will therefore be the same for both.

In the case of vibrations which are not simple, it is easy to see (from comparison with simple vibration) that if the acceleration increases in a greater ratio than the distance from the mean position, the period of vibration will be shortened by increasing the amplitude; but if the acceleration increases in a less ratio than the distance, as in the case of the common pendulum vibrating in an arc of moderate extent, the period is increased by increasing the amplitude.

112. Experimental Investigation of the Motion of Pendulums.—The preceding investigation applies to the simple pendulum; that is to say to a purely imaginary existence; but it can be theoretically demonstrated that every rigid body vibrating about a horizontal axis under the action of gravity (friction and the resistance of the air being neglected), moves in the same manner as a simple pendulum of determinate length called the *equivalent simple pendulum*. Hence the above results can be verified by experiments on actual pendulums.

The discovery of the experimental laws of the motion of pendulums was in fact long anterior to the theoretical investigation. It was the earliest and one of the most important discoveries of Galileo, and dates from the year 1582, when he was about twenty years of age. It is related that on one occasion, when in the cathedral of Pisa, he was struck with the regularity of the oscillations of a lamp suspended from the roof, and it appeared to him that these oscillations, though diminishing in extent, preserved the same duration. He tested the fact by repeated trials, which confirmed him in the belief of its perfect exactness. This law of isochronism can be easily verified. It is only necessary to count the vibrations which take place in a given time with different amplitudes. The numbers will be found to be exactly the same. This will be found to hold good even when some of the vibrations compared are so small that they can only be observed with a telescope.

By employing balls suspended by threads of different lengths, Galileo discovered the influence of length on the time of vibration. He ascertained that when the length of the thread increases, the time of vibration increases also; not, however, in proportion to the length simply, but to its square root.

113. Cycloidal Pendulum.—It is obvious from § 64 that the effective component of gravity upon a particle resting on a smooth inclined plane is proportional to the sine of the inclination. The acceleration of a particle so situated is in fact $g \sin \alpha$, if α denote the inclination of the plane. When a particle is guided along a smooth curve its acceleration is expressed by the same formula, α now denoting the inclination of the curve at any point to the horizon. This inclination varies from point to point of the curve, so that the acceleration $g \sin \alpha$ is no longer a constant quantity. The motion of a common pendulum corresponds to the motion of a particle which is guided to move in a circular arc; and if x denote distance from the lowest point, measured along the arc, and r the radius of the circle (or the length of the pendulum), the acceleration at any point is $g \sin \frac{x}{r}$.

This is sensibly proportional to x so long as x is a small fraction of r; but in general it is not proportional to x, and hence the vibrations are not in general isochronous.

To obtain strictly isochronous vibrations we must substitute for the circular arc a curve which possesses the property of having an inclination whose sine is simply proportional to distance measured along the curve from the lowest point. The curve which possesses this property is the cycloid. It is the curve which is traced by a point in the circumference of a circle which rolls along a straight line. The cycloidal pendulum is constructed by suspending an ivory ball or some other small heavy body by a thread between two cheeks (Fig. 47), on which the thread winds as the ball swings to either side. The cheeks must themselves be the two halves of a cycloid whose length is double that of the thread, so that each cheek has the same length as the thread. It can be demonstrated[1] that under these circumstances the path of the ball will be a cycloid identical with that to which the cheeks belong. Neglecting friction and the rigidity of the thread, the acceleration in this case is proportional to distance measured along the cycloid from its lowest point, and hence the time of vibration will be strictly the same for large as for small amplitudes. It will, in fact, be the same as that of a simple pendulum having the same length as the cycloidal pendulum and vibrating in a small arc.

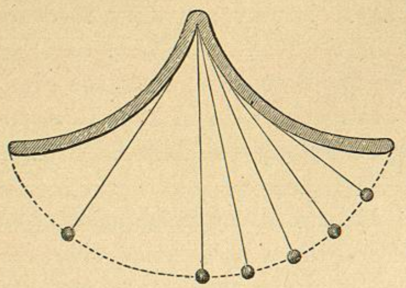

Fig 47.—Cycloidal Pendulum.

Attempts have been made to adapt the cycloidal pendulum to clocks, but it has been found that, owing to the greater amount of friction, its rate was less regular than that of the common pendulum. It may be remarked, that the spring by which pendulums are often suspended has the effect of guiding the pendulum bob in a curve which is approximately cycloidal, and thus of diminishing the irregularity of rate resulting from differences of amplitude.

114. Moment of Inertia.—Just as the mass of a body is the measure of the force requisite for producing unit acceleration when the movement is one of pure translation; so the *moment of inertia* of a rigid body turning about a fixed axis is the measure of the couple requisite for producing unit acceleration of angular velocity. We suppose angle to be measured by $\frac{\text{arc}}{\text{radius}}$, so that the angle turned by the body is equal to the arc described by any point of it divided by the distance of this point from the axis; and the angular velocity of the body will be the velocity of any point divided by its distance from the axis. The moment of inertia of the body round the axis is numerically equal to the couple which would produce unit change of angular velocity in the body in unit time. We shall now show how to express the moment of inertia in terms of the masses of the particles of the body and their distances from the axis.

