# Capítulo 8

# Controlador Óptimo en Sistemas Polinomiales

## 8.1. Problema del Controlador Óptimo

### 8.1.1. Planteamiento del Problema

Sea $(\Omega, F, P)$ un espacio de probabilidad completo con una familia creciente y continua por la derecha de $\sigma$-algebras $F_t, t \geq 0$, y sean $(W_1(t), F_t, t \geq 0)$ y $(W_2(t), F_t, t \geq 0)$ procesos de Wiener $F_t$-adaptados. Considere el proceso aleatorio no observable $F_t$-medible $x(t)$ gobernado por la ecuación polinomial de tercer grado

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + G(t)u(t)dt + b(t)dW_1(t), x(t_0) = x_0, \quad (8.1)$$

y el proceso de salida (observación)

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t). \quad (8.2)$$

Aquí, $x(t) \in R^n$ es el vector de estado no observable, para el cual los componentes del segundo y tercer grados están dados en la siguiente forma: $x^2(t) = [x_1^2(t)\ x_2^2(t)\ x_3^2(t)\ \ldots\ x_n^2(t)]^T$, $x^3(t) = [x_1^3(t)\ x_2^3(t)\ x_3^3(t)\ \ldots\ x_n^3(t)]^T$, $u(t) \in R^p$ es la variable de control, $y(t) \in R^m$ es el proceso de observaciones, y los procesos de Wiener independientes $W_1(t)$ y $W_2(t)$ representan disturbios aleatorios en las ecuaciones de estado y de observaciones, los cuales son independientes del vector inicial Gaussiano $x_0$. $A(t)$ es una matriz no cero y $B(t)B^T(t)$ es una matriz positiva definida. Además, la función de costo cuadrático a ser minimizada $J$, está dada por

$$\begin{aligned} J \;=\; & E[\frac{1}{2}[x(T) - z_0]^T \Phi [x(T) - z_0] + \\ & \frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} x^T(s)L(s)x(s)ds], \end{aligned} \tag{8.3}$$

donde $z_0$ es un vector dado, $K$ es una matriz positiva definida y $\Phi$, $L$ son matrices simétricas definidas no-negativas, $T > t_0$ es un cierto instante de tiempo, el símbolo $E[f(x)]$ denota la esperanza (media) de una función $f$ de una variable aleatoria $x$, y $a^T$ denota la transpuesta de un vector (matriz) $a$.

El problema de control óptimo consiste en encontrar el control $u^*(t)$, $t \in [t_0, T]$, que minimice el criterio $J$ a lo largo de la trayectoria $x^*(t)$, $t \in [t_0, T]$, generada al sustituir $u^*(t)$ en la ecuación de estado (8.1).

### 8.1.2. Principio de Separación para Sistemas Polinomiales

Así como para los sistemas estocásticos lineales, el principio de separación también es válido para un sistema estocástico dado por una ecuación polinomial de tercer grado, con observaciones lineales, y criterio cuadrático. El principio de separación ya ha sido enunciado en secciones anteriores, pero se hará mención del mismo para facilitar la lectura y comprensión del texto. Reemplazando el estado del sistema no observable $x(t)$ por su estimado óptimo $m(t)$ dado por la ecuación (5.15)

$$\begin{aligned} dm(t) &= (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + \\ &\quad a_3(t)(3p(t) * m(t) + m^3(t))dt + G(t)u(t) + \\ &\quad P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt), \end{aligned} \quad (8.4)$$

con la condición inicial $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$. Aquí, $m(t)$ es el mejor estimado del proceso no-observable $x(t)$ en el tiempo $t$ basado en el proceso de observación $Y(t) = \{y(s), t_0 \leq s \leq t\}$, el cual está dado por la esperanza condicional $m(t) = E(x(t) \mid F_t^Y)$, $m(t) = [m_1(t)\ m_2(t)\ \dots\ m_n(t)]$; $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T | Y(t)] \in \mathbb{R}^n$ es la matriz de covarianza del error; $p(t) \in \mathbb{R}^n$ es el vector cuyos componentes son las varianzas de los componentes de $x(t) - m(t)$, i.e., los elementos de la diagonal de $P(t)$; $m^2(t)$ y $m^3(t)$ son definidos como vectores de los cuadrados y cubos de los componentes de $m(t)$: $m^2(t) = [m_1^2(t)\ m_2^2(t)\ \dots\ m_n^2(t)]^T$, $m^3(t) = [m_1^3(t)\ m_2^3(t)\ \dots\ m_n^3(t)]^T$; $P(t)m(t)$ es el producto convencional de la matriz $P(t)$ por un vector $m(t)$; y $p(t) * m(t)$ es el producto de dos vectores dado como el producto entre sus componentes: $p(t) * m(t) = [p_1(t)m_1(t)\ p_2(t)m_2(t)\ \dots\ p_n(t)m_n(t)]^T$. El mejor estimado $m(t)$ minimiza el criterio

$$H = E[(x(t) - m(t))^T(x(t) - m(t))], \quad (8.5)$$

con respecto a la elección del estimado $m$ como una función de las observaciones $y(t)$, en todo momento de tiempo $t$ ([62]).

La ecuación complementaria para la matriz de varianza $P(t)$ toma la forma (5.18)

$$\begin{aligned} dP(t) &= (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t) * P(t) + \\ &\quad 2(P(t) * m^T(t))a_2^T(t) + 3a_3(t)(p(t) * P(t)) + \end{aligned} \quad (8.6)$$

$$3(p(t) * P(t))^T a_3^T(t) + 3a_3(t)(m^2(t) * P(t)) +$$
$$3(P(t) * (m^2(t))^T)a_3^T(t) + (b(t)b^T(t)) -$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt,$$

con la condición inicial $P(t_0) = E((x(t_0)-m(t_0))(x(t_0)-m(t_0))^T|y(t_0))$, donde el producto $m(t) * P(t)$ entre un vector $m(t)$ y una matriz $P(t)$ es definido como en la Sección 5.3.

Es posible verificar (como en [56]) que el problema de control óptimo (8.1) y la función de costo (8.3) es equivalente al problema de control óptimo para el estimado (8.4) y la función de costo $J$, representada como:

$$\begin{aligned} J &= E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0] + \\ &\frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds \\ &+\frac{1}{2}\int_{t_0}^{T} tr[P(s)L(s)]ds + tr[P(T)\Phi]\}, \end{aligned} \tag{8.7}$$

donde $tr[A]$ denota la traza de la matriz $A$. $\Phi, K, L$ son matrices simétricas, $K$ es una matriz positiva definida y $\Phi$ y $L$ son matrices no negativas.

Dado que la última parte de $J$ es independiente del control $u(t)$ y del estado $x(t)$, la función de costo reducida $M$ a ser minimizada toma la forma:

$$\begin{aligned} M &= E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0] + \\ &\frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds\}. \end{aligned} \tag{8.8}$$

En conclusión, el principio de separación para sistemas polinomiales de tercer grado establece que la solución del problema original de control óptimo especificada para (8.1),(8.3) puede encontrarse resolviendo el problema de control óptimo dado por (8.4),(8.8). Además, el valor mínimo del criterio $J$ debe ser determinado usando (8.7).

### 8.1.3. Solución al Problema de Control Óptimo

Teniendo como base la solución al problema de control obtenido en el capítulo previo en el caso de un estado del sistema observable, gobernado por una ecuación polinomial de tercer grado, los siguientes resultados son válidos para el problema de control óptimo (8.4),(8.8), donde el estado del sistema (el estimado $m(t)$) es completamente disponible, y observable. La ley de control óptima está dada por:

$$u^*(t) = K^{-1}(t)G^T(t)Q(t)m(t), \tag{8.9}$$

donde la matriz $Q(t)$ es la solución de la siguiente ecuación (7.4), dual a la ecuación de la varianza

$$\begin{aligned} dQ(t) &= (-a_1^T(t)Q(t) - Q(t)a_1^T(t) - 2a_2^T(t)Q(t) * m^T(t) - \\ &\quad 2m(t) * Q(t)a_2(t) - 3a_3^T(t)Q(t) * q^T(t) - \\ &\quad 3q(t) * Q(t)a_3(t) - 3a_3^T(t)Q(t) * ((m^2)^T(t)) - \\ &\quad 3(m^2(t) * Q(t))a_3(t) + L(t) - Q(t)G(t)K^{-1}(t)G^T(t)Q(t))dt, \end{aligned} \tag{8.10}$$

con la condición terminal $Q(T) = \Phi$. La operación binaria $*$ ha sido introducida en la Sección 5.3, y $q(t) = [q_1(t)\ q_2(t)\ ...\ q_n(t)]^T$ denota el vector formado por los elementos de la diagonal de la matriz $Q(t)$. En el proceso de obtención de la ecuación (8.10), ha sido tomado en cuenta que el último término en la ecuación (8.4),

$$P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt),$$

es un ruido blanco Gaussiano.

