CHAPTER XXIII.

EFFLUX OF LIQUIDS.—TORRICELLI'S THEOREM.

265. If an opening is made in the side of a vessel containing water, the liquid escapes with a velocity which is greater as the surface of the liquid in the vessel is higher above the orifice, or to employ the usual phrase, as the *head* of liquid is greater. This point in the dynamics of liquids was made the subject of experiments by Torricelli, and the result arrived at by him was that the velocity of efflux is equal to that which would be acquired by a body falling freely from the upper surface of the liquid to the centre of the orifice. If h be this height, the velocity of efflux is given by the formula

$$v = \sqrt{2gh}.$$

This is called Torricelli's theorem. It supposes the orifice to be small compared with the horizontal section of the vessel, and to be exposed to the same atmospheric pressure as the upper surface of the liquid in the vessel.

It may be deduced from the principle of conservation of energy; for the escape of a mass m of liquid involves a loss mgh of energy of position, and must involve an equal gain of energy of motion. But the gain of energy of motion is $\frac{1}{2}mv^2$; hence we have

$$\tfrac{1}{2}mv^2 = mgh, \; v^2 = 2gh.$$

The form of the issuing jet will depend, to some extent, on the form of the orifice. If the orifice be a round hole with sharp edges, in a thin plate, the flow through it will not be in parallel lines, but the outer portions will converge towards the axis, producing a rapid narrowing of the jet. The section of the jet at which this convergence ceases and the flow becomes sensibly parallel, is called the *contracted vein* or *vena contracta*. The pressure within the jet at this part is atmospheric, whereas in the converging part it is greater

than atmospheric; and it is to the contracted vein that Torricelli's formula properly applies, v denoting the velocity at the contracted vein, and h the depth of its central point below the free surface of the liquid in the vessel.

266. Area of Contracted Vein. Froude's Case.—A force is equal to the momentum which it generates in the unit of time. Let A denote the area of an orifice through which a liquid issues horizontally, and a the area of the contracted vein. From the equality of action and reaction it follows that the resultant force which ejects the issuing stream is equal and opposite to the resultant horizontal force exerted on the vessel. The latter may be taken as a first approximation to be equal to the pressure which would be exerted on a plug closing the orifice, that is to ghA if the density of the liquid be taken as unity.

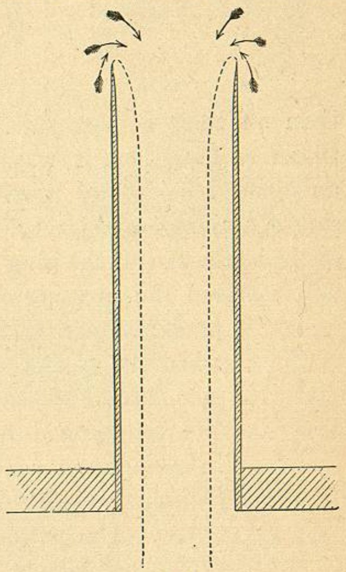

Fig. 167.

The horizontal momentum generated in the water in one second is the product of the velocity v and the mass ejected in one second. The volume ejected in one second is va. This is equal to the mass, since the density is unity, and hence the momentum is v^2a, that is, $2gha$. Equating this last expression for the momentum to the foregoing expression for the force, we have

$$2gha = gh\text{A}$$
$$a = \tfrac{1}{2}\text{A},$$

that is, the area of the contracted vein is half the area of the orifice.

Mr. Froude has pointed out that this reasoning is strictly correct when the liquid is discharged through a cylindrical pipe projecting inwards into the vessel and terminating with a sharp edge (Fig. 167); and he has verified the result by accurate experiments in which the jet was discharged vertically downwards. The direction of flow in different parts of the jet is approximately indicated by the arrows and dotted lines in the figure; and, on a larger, scale by those in Fig. 168, in which the sections of the orifice and of the contracted vein are also indicated by the lines marked D and d. We may remark that since liquids press equally in all directions, there can be no material difference between the velocities of a vertical and of a horizontal jet at the same depth below the free surface.