[1] Since the evolute of the cycloid is an equal cycloid.

Let m denote the mass of any particle, r its distance from the axis, and ϕ the angular acceleration. Then $r\phi$ is the acceleration of the particle m, and the force which would produce this acceleration by acting directly on the particle along the line of its motion is $mr\phi$. The moment of this force round the axis would be $mr^2\phi$ since its arm is r. The aggregate of all such moments as this for all the particles of the body is evidently equal to the couple which actually produces the acceleration of the body. Using the sign Σ to denote "the sum of such terms as," and observing that ϕ is the same for the whole body, we have

$$\text{Applied couple} = \Sigma\,(mr^2\phi) = \phi\,\Sigma\,(mr^2). \qquad (1)$$

When ϕ is unity, the applied couple will be equal to $\Sigma\,(mr^2)$, which is therefore, by the foregoing definition, the moment of inertia of the body round the axis.

115. Moments of Inertia Round Parallel Axes.—The moment of inertia round an axis through the centre of mass is always less than that round any parallel axis.

For if r denote the distance of the particle m from an axis not passing through the centre of mass, and x and y its distances from two mutually rectangular planes through this axis, we have $r^2=x^2+y^2$.

Now let two planes parallel to these be drawn through the centre of mass; let ξ and η be the distances of m from them, and ρ its distance from their line of intersection, which will clearly be parallel to the given axis. Also let a and b be the distances respectively between the two pairs of parallel planes, so that a^2+b^2 will be the square of the distance between the two parallel axes, which distance we will denote by h. Then we have

$$x = \xi \pm a$$
$$y = \eta \pm b$$
$$x^2 = a^2 + \xi^2 \pm 2a\,\xi, \qquad y^2 = b^2 + \eta^2 \pm 2b\,\eta.$$

$$\begin{aligned}\Sigma\,(mr^2) &= \Sigma\left\{m\,(a^2+b^2)\right\} + \Sigma\left\{m\,(\xi^2+\eta^2)\right\} \\ &\quad \pm 2a\,\Sigma\,(m\xi) \pm 2b\,\Sigma\,(m\eta) \\ &= h^2\,\Sigma m + \Sigma\,(m\rho^2) \pm 2a\,\bar{\xi}\,\Sigma m \pm 2b\,\bar{\eta}\,\Sigma m.\end{aligned}$$

where $\bar{\xi}$ and $\bar{\eta}$ are the values of ξ and η for the centre of mass. But these values are both zero, since the centre of mass lies on both the planes from which ξ and η are measured. We have therefore

$$\Sigma\,(mr^2) = h^2\,\Sigma m + \Sigma\,(m\rho^2), \qquad (2)$$

that is to say, the moment of inertia round the given axis exceeds the moment of inertia round the parallel axis through the centre of mass by the product of the whole mass into the square of the distance between the axes.

116. Application to Compound Pendulum.—The application of this principle to the compound pendulum leads to some results of great interest and importance.

Let M be the mass of a compound pendulum, that is, a rigid body free to oscillate about a fixed horizontal axis. Let h, as in the preceding section, denote the distance of the centre of mass from this axis; let θ denote the inclination of h to the vertical, and ϕ the angular acceleration.

Then, since the forces of gravity on the body are equivalent to a single force Mg, acting vertically downwards at the centre of mass, and therefore having an arm $h \sin \theta$ with respect to the axis, the moment of the applied forces round the axis is $\mathrm{M}gh \sin \theta$; and this must, by § 114, be equal to $\phi \Sigma\ (mr^2)$. We have therefore

$$\frac{\Sigma\ (mr^2)}{\mathrm{M}h} = \frac{g \sin \theta}{\phi}. \qquad (3)$$

If the whole mass were collected at one point at distance l from the axis, this equation would become

$$\frac{\mathrm{M}l^2}{\mathrm{M}l} = l = \frac{g \sin \theta}{\phi}; \qquad (4)$$

and the angular motion would be the same as in the actual case if l had the value

$$l = \frac{\Sigma\ mr^2}{\mathrm{M}h}. \qquad (5)$$

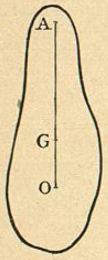

Fig. 48.

l is evidently the length of the equivalent simple pendulum.