Sustituyendo la ley de control óptimo (8.9) en la ecuación (8.4) para el estado reconstruido del sistema $m(t)$, se obtiene la siguiente ecuación para el estimado del estado óptimamente controlado:

$$
\begin{aligned}
dm(t) &= (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + \\
&\quad a_3(t)(3p(t) * m(t) + m^3(t))dt + G(t)(K(t))^{-1}G^T(t)Q(t)m(t)dt + \\
&\quad P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt), \\
&\quad m(t_0) = E(x(t_0)|F_{t_0}^Y).
\end{aligned} \tag{8.11}
$$

Así, la ecuación del estimado del estado óptimamente controlado (8.11), la ecuación de la matriz de ganancia (8.10), la ley de control óptima (8.9), y la ecuación de la varianza (8.6), forman la solución completa del problema del controlador para estados no observables de sistemas polinomiales de tercer grado.

## 8.2. Aplicación del Controlador Polinomial Óptimo a un Sistema Automotriz

### 8.2.1. Planteamiento del problema

Esta sección presenta la aplicación del controlador para un estado polinomial de tercer grado con observaciones lineales y función de costo cuadrática para controlar las variables de estado no observables, y ángulos de orientación y de giro del volante, en un modelo cinemático no lineal de un carro en movimiento [63], el cual ya ha sido presentado en secciones anteriores, pero se repetirá el planteamiento para facilitar la lectura y comprensión del texto. Las ecuaciones de estado para este sistema están representadas por:

$$
\begin{aligned}
dx(t) &= v\cos\phi(t)dt, \\
dy(t) &= v\sin\phi(t)dt,
\end{aligned} \tag{8.12}
$$

$$
\begin{aligned}
d\phi(t) &= (v/l)\tan\delta(t)dt, \\
d\delta(t) &= u(t)dt.
\end{aligned}
$$

Aquí, $x(t)$ y $y(t)$ son las coordenadas cartesianas del centro de masa del carro, $\phi(t)$ es el ángulo de orientación, $v$ es la velocidad, $l$ es la longitud entre los dos ejes del carro, $\delta(t)$ es el ángulo del volante, y $u(t)$ es la variable de control (velocidad angular del volante). Se suponen condiciones iniciales cero para todas las variables.

El proceso de observación para las variables no observables $\phi(t)$ y $\delta(t)$ es dado por las observaciones lineales directas, las cuales contienen disturbios independientes e idénticamente distribuidos, modelados como ruidos blancos Gaussianos. Las ecuaciones correspondientes a las observaciones son

$$
\begin{aligned}
dz_\phi(t) &= \phi(t)dt + w_1(t)dt, \\
dz_\delta(t) &= \delta(t)dt + w_2(t)dt,
\end{aligned}
\tag{8.13}
$$

donde $z_\phi(t)$ es la variable de observación para $\phi(t)$, $z_\delta(t)$ es la variable de observación para $\delta(t)$, y $w_1(t)$ y $w_2(t)$ son ruidos blancos Gaussianos independientes uno del otro. Los valores asignados para la velocidad y la longitud entre los ejes son $v = 17m/min$, $l = 2m$, los cuales corresponden a un modelo de carro de tamaño estándar. En otras palabras, el problema es lograr el giro máximo de las ruedas de su posición inicial, usando la mínima energía para dirigir el volante. Por razones de economizar combustible y reducir la contaminación del aire, el peso del término de control en el criterio, se considera diez veces mayor que el peso del término del estado terminal. El criterio correspondiente $J$ a ser minimizado toma la forma

$$
J = [\phi(t) - \phi^*]^2 + 10\int_0^T u^2(t)dt, \tag{8.14}
$$

donde $T = 0.3min$, y $\phi^* = 10rad$ es un valor grande de $\phi(t)$ inalcanzable en el tiempo $T$.

La aplicación de los algoritmos del controlador obtenido se hace para el sistema no lineal (8.12), observaciones lineales (8.13), y criterio cuadrático (8.14), usando la expansión de Taylor para las últimas dos ecuaciones en (8.12) en el origen, hasta el tercer grado (el cuarto grado no aparece en la serie de Taylor para la tangente)

$$\begin{aligned} d\phi(t) &= ((\frac{v}{l})\delta(t) + (\frac{v}{l})(\frac{\delta^3(t)}{3}))dt, \\ d\delta(t) &= u(t)dt. \end{aligned} \tag{8.15}$$

### 8.2.2. Solución

La solución para el problema del controlador óptimo establecido, es dada como sigue. Dada $K = 1$ y $G^T = [0, 1]$ en (8.14) y (8.15), la ley de control óptimo $u^*(t) = (K(t))^{-1}G^T(t)Q(t)m(t)$ toma la forma

$$u^*(t) = q_{21}(t)m_\phi(t) + q_{22}(t)m_\delta(t), \tag{8.16}$$

y las siguientes ecuaciones para el controlador óptimo (8.9)-(8.11) y (8.6) para el estado polinomial de tercer grado (8.15) sobre observaciones lineales (8.13) y criterio cuadrático (8.14) son:

$$\begin{aligned} dm_\phi &= ((\frac{v}{l})m_\delta + (\frac{v}{3l})(3p_\delta + m_\delta^3) + p_{\phi\phi}(z_\phi - m_\phi) + p_{\phi\delta}(z_\delta - m_\delta))dt, \\ dm_\delta &= (u^*(t) + p_{\delta\phi}(z_\phi - m_\phi) + p_{\delta\delta}(z_\delta - m_\delta))dt, \\ dp_{\phi\phi} &= ((2v/l)p_{\delta\phi}p_{\delta\delta} + \frac{2v}{l}p_{\delta\phi} + \frac{2v}{l}m_\phi^2 p_{\delta\phi} - p_{\phi\phi}^2 - p_{\phi\delta}^2)dt, \\ dp_{\phi\delta} &= (\frac{v}{l}p_{\delta\delta} + \frac{v}{l}m_\delta^2 p_{\delta\delta} - p_{\phi\phi}p_{\phi\delta} - p_{\phi\delta}p_{\delta\delta})dt, \\ dp_{\delta\delta} &= (-p_{\delta\phi}^2 - p_{\delta\delta}^2)dt, \\ dq_{11}(t) &= (-q_{21}^2(t))dt, \\ dq_{12}(t) &= (-\frac{v}{l}q_{11}^2 - q_{12}q_{22} - \frac{v}{l}q_{11} - \frac{v}{l}m_\phi^2 q_{11})dt, \\ dq_{22}(t) &= (-\frac{2v}{l}q_{12} - \frac{2v}{l}q_{12}q_{22} - \frac{2v}{l}m_\delta^2 q_{12} - q_{22}^2)dt. \end{aligned} \tag{8.17}$$

Aquí, $m_\phi$ y $m_\delta$ son los estimados de las variables $\phi$ y $\delta$; $p_{\phi\phi}, p_{\phi\delta}, p_{\delta\delta}$ son elementos de la matriz simétrica de covarianza $P$; y $q_{11}(t), q_{21}(t), q_{22}(t)$ son elementos de la matriz simétrica de ganancia $Q(t)$ formando el control óptimo (8.16). Los siguientes valores iniciales para las variables de entrada son asignados: $m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$. Son disturbios Gaussianos $w_1(t)$ y $w_2(t)$ en (8.13) son realizados como ruidos blancos, tomando el block correspondiente en el *MatLab 6, versión 1.2.* Las condiciones terminales para los elementos de la matriz de ganancia $Q$ están dadas por: $q_{11}(T) = 0.1, q_{12}(T) = 0, q_{22}(T) = 0$, en el tiempo final $T = 0.3min$.