267. Contracted Vein for Orifice in Thin Plate.—When the liquid is simply discharged through a hole cut in the side of the vessel and bounded by a sharp edge, the direction of flow in different parts of the stream is shown by the arrows and dotted lines in Fig. 169. The pressure on the sides, in the neighbourhood of the orifice, is less than that due to the depth, because the curved form of the lines of flow implies (on the principles of centrifugal force) a smaller pressure on their concave than on their convex side. The pressure around the orifice is therefore less than it would be if the hole were plugged. The unbalanced horizontal pressure on the vessel (if we suppose the side containing the jet to be vertical) will therefore exceed the statical pressure on the plug ghA, since the removal of the plug not only removes the pressure on the plug but also a portion of the pressure on neighbouring parts. This unbalanced force, which is greater than ghA, is necessarily equal to the momentum generated per second in the liquid, which is still represented by the expression v^2a or $2gha$; hence $2gha$ is greater than ghA, or a is greater than $\frac{1}{2}A$. Reasoning similar to this applies to all ordinary forms of orifice. The peculiarity of the case investigated by Mr. Froude consists in the circumstance that the pressure on the parts of the vessel in the neighbourhood of the orifice is normal to the direction of the jet, and any changes in its amount which may be produced by unplugging the orifice have therefore no influence upon the pressures on the vessel in or opposite to the direction of the jet.[1]

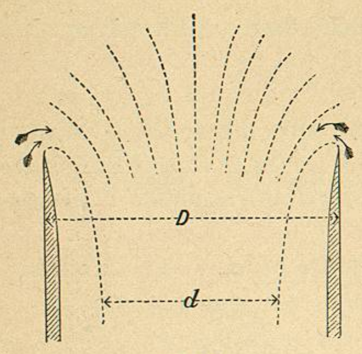

Fig. 168.

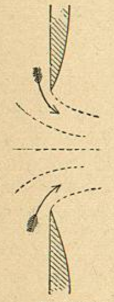
Fig. 169.

268. Apparatus for Illustration.—In the preceding investigations,

[1] This section and the preceding one are based on two communications read before the Philosophical Society of Glasgow, February 23d and March 31st, 1876; one being an extract from a letter from Mr. Froude to Sir William Thomson, and the other a communication from Professor James Thomson, to whom we are indebted for the accompanying illustrations.

no account is taken of friction. When experiments are conducted on too small a scale, friction may materially diminish the velocity; and further, if the velocity be tested by the height or distance to which the jet will spout, the resistance of the air will diminish this height or distance, and thus make the velocity appear less than it really is.

Fig. 170 represents an apparatus frequently employed for illustrating some of the consequences of Torricelli's theorem. An upright cylindrical vessel is pierced on one side with a number of orifices in the same vertical line, which can be opened or closed at pleasure. A tap placed above the vessel supplies it with water, and, with the help of an overflow pipe, maintains the surface at a constant level, which is as much above the highest orifice as each orifice is above that next below it. The liquid which escapes is received in a trough, the edge of which is graduated. A travelling piece with an index