117. Convertibility of Centres.—Again, if we introduce a length k such that $\mathrm{M}k^2$ is equal to $\Sigma\ (m\rho^2)$, that is, to the moment of inertia round a parallel axis through the centre of mass, we have

$$\Sigma\ (mr^2) = \Sigma\ (m\rho^2) + h^2\ \Sigma m = \mathrm{M}k^2 + \mathrm{M}h^2,$$

and equation (5) becomes

$$l = \frac{k^2 + h^2}{h} = \frac{k^2}{h} + h, \qquad (6)$$

$$\text{or } k^2 = (l - h)\ h. \qquad (7)$$

In the annexed figure (Fig. 48) which represents a vertical section through the centre of mass, let G be the centre of mass, A the "centre

of suspension," that is, the point in which the axis cuts the plane of the figure, and O the "centre of oscillation," that is, the point at which the mass might be collected without altering the movement. Then, by definition, we have

$$l = \mathrm{AO},\ h = \mathrm{AG}, \text{ therefore } l - h = \mathrm{GO},$$

so that equation (7) signifies

$$k^2 = \mathrm{AG}\ .\ \mathrm{GO}. \qquad (8)$$

Since k^2 is the same for all parallel axes, this equation shows that when the body is made to vibrate about a parallel axis through O, the centre of oscillation will be the point A. That is to say; *the centres of suspension and oscillation are interchangeable, and the product of their distances from the centre of mass is* k^2.

118. If we take a new centre of suspension A′ in the plane of the figure, the new centre of oscillation O′ will lie in the production of A′G, and we must have

$$\mathrm{A'G}\ .\ \mathrm{GO'} = k^2 = \mathrm{AG}\ .\ \mathrm{GO}.$$

If A′G be equal to AG, GO′ will be equal to GO, and A′O′ to AO, so that the length of the equivalent simple pendulum will be unchanged. *A compound pendulum will* therefore *vibrate in the same time about all parallel axes which are equidistant from the centre of mass.*

When the product of two quantities is given, their sum is least when they are equal, and becomes continually greater as they depart further from equality. Hence the length of the equivalent simple pendulum AO or AG + GO is least when

$$\mathrm{AG} = \mathrm{GO} = k,$$

and increases continually as the distance of the centre of suspension from G is either increased from k to infinity or diminished from k to zero. Hence, when a body vibrates about an axis which passes very nearly through its centre of gravity, its oscillations are exceedingly slow.

119. Kater's Pendulum.—The principle of the convertibility of centres, established in § 117, was discovered by Huygens, and affords the most convenient practical method of constructing a pendulum of known length. In Kater's pendulum there are two parallel knife-edges about either of which the pendulum can be made to vibrate, and one of them can be adjusted to any distance from the other. The pendulum is swung first upon one of these edges and then upon the other, and, if any difference is detected in the times of vibration, it is corrected by moving the adjustable edge. When the difference has been completely destroyed, the distance between the two edges is the length of the equivalent simple pendulum. It is necessary, in any arrangement of this kind, that the two knife-edges should be in a plane passing through the centre of gravity; also that they should be on opposite sides of the centre of gravity, and at unequal distances from it.

120. Determination of the Value of g.—Returning to the formula for the simple pendulum $T=\pi\sqrt{\frac{l}{g}}$, we easily deduce from it $g=\frac{\pi^2 l}{T^2}$, whence it follows that the value of g can be determined by making a pendulum vibrate and measuring T and l. T is determined by counting the number of vibrations that take place in a given time; l can be calculated, when the pendulum is of regular form, by the aid of formulæ which are given in treatises on rigid dynamics, but its value is more easily obtained by Kater's method, described above, founded on the principle of the convertibility of the centres of suspension and oscillation.

It is from pendulum observations, taken in great numbers at different parts of the earth, that the approximate formula for the intensity of gravity which we have given at § 91 has been deduced. Local peculiarities prevent the possibility of laying down any general formula with precision; and the exact value of g for any place can only be ascertained by observations on the spot.

CHAPTER IX.

CONSERVATION OF ENERGY.

121. Definition of Kinetic Energy.—We have seen in § 93 that the work which must be done upon a mass of m grammes to give it a velocity of v centimetres per second is $\frac{1}{2}mv^2$ ergs. Though we have proved this only for the case of falling bodies, with gravity as the working force, the result is true universally, as is shown in advanced treatises on mathematical physics. It is true whether the motion be rectilinear or curvilinear, and whether the working force act in the line of motion or at an angle with it.

If the velocity of a mass increases from v_1 to v_2, the work done upon it in the interval is $\frac{1}{2}m\ (v_2^2-v_1^2)$; in other words, is the increase of $\frac{1}{2}mv^2$.

On the other hand, if a force acts in such a manner as to oppose the motion of a moving mass, the force will do negative work, the amount of which will be equal to the decrease in the value of $\frac{1}{2}mv^2$.

For example, during any portion of the ascent of a projectile, the diminution in the value of $\frac{1}{2}mv^2$ is equal to gm multiplied by the increase of height; and during any portion of its descent the increase in $\frac{1}{2}mv^2$ is equal to gm multiplied by the decrease of height.

The work which must have been done upon a body to give it its actual motion, supposing it to have been initially at rest, is called the *energy of motion* or the *kinetic energy* of the body. It can be computed by multiplying *half the mass by the square of the velocity.*

122. Definition of Static or Potential Energy.—When a body of mass m is at a height s above the ground, which we will suppose level, gravity is ready to do the amount of work gms upon it by making it fall to the ground. A body in an elevated position may therefore be regarded as a reservoir of work. In like manner a wound-up clock, whether driven by weights or by a spring, has