Así, el sistema compuesto por las dos últimas ecuaciones de (8.12) y las ecuaciones (8.17) debe ser resuelto con las condiciones iniciales $m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$, y las condiciones terminales $q_{11}(T) = 0.1, q_{12}(T) = 0, q_{22}(T) = 0$. Este problema de frontera es resuelto numéricamente usando un método iterativo, pasando del sistema en tiempo directo al sistema en tiempo inverso, como fue realizado en la sección de la aplicación del control óptimo a un sistema automotriz (Sección 8.2). Las gráficas de la simulación para el caso polinomial de tercer grado son mostradas en la Figura 8.1. La ley de control óptimo en el caso lineal es la misma que en (8.16), pero las ecuaciones del controlador lineal óptimo están dadas por:

$$
\begin{aligned}
dm_\phi &= (\frac{v}{l} m_\delta + p_{\phi\phi}(z_\phi - m_\phi) + p_{\phi\delta}(z_\delta - m_\delta))dt, \qquad (8.18)\\
dm_\delta &= (u^*(t) + p_{\delta\phi}(z_\phi - m_\phi) + p_{\delta\delta}(z_\delta - m_\delta))dt,\\
dp_{\phi\phi} &= (\frac{2v}{l} p_{\delta\phi} - p_{\phi\phi}^2 - p_{\phi\delta}^2)dt,\\
dp_{\phi\delta} &= (\frac{v}{l} p_{\delta\delta} - p_{\phi\phi} p_{\phi\delta} - p_{\phi\delta} p_{\delta\delta})dt,\\
dp_{\delta\delta} &= (-p_{\delta\phi}^2 - p_{\delta\delta}^2)dt.\\
dq_{11}(t) &= (-q_{21}^2(t))dt,\\
dq_{12}(t) &= (-q_{12} q_{22} - \frac{v}{l} q_{11})dt,
\end{aligned}
$$

$$dq_{22}(t) = (-\frac{2v}{l}q_{12} - q_{22}^2)dt.$$

Se puede observar que en el caso lineal solo se requiere un paso en el sistema inverso para $q$'s, porque las ecuaciones para $q$'s en (8.18) no dependen de $\phi$, $\delta$, $m_\phi$, ni $m_\delta$, y los valores iniciales para $q$'s en $t = 0$ pueden ser obtenidos después de un pase por el sistema inverso (con el tiempo $(-t)$). Las gráficas de la simulación para el caso lineal se muestran en la Figura 8.2. Así, dos conjuntos de gráficas son obtenidos: 1. Gráficas de las variables $\phi$ y $\delta$ que satisfacen las ecuaciones del sistema polinomial (8.15) y el controlador usando el regulador lineal óptimo definido por (8.16), (8.18); gráficas de los estimados $m_\phi$ y $m_\delta$ que satisfacen el sistema (8.18) y el controlador usando el regulador óptimo lineal definido por (8.16), (8.18); gráficas de los valores correspondientes del criterio $J$; gráficas de los valores correspondientes del control óptimo $u^*$ (Figura 8.1).

2. Gráficas de las variables $\phi$ y $\delta$ que satisfacen el sistema polinomial (8.15) y el controlador usando el regulador óptimo polinomial de tercer grado definido por (8.16), (8.17); gráficas de los estimados $m_\phi$ y $m_\delta$ que satisfacen el sistema (8.17) y el controlador usando el regulador óptimo polinomial de tercer grado definido por (8.16), (8.17); gráficas de los valores correspondientes del criterio $J$; gráficas de los valores correspondientes del control óptimo $u^*$ (Figura 8.2).

Los valores obtenidos de la variable controlada $\phi$ y del criterio $J$ son comparados para el controlador óptimo polinomial de tercer grado y el controlador óptimo lineal en el tiempo terminal $T = 0.3min$ en la siguiente tabla (correspondiente a las Figuras 8.1 y 8.2).

| Controlador Lineal | Controlador Polinomial de Tercer Orden |
|---|---|
| $\phi(0.3) = 0.054rad$ | $\phi(0.3) = 0.084rad$ |
| $J = 98.971$ | $J = 98.45$ |

Los resultados de la simulación demuestran que el valor de la variable controlada $\phi$ en el punto terminal $T = 0.3min$ es mayor por una y media veces en el controlador polinomial con respecto al controlador lineal, y la diferencia entre los valores iniciales y finales del criterio es más que una y media veces mayor en el controlador polinomial de tercer orden con respecto al controlador lineal. Por tanto, mediante esta simulación queda demostrada la eficacia del algoritmo del controlador polinomial de tercer grado, con respecto al controlador lineal.

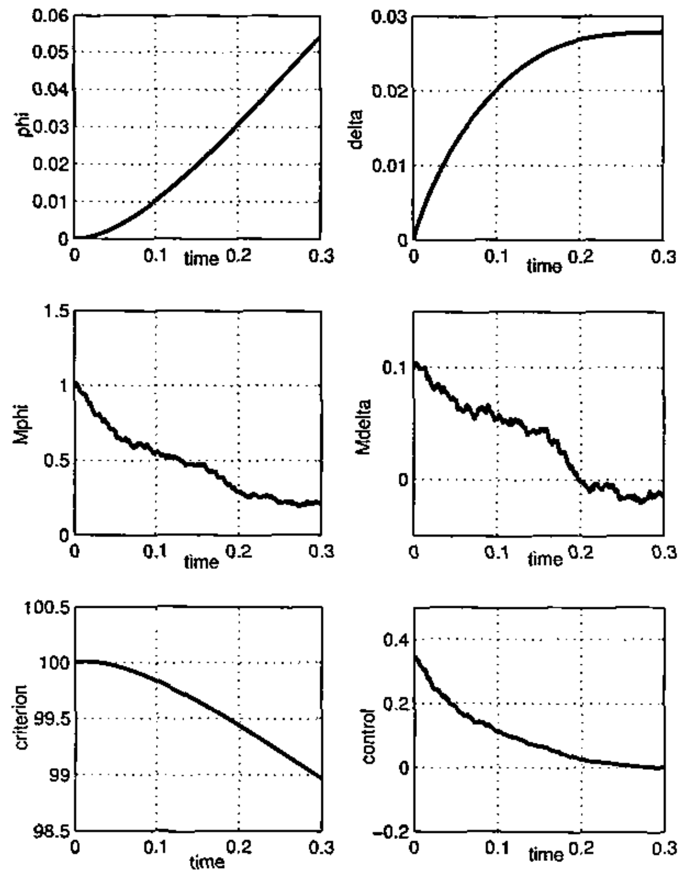


Figura 8.1: Gráficas del controlador lineal formado por las ecuaciones (8.18),(8.16). $phi = \phi, delta = \delta, Mphi = m_\phi, Mdelta = m_\delta, criterion = J, control = u^*$.

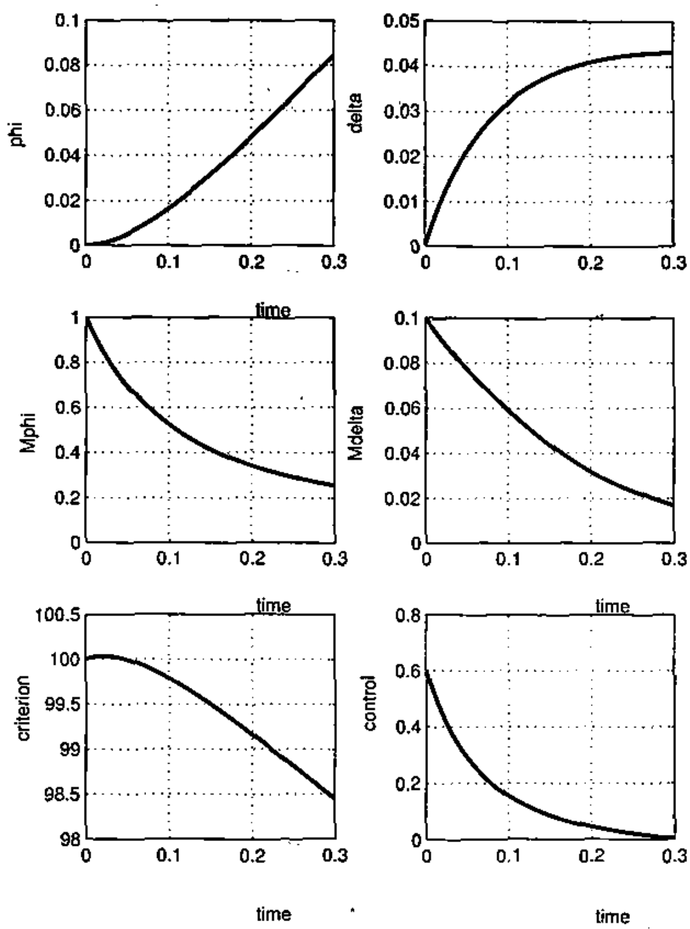


Figura 8.2: Gráficas del controlador óptimo polinomial de tercer grado, correspondiente a las ecuaciones (8.17) y (8.16). $phi = \phi, delta = \delta, Mphi = m_\phi, Mdelta = m_\delta, criterion = J, control = u^*$.