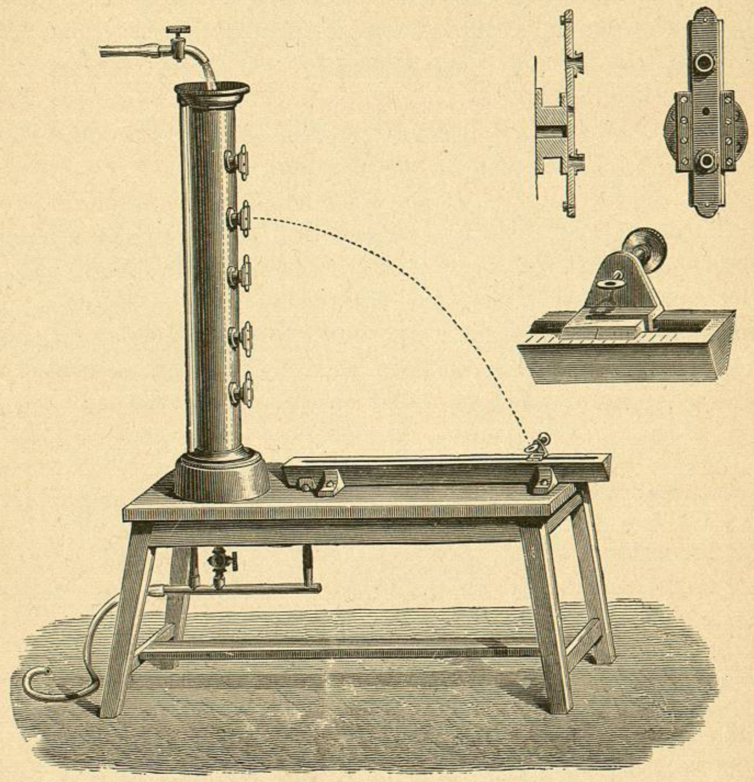
Fig. 170.—Apparatus for verifying Torricelli's Theorem.

line engraved on it slides along the trough; it carries, as shown in one of the separate figures, a disc pierced with a circular hole, and capable of being turned in any direction about a horizontal axis passing through its centre. In this way the disc can always be placed in such a position that its plane shall be at right angles to the liquid jet, and that the jet shall pass freely and exactly through its centre. The index line then indicates the range of the jet with considerable precision. This range is reckoned from the vertical plane containing the orifices, and is measured on the horizontal plane passing through the centre of the disc. The distance of this latter plane below the lowest orifice is equal to that between any two consecutive orifices.

The jet, consisting as it does of a series of projectiles travelling in the same path, has the form of a parabola.

Let a be the range of the jet, b the height of the orifice above the centre of the ring, and v the velocity of discharge, which we assume to be horizontal. Then if t be the time occupied by a particle of the liquid in passing from the orifice to the ring, we have to express that a is the distance due to the horizontal velocity v in the time t, and that b is the vertical distance due to gravity acting for the same time. We have therefore

$$a = vt$$
$$b = \tfrac{1}{2}gt^2$$
$$\text{whence } t^2 = \frac{a^2}{v^2} = \frac{2b}{g},\ v^2 = \frac{ga^2}{2b}.$$

But according to Torricelli's theorem, if h be the height of the surface of the water above the orifice, we have $v^2 = 2gh$; and comparing this with the above value of v^2 we deduce

$$\frac{a^2}{2b} = 2h,\ a^2 = 4bh.$$

One consequence of this last formula is, that if the values of b and h be interchanged, the value of a will remain unaltered. This amounts to saying that the highest orifice will give the same range as the lowest, the highest but one the same as the lowest but one, and so on; a result which can be very accurately verified.

If we describe a semicircle on the line $b+h$, the length of an ordinate erected at the point of junction of b and h is $\sqrt{bh}$, and since $a = \sqrt{4bh} = 2\sqrt{bh}$, it follows that the range is double of this ordinate. This is on the hypothesis of no friction. Practically it is less than double. The greatest ordinate of the semicircle is the central one, and accordingly the greatest range is given by the central orifice.

269. Efflux from Air-tight Space.—When the air at the free surface of the liquid in a vessel is at a different pressure from the air into which the liquid is discharged, we must express this difference of pressures by an equivalent column of the liquid, and the velocity of efflux will be that due to the height of the surface above the orifice increased or diminished by this column. Efflux will cease altogether when the pressure on the free surface, together with that due to the height of the free surface above the orifice, is equal to the pressure outside the orifice; or if efflux continue under such circumstances it can only do so by the admission of bubbles of air. This explains the action of vent-pegs.