# Capítulo 9

# Conclusiones, Aportaciones y Recomendaciones para Trabajos Futuros

## 9.1. Conclusiones

Se han obtenido matemáticamente los algoritmos del regulador óptimo para sistemas de Itô-Volterra con entradas de control continuas o discontinuas, partiendo del principio de dualidad para el caso de las ecuaciones de Itô-Volterra con observaciones lineales. Utilizando el principio de separación, y los algoritmos de filtrado obtenidos previamente, se obtuvo matemáticamente el controlador óptimo para los sistemas de Itô-Volterra.

Se obtuvieron los algoritmos de filtrado y control óptimos para ecuaciones de estado polinomiales de tercer y cuarto grado, con observaciones lineales. Se ha mostrado la eficiencia de los algoritmos de filtrado y control, obtenidos matemáticamente, para sistemas polinomiales de tercero y cuarto grados, con observaciones continuas, comparándolos con los algoritmos de filtrado de Kalman-Bucy ya existentes, mediante una simulación en *MatLab 6, versión 1.2.*, aclarando que en esta simulación, el ruido blanco es considerado como una señal de banda ancha finita, por lo cual es una aproximación. En forma similar al caso de Itô-Volterra, se obtuvieron mediante procedimientos matemáticos, los algoritmos del controlador óptimo para ecuaciones de estado polinomiales de tercer grado. Aplicándolos, a un fenómeno físico, mediante simulación en *MatLab 6, versión 1.2.*, se compararon los algoritmos del controlador polinomial obtenido en este trabajo, con los algoritmos del controlador lineal, obteniendo mejores resultados con el controlador polinomial. Como un caso general, se trabajó con los algoritmos de filtrado óptimo para el caso bilineal, y teniendo el deseo de verificarlos en un número mayor a dos ecuaciones, se llevó a cabo su aplicación a un modelo matemático de un reactor de polimerización, con el propósito de mostrar su eficacia respecto a los algoritmos lineales ya existentes. Lo anterior se consiguió mediante la simulación en *MatLab 6, versión 1.2.* Queda como trabajo a futuro la verificación de la eficacia de los algoritmos obtenidos mediante su aplicación a diversos fenómenos físicos que se presentan en la naturaleza.

## 9.2. Aportaciones

Las aportaciones se pueden enlistar en la siguiente forma:

a) Diseño de algoritmos de filtrado óptimo para:

- Sistemas de Itô-Volterra y observaciones lineales continuas.
- Sistemas de Itô-Volterra y observaciones lineales discontinuas.
- Ecuaciones de estado polinomiales de tercer y cuarto grados y observaciones lineales continuas.

- Ecuaciones de estado bilineales y observaciones lineales continuas.

b) Diseño de algoritmos de control óptimo para:

- Sistemas de Itô-Volterra y entradas de control lineales continuas.
- Sistemas de Itô-Volterra y entradas de control lineales discontinuas.
- Ecuaciones de estado polinomiales de tercer grado y entradas de control lineales continuas.

c) Diseño del controlador para sistemas que representan procesos no observables para:

- Sistemas de Itô-Volterra con observaciones y entradas de control lineales continuas.
- Sistemas de Itô-Volterra con observaciones y entradas de control lineales discontinuas.
- Ecuaciones de estado polinomiales de tercer grado con observaciones y entradas de control lineales continuas.

d) Los problemas técnicos resueltos en este trabajo son los siguientes:

- Obtención del control óptimo del movimiento de un misil con motores jet e impulsivos.
- Obtención de controlador óptimo del movimiento de un misil con motores jet e impulsivos y velocidad no observable.
- Obtención de las ecuaciones de filtrado óptimo referentes al movimiento angular de un automóvil.

- Obtención de las ecuaciones de control óptimo referentes al movimiento angular de un automóvil.
- Obtención de las ecuaciones del controlador óptimo referentes al movimiento angular de un automóvil.
- Obtención de las ecuaciones de filtrado óptimo para la estimación de un proceso de polimerización.

## 9.3. Recomendaciones para Trabajos Futuros

Existen múltiples casos a desarrollar en las áreas de filtrado y control, dada la diversidad de los procesos de la naturaleza. Algunos de ellos pueden ser:

- Corroboración de los algoritmos de filtrado y control obtenidos, mediante su aplicación en diversos fenómenos físicos.
- Desarrollo de filtro y control óptimo para sistemas de otros grados polinomiales superiores, con observaciones continuas.
- Desarrollo de filtro y control óptimo para sistemas de otros grados polinomiales superiores, con observaciones discontinuas.
- Desarrollo de filtro y control óptimo para ecuaciones de estado con términos no lineales y no polinomiales, con funciones de tipos exponencial, logarítmico, trigonométrico, etc., con observaciones continuas y discontinuas.
- Aplicación de los algoritmos obtenidos a problemas técnicos de diversas áreas de Ingeniería.

- Verificación de los algoritmos de filtrado bilineales, interdisciplinariamente, esto es con un especialista en las dos ingenierías, de control y química, para el proceso de polimerización presentado por Ogunnaike [64], utilizando métodos químicos para elegir las ecuaciones afines, y los valores de los parámetros, y los algoritmos obtenidos en este trabajo.

La verificación y corroboración de los algoritmos de filtrado obtenidos permitirá la consolidación de los mismos,y esto, junto con el desarrollo de los algoritmos de filtrado y control para otras condiciones específicas que presenta la naturaleza, proveedrá de herramientas para resolver diversos problemas técnicos de las Ingenierías Química, Automotriz, Eléctrica, etc.

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# OPTIMAL CONTROL IN ITO-VOLTERRA SYSTEMS

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**Abstract.** This paper presents solutions of the optimal linear-quadratic control problems for stochastic integral Ito-Volterra systems with continuous/discontinuous states. The obtained solutions are based on applying the duality principle for Volterra systems to the known solutions of the dual filtering problems for Ito-Volterra states over continuous/ discontinuous observations. The optimal control laws and the gain matrix equations are first derived in the general case an Ito-Volterra state equation and then simplified in the case of a dynamic plant governed by a differential state equation. The technical example illustrating application of the obtained results is finally given.
**AMS subject classification:** 49K22, 93E20

## 1 Introduction

The optimal control and filtering problems for dynamic systems with delays, which represent a particular case of discontinuous integral systems, have been studied in a number of publications from various viewpoints (see, for example, [9], [10], [1] for dynamic systems and [17] for a particular case of integral Volterra ones). This attention is directly related to common use of dynamic systems with delays in global economy concepts [12], marketing models [13], technical systems [7], and others. Since the class of integral Volterra systems includes the class of retarded dynamic ones, the study of integral systems becomes a significant part of the control theory. Nevertheless, the integral Volterra systems have been of independent interest in the deterministic environment, as well as in the stochastic one (see [2]).