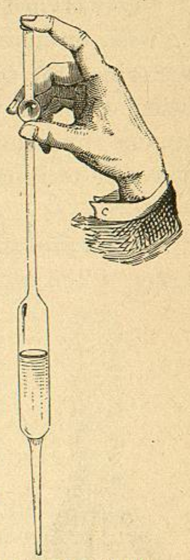

Fig 171.—Pipette.

Pipette.—This is a glass tube (Fig. 171) open at both ends, and terminating below in a small tapering spout. If water be introduced into the tube, either by aspiration or by direct immersion in water, and if the upper end be closed with the finger, the efflux of the liquid will cease almost instantly. On admitting the air above, the efflux will begin again, and can again be stopped at pleasure.

The Magic Funnel.—This funnel is double, as is shown in Fig. 172. Near the handle is a small opening by which the space between the two funnels communicates with the external air. Another opening connects this same space with the tube of the inner funnel. If the interval between the two funnels be filled with any liquid, this liquid will run out or will cease to flow according as the upper hole is open or closed. The opening and closing of the hole can be easily effected with the thumb of the hand holding the funnel without the knowledge of the spectator. This device has been known from very early times.

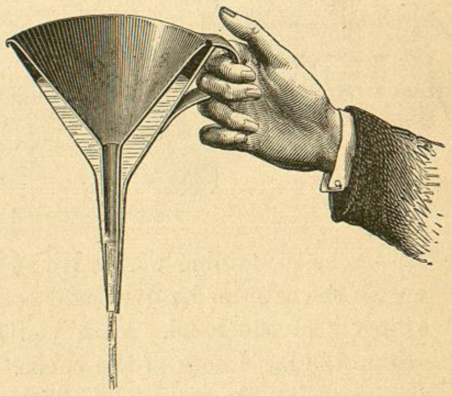

Fig. 172.—Magic Funnel.

The instrument may be used in a still more curious manner. For this purpose the space inside is secretly filled with highly-coloured wine, which is prevented from escaping by closing the opening above.

Water is then poured into the central funnel, and escapes either by itself or mixed with wine, according as the thumb closes or opens the orifice for the admission of air. In the second case, the water being coloured with the wine, it will appear that wine alone is issuing from the funnel; thus the operator will appear to have the power of making either water or wine flow from the vessel at his pleasure.

The Inexhaustible Bottle.—The inexhaustible bottle (Fig. 173) is a toy of the same kind. It is an opaque bottle of sheet-iron or

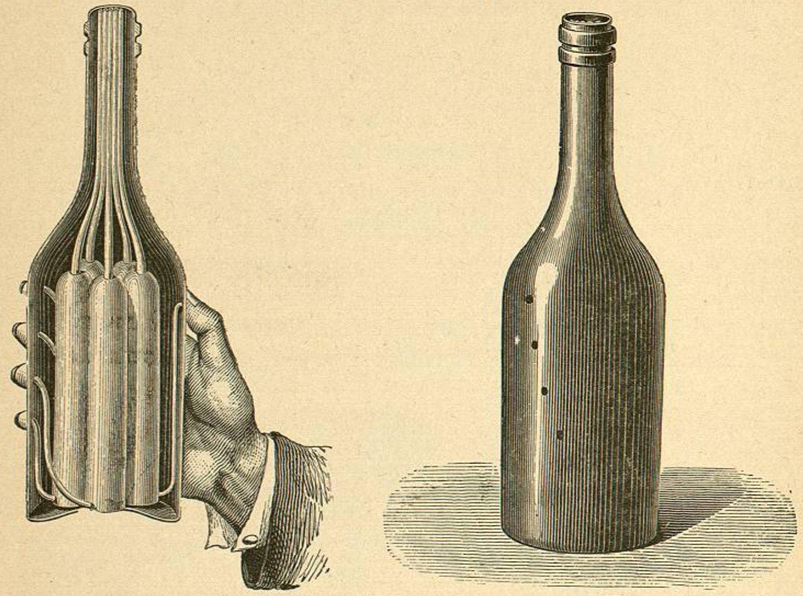

Fig. 173.—Inexhaustible Bottle.

gutta-percha, containing within it five small vials. These communicate with the exterior by five small holes, which can be closed by the five fingers of the hand. Each vial has also a small neck which passes up the large neck of the bottle. The five vials are filled with five different liquids, any one of which can be poured out at pleasure by uncovering the corresponding hole.