This paper presents solutions of the optimal linear-quadratic control problems for stochastic integral Ito-Volterra systems with continuous and then discontinuous states. There are a number of papers investigating the control problems for continuous system states given by stochastic differential equations (see [11, 19] and bibliography therein) or bivariate Volterra ones [17], or deterministic continuous and discontinuous system states governed by Volterra equations [3, 4]. However, the problems have not been treated yet in the case of integral stochastic systems governed by Ito-Volterra equations. The solution presented in the paper is based on applying the duality principle for Volterra systems (substantiated in [3]) to the known solutions of the dual filtering problems for Ito-Volterra states over continuous and then discontinuous observations. The duality principle enables one to use the optimal gain matrix structure in the dual filtering problem for the optimal gain matrix in the control one, as it was done for differential stochastic systems [14, 15]. As a result, the optimal control law and the gain matrix formula are first derived in the general case an Ito-Volterra state equation, where the gain matrix constituent satisfying a Riccati equation depends on two time variables, as the cross-correlation matrix in the dual filtering problem does (see [5]). The gain matrix formula is then simplified in the case of a dynamic plant (the internal part of a system) governed by a differential state equation, where the gain matrix constituent satisfying a Riccati equation depends on only one time variable, similarly to the variance in the dual filtering problem (see [6]). The obtained results for discontinuous system states, where the optimal control problem is dual to the optimal filtering problem over discontinuous observations ([5, 6]), consist of the discontinuous control law and the corresponding Riccati equation with integration with a discontinuous measure, which allows discontinuous solutions. In particular, the obtained results enable one to compute jumps of the optimal control parameters (the gain matrix, optimal control law, and optimally controlled state) that can appear due to discontinuities in system behavior.

The secondary goal of this paper is to reveal more functional capabilities of the duality principle as a means for solving the optimal control (or, vice versa, filtering) problems. Indeed, the duality principle applicability to linear dynamic systems is well known (see [8, 18]) and its applicability to linear integral Volterra (Ito-Volterra) systems is investigated in [3, 4] and this paper. However, it seems that the more advanced conjecture is valid: the duality principle should be valid in all cases of linear and nonlinear systems, where the optimal solution to control or filtering problem exists. Taking this working hypothesis into account makes the duality principle a quite powerful tool for designing the optimal control and filtering algorithms.

The paper is organized as follows. The Section 2 presents the optimal control statement and its solution (the optimal control law and gain matrix equation) for a continuous Ito-Volterra system, based on applying the duality principle to the solution of the dual filtering problem. The optimal control problem for a discontinuous Ito-Volterra system is stated and solved in Section 3. The obtained results are then simplified in the case of a dynamic plant (the internal part of a system) governed by a differential equation. In particular, the relations enabling one to compute jumps of the optimal control parameters are obtained. Finally, Section 4 presents the technical example illustrating application of the obtained results to solution of the optimal control problem of launching a missile with continuous and impulsive jet motors to the maximal possible altitude with the minimal fuel consumption, if the velocity dynamics is affected by equally distributed and independent stochastic disturbances.

# 2 Optimal Control in Continuous Ito-Volterra Systems

## 2.1 Problem statement

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ be an $F_t$-adapted Wiener process. Let us consider the $F_t$-measurable random process $x(t)$ governed by the Ito-Volterra equation

$$x(t) = x(t_0)+\int_{t_0}^{t}(a_0(t,s)+a(t,s)x(s)+b(t,s)u(t,s))ds+\int_{t_0}^{t}g(s)dW_1(s). \quad (1)$$

Here $x(t) \in R^n$ is the state vector, $u(t,s) \in R^p$ is the control variable, the Wiener process $W_1(t)$ represents a random disturbance, and the initial Gaussian vector $x(t_0)$ is independent of $W_1(t)$. The quadratic cost function $J$ to be minimized is defined as follows

$$J = E[\frac{1}{2}[x(T)-x_0]^T \Psi^{-1} [x(T)-x_0] \quad (2)$$

$$+\frac{1}{2}\int_{t_0}^{T} u^T(t,s)R(s)u(t,s)ds + \frac{1}{2}\int_{t_0}^{T} x^T(s)Q(s)x(s)ds],$$

where $x_0$ is a given vector, $\Psi$, $R$, $Q$ are positive (nonnegative) definite symmetric matrices, $T > t_0$ is a certain time moment, the symbol $E[f(x)]$ means the expectation (mean) of a function $f$ of a random variable $x$, and $a^T$ denotes transpose to a vector (matrix) $a$.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1).

## 2.2 Duality principle

For dynamic systems governed by differential equations, solution of the optimal control problem can be obtained using the solution of the optimal filtering problem and the duality principle [8, 18]. Thus, it would be helpful to introduce the duality principle for integral stochastic systems, as done below.

Consider the integral Volterra systems:

$$x(t) = x(t_0) + \int_{t_0}^{t}(a(t,s)x(s) + b(t,s)u(t,s))ds, \quad (3)$$

$$y(t) = \int_{t_0}^{t} c(t,s)x(s)ds + \int_{t_0}^{t} d(t,s)u(t,s)ds$$

and

$$z(t) = z(t_0) + \int_{t_0}^{t} -a^T(t,s)z(s)ds + \int_{t_0}^{t} c^T(t,s)v(t,s)ds, \tag{4}$$

$$\gamma(t) = \int_{t_0}^{t} b^T(t,s)z(s)ds + \int_{t_0}^{t} d^T(t,s)v(t,s)ds.$$

The duality principle claims that the system (3) is controllable (observable) at $t_0$, if and only if the system (4) is observable (controllable) at $t_0$.

The proof of the duality principle for integral systems [3] is based on the fact that there exists the transition matrix $\Phi(t,t_0)$; $t, t_0 \in (-\infty, \infty)$, such that

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)b(t,\tau)u(t,\tau)d\tau.$$

## 2.3 Dual filtering problem solution

The suggested solution to the optimal control problem for integral stochastic systems is based on applying the duality principle to the optimal filtering problem solution obtained in [5]. Indeed, let us consider the filtering problem, dual to the optimal control one given by (1),(2), for the state equation

$$z(t) = z(t_0) + \int_{t_0}^{t} (a_0^T(t,s) - a^T(t,s)z(s))ds + \int_{t_0}^{t} Q^{1/2}(s)dW_3(s) \tag{5}$$

and the observation equation

$$y(t) = \int_{t_0}^{t} (b^T(t,s)x(s))ds + \int_{t_0}^{t} R^{1/2}(s)dW_4(s), \tag{6}$$

where $W_3(s)$ and $W_4(s)$ are independent Wiener processes which are in turn independent of an initial Gaussian vector $z(t_0)$.

The filtering problem is to find the best estimate for the Ito-Volterra process $x(t)$ at time $t$ based on the observation process $Y(t) = \{y(s), t_0 \leq s \leq t\}$, that is the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$. Denote the correlation function of the best estimate as $P(t) = E((x(t) - m(t)(x(t) - m(t))^T \mid F_t^Y)$.

As shown in [5] and the previous papers [16, 21], it is impossible to obtain a closed system of filtering equations for variables $m(t)$ and $P(t)$ due to the Volterra nature of the equations (5) and (6). Designing a closed filter requires introducing the additional cross-correlation function $f(t,s)$ characterizing a deviation of the best estimate $m(t)$ from the real state $x(t)$:

$$f(t,s) = E((z_s^t - m_s^t)(z(s) - m(s))^T \mid F_{t,s}^Y),$$

where

$$z_s^t = z(t_0) + \int_{t_0}^{s} (a_0^T(t,r) - a^T(t,r)z(r))dr + \int_{t_0}^{s} Q^{1/2}(r)dW_3(r),$$

$F^Y_{t,s}$ is the $\sigma$-algebra generated by the stochastic process $y^t_s$

$$y^t_s = \int_{t_0}^{s} b^T(t,s)z(s)ds + \int_{t_0}^{s} R^{1/2}(s)dW_4(s),$$

$$\text{and} \quad m^t_s = E(z^t_s \mid F^Y_{t,s}).$$

The optimal filter for the state vector (5) over the continuous observation process (6) is given [5] by the following equations for the optimal estimate $m(t)$, its correlation function $P(t)$, and the cross-correlation function $f(t,s)$

$$m(t) = m(t_0) + \int_{t_0}^{t} (a_0^T(t,s) - a^T(t,s)m(s))ds + \tag{7}$$

$$\int_{t_0}^{t} f(t,s)b(t,s)(R(s))^{-1}[dy(s) - b^T(t,s)m(s)ds].$$

$$P(t) = P(t_0) + \int_{t_0}^{t} [-a^T(t,s)f^T(t,s) - f(t,s)a(t,s) + Q(s)]ds - \tag{8}$$

$$\int_{t_0}^{t} f(t,s)b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s)ds,$$

$$f(t,s) = P(t_0) + \int_{t_0}^{s} [-a^T(s,r)f^T(t,r) - f(s,r)a(t,r) + Q(r)]dr - \tag{9}$$

$$\int_{t_0}^{s} [f(t,r)b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r) +$$

$$f(s,r)b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r) -$$

$$(1/2)f(t,r)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r) -$$

$$(1/2)f(s,r)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r)]dr,$$

where $m(t_0) = E(z(t_0) \mid F^Y_{t_0}) = x_0$ and $P(t_0) = E((z(t_0) - m(t_0)(z(t_0) - m(t_0))^T \mid F^Y_{t_0})$ are the initial conditions.