270. Intermittent Fountain.—The intermittent fountain is an apparatus analogous to the preceding, except that the interruptions in the efflux are produced automatically by the action of the instru-

ment, without the intervention of the operator. It consists of a globe V (Fig. 174), which can be closed air-tight by means of a stopper, and is in communication with efflux tubes *a*, which discharge into a basin B, having a small hole *o* in its bottom for permitting the water to escape into a lower basin C. A central tube *t*, open at both ends, extends nearly to the top of the globe, and nearly to the bottom of the basin B.

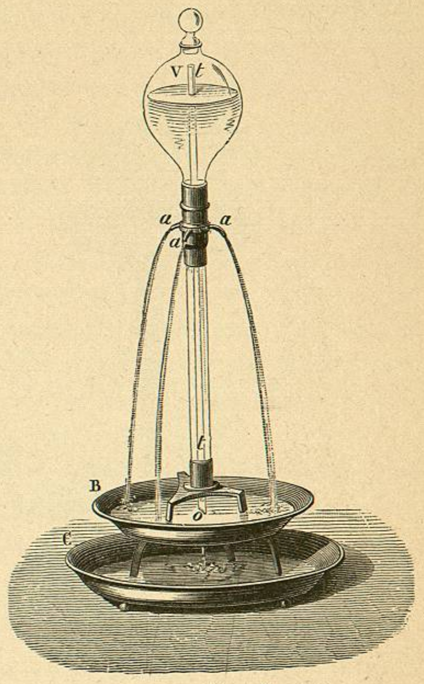

Fig. 174.—Intermittent Fountain.

Suppose the globe to be filled with water, the basins being empty. Then the water will flow from the efflux tubes *a*, while air will pass up through the central tube. As the water issues from the efflux tubes much faster than it escapes through the opening *o*, the level rises in the basin B till the lower end of the tube *t* is covered. The pressure of the air in the upper part of the globe then rapidly diminishes, and the efflux from the tubes *a* is stopped. But as the water continues to escape from the basin B through the opening *o*, the bottom of the tube *t* is again uncovered, the liquid again issues from the efflux tubes, and the same changes are repeated.

271. Siphon.—The siphon is an instrument in which a liquid, under the combined action of its own weight and atmospheric pressure, flows first up-hill and then down-hill, but always in such a way as to bring about a lowering of the centre of gravity of the whole liquid mass.

In its simplest form, it consists of a bent tube, one end of which is immersed in the liquid to be removed, while the other end either discharges into the air, at a lower level than the surface of the liquid in the vessel, as in Fig. 175, or dips into the liquid of a receiving vessel, the surface of this liquid being lower than that of the liquid in the discharging vessel.

We shall discuss the latter case, and shall denote the difference of levels of the two surfaces by h, while the height of a column of the liquid equivalent to atmospheric pressure will be denoted by H.

Let the siphon be full of liquid, and imagine a diaphragm to be drawn across it at any point, so as to prevent flow. Let this dia-

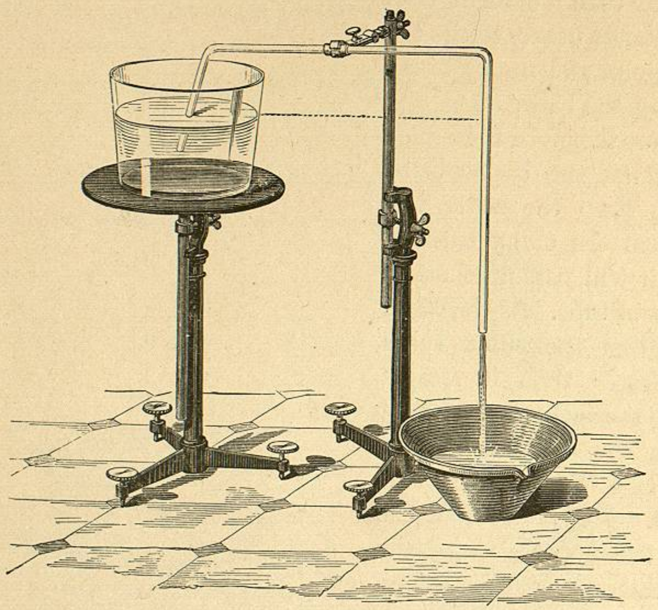

Fig. 175.—Siphon.

phragm be at a height x above the higher of the two free surfaces, and at a height y above the lower, so that we have

$$y - x = h.$$

The pressure on the side of the diaphragm next the higher free surface will be $H - x$, (pressure being expressed in terms of the equivalent liquid column,) and the pressure on the other side of the diaphragm will be $H - y$, which is less than the former by $y - x$, that is by h. The diaphragm therefore experiences a resultant force due to a depth h of the liquid, urging it from the higher to the lower free surface, and if the diaphragm be removed, the liquid will be propelled in this direction.

In practice, the two legs of the siphon are usually of unequal length, and the flow is from the shorter to the longer; but this is by no means essential, for by a sufficiently deep immersion of the long

leg, the direction of flow may be reversed. The direction of flow depends not on the lengths of the legs, but on the levels of the two free surfaces.

If the liquid in the discharging vessel falls below the end of the siphon, or if the siphon is lifted out of it, air enters, and the siphon is immediately emptied of liquid. If the liquid in the receiving vessel is removed, so that the discharging end of the siphon is surrounded by air, as in the figure, the flow will continue, unless air bubbles up the tube and breaks the liquid column. This interruption is especially liable to occur in large tubes. It can be prevented by bending the end of the siphon round, so as to discharge the liquid in an ascending direction. To adapt the foregoing investigation to the case of a siphon discharging into air, we have only to substitute the level of the discharging end for the level of the lower free surface, so that y will denote the depth of the discharging end below the diaphragm, and h its depth below the surface of the liquid which is to be drawn off.

As the ascent of the liquid in the siphon is due to atmospheric pressure on the upper free surface, it is necessary that the highest point of the siphon (if intended for water) should not be more than about 33 feet above this surface.

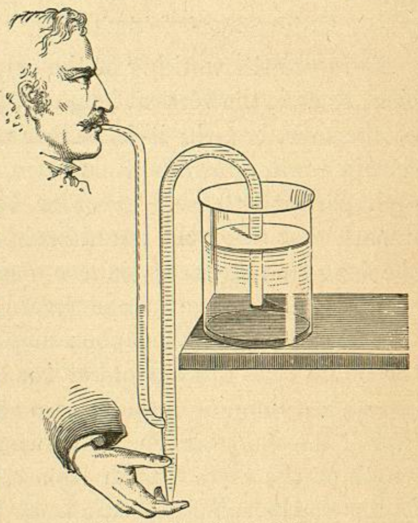

Fig. 176.—Starting the Siphon.

272. Starting the Siphon.—In order to make a siphon begin working, we must employ means to fill it with the liquid. This can sometimes be done by dipping it in the liquid, and then placing it in position while the ends are kept closed; or by inserting one end in the liquid which we wish to remove, and sucking at the other. It is usually more convenient to apply suction by means of a side tube, as in Fig. 176, this tube being sometimes provided with an enlargement to prevent the liquid from entering the mouth. One end of the siphon is inserted in the liquid which is to be removed, while the other end is stopped, and the operator applies suction at