Thus, the solution to the optimal control problem defined by (1),(2) can be now obtained using the expression for the optimal filter gain matrix in (7) and the cross-correlation equation (9).

## 2.4 Optimal control problem solution

Since the filter gain matrix in (7) is equal to

$$M_f(t,s) = f(t,s)b(t,s)(R(s))^{-1},$$

the dual gain matrix in the optimal control problem takes the form of its transpose

$$M_c(t,s) = R^{-1}(s)b^T(t,s)f^T(t,s).$$

Hence, the optimal control law in the problem (1), (2) is given by

$$u^*(t,s) = R^{-1}(s)b^T(t,s)f^T(t,s)x(s), \tag{10}$$

where $f(t,s)$ is the solution of the integral Riccati equation

$$f(t,s) = P(t_0) + \int_{t_0}^{s} [-a^T(s,r)f^T(t,r) - f(s,r)a(t,r) + Q(r)]dr - \tag{11}$$

$$\int_{t_0}^{s} [f(t,r)b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r) +$$

$$f(s,r)b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r) -$$

$$(1/2)f(t,r)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r) -$$

$$(1/2)f(s,r)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r)]dr,$$

with the terminal condition $f(T,T) = P(T) = \Psi^{-1}$.

Finally, upon substituting the optimal control (10) into the state equation (1), the optimally controlled state equation is obtained

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(t,s) + a(t,s)x(s) + \tag{12}$$

$$b(t,s)R^{-1}(s)b^T(t,s)f^T(t,s)x(s))ds + \int_{t_0}^{t} g(s)dW_1(s).$$

# 3 Optimal Control in Discontinuous Ito-Volterra Systems

## 3.1 Problem statement

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ be an $F_t$-adapted Wiener process. Let us consider the $F_t$-measurable random process $x(t)$ governed by the Ito-Volterra equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(t,s) + a(t,s)x(s)ds + \tag{13}$$

$$\int_{t_0}^{t} b(t,s)u(t,s))dv(s) + \int_{t_0}^{t} g(s)dW_1(s).$$

Here $x(t) \in R^n$ is the state vector, $u(t,s) \in R^p$ is the control variable, $v(s)$ is a scalar bounded variation function, the Wiener process $W_1(t)$ represents a random disturbance, and the initial Gaussian vector $x(t_0)$ is independent of $W_1(t)$. The quadratic cost function $J$ to be minimized is defined as follows

$$J = E[\frac{1}{2}[x(T) - x_0]^T \Psi^{-1} [x(T) - x_0] \tag{14}$$

$$+\frac{1}{2}\int_{t_0}^{T} u^T(t,s)R(s)u(t,s)dv(s) + \frac{1}{2}\int_{t_0}^{T} x^T(s)Q(s)x(s)ds],$$

where $x_0$ is a given vector, $\Psi$, $R$, $Q$ are positive (nonnegative) definite symmetric matrices, $T > t_0$ is a certain time moment.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (13). The state trajectory $x(t)$ may be discontinuous due to discontinuity of the integral with discontinuous function $w(t)$ in the right-hand side of (24). This model of system states enables one to consider sharp changes (jumps) in system position, as well as its gradual continuous movement.

## 3.2 Dual filtering problem solution

Using the same technique as in Section 2, the suggested solution to the optimal discontinuous control problem for integral stochastic systems is based again on applying the duality principle to the optimal discontinuous filtering problem solution obtained in [5]. In this case, the filtering problem over discontinuous observations, dual to the optimal discontinuous control problem (13), (14), is formulated for the unobserved state

$$z(t) = z(t_0) + \int_{t_0}^{t} (a_0^T(t,s) - a^T(t,s)z(s))ds + \int_{t_0}^{t} Q^{1/2}(s)dW_3(s) \tag{15}$$

and the discontinuous observation process

$$y(t) = \int_{t_0}^{t} (b^T(t,s)x(s))dv(s) + \int_{t_0}^{t} R^{1/2}(s)dW_4(v(s)), \tag{16}$$

where $W_3(s)$ and $W_4(s)$ are independent Wiener processes which are in turn independent of an initial Gaussian vector $z(t_0)$. The filtering objective is the same as in Subsection 2.3.

As a result, the following filtering equations for the optimal estimate $m(t)$, its correlation function $P(t)$, and the cross-correlation function $f(t,s)$ (all notation is the same in Subsection 2.3) have been obtained in [5], applying the filtering procedure [20] for deriving the filtering equations over discontinuous observations from the known filtering equations over continuous ones to the equations (8)–(10):

$$m(t) = m(t_0) + \int_{t_0}^{t} (a_0^T(t,s) - a^T(t,s)m(s))ds+ \tag{17}$$

$$\int_{t_0}^{t} f(t,s)b(t,s)(R(s))^{-1}[dy(s) - b^T(t,s)m(s)dv(s)],$$

$$P(t) = P(t_0) + \int_{t_0}^{t} [-a^T(t,s)f^T(t,s) - f(t,s)a(t,s) + Q(s)]ds - \tag{18}$$
$$\int_{t_0}^{t} f(t,s)b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s)dv(s),$$

$$f(t,s) = P(t_0) + \int_{t_0}^{s} [-a^T(s,r)f^T(t,r) - f(s,r)a(t,r) + Q(r)]dr - \tag{19}$$
$$\int_{t_0}^{s} [f(t,r)b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r)+$$
$$f(s,r)b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r)-$$
$$(1/2)f(t,r)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r)-$$
$$(1/2)f(s,r)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r)]dv(r),$$

where $m(t_0) = E(z(t_0) \mid F_{t_0}^Y) = x_0$ and $P(t_0) = E((z(t_0) - m(t_0)(z(t_0) - m(t_0))^T \mid F_{t_0}^Y)$ are the initial conditions. The functions $m(t)$ and $P(t)$ have jumps at the discontinuity points of the function $v(t)$, and the function $f(t,s)$ is continuous in $t$ and has jumps in the second time argument $s$ at the same discontinuity points of $v(t)$.

### 3.3 Optimal control problem solution

Based on the duality of the filtering and control gain matrices, we conclude that the optimal control law is given by the same expression

$$u^*(t,s) = R^{-1}(s)b^T(t,s)f^T(t,s)x(s), \tag{20}$$

and $f(t,s)$ is the solution of the integral Riccati equation

$$f(t,s) = P(t_0) + \int_{t_0}^{s} [-a^T(s,r)f^T(t,r) - f(s,r)a(t,r) + Q(r)]dr - \tag{21}$$
$$\int_{t_0}^{s} [f(t,r)b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r)+$$
$$f(s,r)b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r)-$$
$$(1/2)f(t,r)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r)-$$
$$(1/2)f(s,r)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r)]dv(r),$$

with the terminal condition $f(T,T) = P(T) = \Psi^{-1}$.

Upon substituting the optimal control (20) into the state equation (1), the optimally controlled state equation is obtained

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(t,s) + a(t,s)x(s) + \tag{22}$$

$$b(t,s)R^{-1}(s)b^T(t,s)f^T(t,s)x(s))dv(s) + \int_{t_0}^{t} g(s)dW_1(s).$$

The obtained equations (21)-(22) are integral equations with integration w.r.t. a discontinuous measure generated by a bounded variation function $v(t)$, which do not tell us how to compute jumps of the optimally controlled state $x(t)$ and the gain-forming matrix $f(t,s)$ at the discontinuity points of the function $v(t)$ corresponding to discontinuities in the state $x(t)$. Nevertheless, in accordance with Theorem 3 in [5], the jumps can be computed solving the following system of differential equations, where $x(t-)$ and $f(t,s-)$ are values from the left of the system state $x(t)$ and its cross-correlation $f(t,s)$ at their discontinuity points $t$ and $(t,s)$, respectively:

$$\frac{dx}{dv} = b(t,t)R^{-1}(t)b^T(t,t)f(t,v)x(v),$$

$$x(0) = x(t-), \quad v \in [0, \Delta v(t)],$$

$$\frac{df(t,v)}{dv} = -[f(t,v)b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,v)+$$

$$f(s,v)b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,v)-$$

$$(1/2)f(t,v)b(t,s)(R(s))^{-1}b^T(s,s)f^T(s,v)-$$

$$(1/2)f(s,v)b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,v)],$$

$$f(t,0) = f(t,s-), \quad v \in [0, \Delta v(s)].$$

Subsequent solution yields the following jump expressions

$$\Delta x(t) = b(t,t)R^{-1}(t)b^T(t,t)f(t,t-)\Delta v(t),$$

$$\Delta f(t,s) = -[f(t,s-)[I + (b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)+$$

$$b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$

$$(1/2)b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$

$$(1/2)b(t,s)(R(s))^{-1}b^T(s,s)f(s,s-))\Delta v(s)]^{-1}\times$$

$$b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)+$$

$$f(s,s-)[I + (b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)+$$

$$b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$

$$(1/2)b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$

$$(1/2)b(t,s)(R(s))^{-1}b^T(s,s)f^T(s,s-))\Delta v(s)]^{-1}\times$$
$$b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)f(s,s-)[I+(b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)+$$
$$b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)b(t,s)(R(s))^{-1}b^T(s,s)f^T(s,s-))\Delta v(s)]^{-1}\times$$
$$b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)f(t,s-)[I+(b(s,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)+$$
$$b(t,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)b(s,s)(R(s))^{-1}b^T(t,s)f^T(t,s-)-$$
$$(1/2)b(t,s)(R(s))^{-1}b^T(s,s)f^T(s,s-))\Delta v(s)]^{-1}\times$$
$$b(t,s)(R(s))^{-1}b^T(s,s)f^T(s,s-)]\Delta v(s),$$

where $I$ is the $n\times n$-dimensional identity matrix.

Following [5], the obtained jump expressions can be incorporated into the regulator equations (21)–(22), using the form of the equivalent equations with a measure

$$x(t)=x(t_0)+\int_{t_0}^{t}(a_0(t,s)+a(t,s)x(s)+ \tag{23}$$
$$b(t,s)R^{-1}(s)b^T(t,s)f^T(t,s-)x(s))dv(s)+\int_{t_0}^{t}g(s)dW_1(s).$$

$$f(t,s)=\int_{t_0}^{s}[-a^T(s,r)f^T(t,r)-f(s,r)a(t,r)+Q(r)]dr- \tag{24}$$
$$\int_{t_0}^{s}[f(t,r-)[I+(b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)+$$
$$b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(t,r)(R(r))^{-1}b^T(s,r)f(s,r-))\Delta v(r)]^{-1}\times$$
$$b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)+$$
$$f(s,r-)[I+(b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)+$$
$$b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r-))\Delta v(r)]^{-1}\times$$

$$b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)f(s,r-)[I+(b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)+$$
$$b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r-))\Delta v(r)]^{-1}\times$$
$$b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)f(t,r-)[I+(b(s,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)+$$
$$b(t,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(s,r)(R(r))^{-1}b^T(t,r)f^T(t,r-)-$$
$$(1/2)b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r-))\Delta v(r)]^{-1}\times$$
$$b(t,r)(R(r))^{-1}b^T(s,r)f^T(s,r-)]dv(r),$$

with the terminal condition $f(T,T)=P(T)=\Psi^{-1}$. Here $\Delta v(t)$ is the jump of the bounded variation function $v(t)$ at its discontinuity point $t$, and $x(t-)$ and $f(t,s-)$ are values from the left of the system state $x(t)$ and the gain-forming matrix $f(t,s)$ at their discontinuity points $t$ and $(t,s)$, respectively.

## 3.4 Optimal control for dynamic plant

As shown in this section, the huge equations of Subsection 3.3 can be significantly simplified in the case of a dynamic system, if the state equation (13) has an internal differential part, i.e., is given by

$$x(t)=x(t_0)+\int_{t_0}^{t}(a_0(s)+a(s)x(s)ds+ \tag{25}$$

$$\int_{t_0}^{t}b(t,s)u(t,s))dv(s)+\int_{t_0}^{t}g(s)dW_1(s),$$

and the quadratic cost function $J$ is the same as in (14). Then, the dual filtering problem should be formulated for the unobserved state

$$z(t)=z(t_0)+\int_{t_0}^{t}(a_0^T(s)-a^T(s)z(s))ds+\int_{t_0}^{t}Q^{1/2}(s)dW_3(s) \tag{27}$$

and the discontinuous observation process

$$y(t)=\int_{t_0}^{t}(b^T(t,s)x(s))dv(s)+\int_{t_0}^{t}R^{1/2}(s)dW_4(v(s)). \tag{28}$$

As was proved in [6], in the case of a dynamic equation (27), it is possible to obtain a closed system of the optimal filtering equations with respect to only two variables, the optimal estimate $m(t)$ and its variance $P(t)$, without introducing the cross-correlation $f(t,s)$. Those filtering equations take the form [6]

$$m(t) = m(t_0) + \int_{t_0}^{t} (a_0^T(s) - a^T(s)m(s))ds + \tag{29}$$

$$\int_{t_0}^{t} P(s)b(t,s)(R(s))^{-1}[dy(s) - b^T(t,s)m(s)dv(s)].$$

$$P(t) = P(t_0) + \int_{t_0}^{t} [-a^T(s)P(s) - P(s)a(s) + Q(s)]ds - \tag{30}$$

$$\int_{t_0}^{t} P(s)b(t,s)(R(s))^{-1}b^T(t,s)P(s)dv(s),$$

where $m(t_0) = E(z(t_0) \mid F_{t_0}^Y) = x_0$ and $P(t_0) = E((z(t_0) - m(t_0)(z(t_0) - m(t_0))^T \mid F_{t_0}^Y) = \Psi$ are the initial conditions.

Based again on the duality of the filtering and control gain matrices, we conclude that the optimal control law is given by the expression

$$u^*(t,s) = R^{-1}(s)b^T(t,s)P(s)x(s), \tag{31}$$

and $P(s)$ is the solution of the integral Riccati equation

$$P(t) = P(t_0) + \int_{t_0}^{t} [-a^T(s)P(s) - P(s)a(s) + Q(s)]ds - \tag{32}$$

$$\int_{t_0}^{t} [P(s)b(t,s)(R(s))^{-1}b^T(t,s)P(s)]dv(s),$$

with the terminal condition $P(T) = \Psi^{-1}$.

Upon substituting the optimal control (31) into the state equation (25), the optimally controlled state equation is obtained

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(s) + a(s)x(s) + \tag{33}$$

$$b(t,s)R^{-1}(s)b^T(t,s)P(s)x(s))dv(s) + \int_{t_0}^{t} g(s)dW_1(s).$$

Correspondingly, the jumps of the optimally controlled state $x(t)$ and the gain-forming matrix $P(t)$ at the discontinuity points of $v(t)$ take the more simplified form

$$\Delta x(t) = b(t,t)R^{-1}(t)b^T(t,t)P(t-)\Delta v(t),$$

$$\Delta P(t) = -[P(t-)[I + (b(t,t)(R(t))^{-1}b^T(t,t)P(t-)\Delta v(t)]^{-1} \times$$
$$b(t,t)(R(t))^{-1}b^T(t,t)P(t)]\Delta v(t),$$

which can be incorporated into the following equations with a measure

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(s) + a(s)x(s) + \tag{34}$$

$$b(t,s)R^{-1}(s)b^T(t,s)P(s-)x(s))dv(s) + \int_{t_0}^{t} g(s)dW_1(s).$$

$$P(t) = \int_{t_0}^{t} [-a^T(s)P(s) - P(s)a(s) + Q(s)]ds - \tag{35}$$

$$\int_{t_0}^{t} [P(s-)[I + (b(t,s)(R(s))^{-1}b^T(t,s)P(s-)\Delta v(s)]^{-1} \times$$

$$b(t,s)(R(s))^{-1}b^T(t,s)P(s-)]dv(s),$$

with the terminal condition $P(T) = \Psi^{-1}$.

## 4 Movement of Missile with Impulsive and Jet Motors

Let us consider the optimal control problem for movement of a missile with two motors, impulsive and jet (continuous), whose task is to reach the maximal possible altitude at a certain time moment $T > 0$ with the minimal possible fuel consumption. The missile movement is considered governed by the following equations (cf. [7])

$$h(t) = h_0 + \int_0^t v(s)ds,$$

$$m(t) = m_0 + \int_0^t \frac{P_p(s)}{C(t,s)} ds,$$

$$v(t) = \int_0^t \frac{P_p(s) - Q(h,v)}{m(s)} dw(s) - \int_0^t g ds + \int_0^t r(s)dW(s),$$

where $t_0 = 0$, $v(t)$ is the missile velocity, $v_0 = v(0) = 0$;

$h_0 = h(0) > 0$ is the initial adjusted altitude corresponding the missile position on the earth surface, $h(t)$ is the current adjusted altitude;

$m(s)$ is the missile and fuel mass, $m_0 >> 0$;

$P_p(t)$ is the propulsion force;

$C(t,s) < 0$ is the difference factor of the ideal velocities of the missile at time $t$ and the outflowed fuel at time $s$, which is varying with change of altitude and, consequently, temperature, pressure, gravity acceleration, etc.;

$g$ is the gravity acceleration;

$r(s)dW(s)$ is the stochastic disturbance represented by a Wiener process and arising due to the resulting effect of unknown equally distributed and independent disturbances affecting velocity dynamics; and

$w(s)$ is a bounded variation function which represents functioning of two missile motors, impulsive and jet (continuous): the jet motor expels fuel gradually and the impulsive one does this instantaneously at a certain time moment $t_1$, $0 \le t_1 \le T$. Thus, the motors functioning is described using decomposition of $w(t)$ into its continuous component $w^c(t)$ (continuous jet) and the Heaviside function $\chi(t-t_1)$ with jump at the moment $t_1$ (impulsive motor), i.e., $w(t) = w^c(t) + \chi(t-t_1)$.

It is assumed that the atmosphere resistance force is absent: $Q(h,v) = 0$.

Upon selecting the mass outflow function $u(s) = \frac{\dot{m}(s)}{m(s)} = \frac{d}{ds}[\ln(m(s))]$ as control, the optimal control problem is completely stated for the system state $x(t) = [h(t), v(t)]$ governed by the equation

$$x(t) = x_0 + \int_0^t Ax(s)ds + \int_0^t B(t,s)u(s)dw(s) + \int_0^t Gds + \int_0^t R(s)dW(s),$$

where

$$x(t) = \begin{bmatrix} h(s) \\ v(s) \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B(t,s) = \begin{bmatrix} 0 \\ C(t,s) \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ -g \end{bmatrix}, R(s) = \begin{bmatrix} 0 \\ r(s) \end{bmatrix}, u(s) = \frac{\dot{m}(s)}{m(s)} = \frac{d}{ds}[\ln(m(s))],$$

$x_0 = [h_0, 0]$, and the cost function to be minimized

$$J = \frac{1}{2}\left[x(T) - \begin{bmatrix} h^* \\ 0 \end{bmatrix}\right]^T \psi \left[x(T) - \begin{bmatrix} h^* \\ 0 \end{bmatrix}\right]$$

$$+\frac{1}{2}\int_0^t u^2(s)dw(s) \to \min_{u(\cdot)},$$

where

$\psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $h^* >> h_0$, and $T > 0$ is a certain time moment.

In accordance with (31), the optimal control is given by

$$u^*(t,s) = \begin{bmatrix} 0 & C(t,s) \end{bmatrix} P(s) \begin{bmatrix} h(s) \\ v(s) \end{bmatrix}.$$

Note that the initial adjusted altitude $h_0 > 0$ is determined from the conditions $v(0) = 0$ and $\dot{v}(0) = 0$ (there is equilibrium of the missile on the earth surface at the initial time moment), which, upon substituting the optimal control $u^*(t,s)$ into the velocity equation, yield $0 = C(t_0,t_0)u^*(t_0,t_0) - g = C(0,0)u^*(0,0) - g$. Thus, the initial adjusted altitude $h_0 > 0$ is determined from the equation

$$g = C(0,0)\begin{bmatrix} 0 & C(0,0) \end{bmatrix} P(0) \begin{bmatrix} h_0 \\ 0 \end{bmatrix}.$$

In accordance with (34)–(35), the equations for an optimal trajectory $x(t)$ and the matrix $P(t)$ take the forms

$$P(t) = P(0) - \int_0^t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P(s)ds$$

$$- \int_0^t P(s) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ds - \int_0^t P(s-) \times \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ C(t,s) \end{bmatrix} [\, 0 \quad C(t,s) \,] P(s-)\Delta w(s)\}^{-1}$$

$$\times \begin{bmatrix} 0 \\ C(t,s) \end{bmatrix} [\, 0 \quad C(t,s) \,] P(s-)dw(s),$$

with the terminal condition $P(T) = \psi$, and

$$x(t) = x_0 + \int_0^t \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(s) + G \right\} ds + \int_0^t \begin{bmatrix} 0 \\ C(t,s) \end{bmatrix}$$

$$\times [\, 0 \quad C(t,s) \,] P(s-)x(s)dw(s) + \int_0^t \begin{bmatrix} 0 \\ r(s) \end{bmatrix} dW(s),$$

with the initial condition $x(0) = \begin{bmatrix} h_0 \\ 0 \end{bmatrix}$, and their jumps at the point $t_1$, where the impulsive motor is applied, are equal to

$$\Delta P(t_1) = P(t_1-)\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ C(t_1,t_1) \end{bmatrix}$$

$$\times [\, 0 \quad C(t_1,t_1) \,] P(t_1-)\Delta w(t_1)\}^{-1}$$

$$\times \begin{bmatrix} 0 \\ C(t_1,t_1) \end{bmatrix} [\, 0 \quad C(t_1,t_1) \,] P(t_1-)\Delta w(t_1),$$

$$\Delta x(t_1) = \begin{bmatrix} 0 \\ C(t_1,t_1) \end{bmatrix} [\, 0 \quad C(t_1,t_1) \,] P(t_1-)x(t_1)\Delta w(t_1).$$

Thus, the complete algorithm for solving this optimal control problem is described as follows:

- the equation for the matrix $P(t)$ with the terminal condition $P(T) = \psi$ and the jump $\Delta P(t_1)$ at the point $t_1$ is solved;
- the initial condition $P(0)$ is thus determined;
- the initial adjusted altitude $h_0$ is calculated;
- substituting $u^*(t,s)$ into the state equations and solving them with initial conditions $h(0) = h_0$ and $v(0) = 0$ yields the optimal trajectories $[h(t), v(t)] = x(t)$, where the velocity $v(t)$ has a jump at the point $t_1$, and the adjusted altitude $h(t)$ is continuous;
- the desirable maximal altitude is determined as $h(T) - h_0$.

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# OPTIMAL CONTROL IN UNOBSERVABLE ITO-VOLTERRA SYSTEMS

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**Abstract.** This paper presents solution of the optimal linear-quadratic controller problem for unobservable Ito-Volterra systems with continuous/discontinuous states over continuous/discontinuous observations. As a result, the system of the optimal controller equations is obtained, including a linear integral equation for the optimally controlled estimate and two integral Riccati equations for its cross-correlation function and a constituent of the optimal regulator gain matrix. Those equations are then simplified in the case of a dynamic plant (the internal part of a state equation) governed by a differential equation.
**Keywords.** Ito-Volterra system, optimal controller, filtering.
**AMS (MOS) subject classification:** 49K22, 93E20.

## 1 Introduction

This paper presents solution to the optimal linear-quadratic controller problem for unobservable Ito-Volterra systems with continuous/discontinuous states over continuous/discontinuous observations. Due to the separation principle for integral systems, which is stated analogously to that for dynamic differential ones [5], the initial continuous problem is split into the optimal minmax filtering problem for Ito-Volterra systems over continuous observations (see [1]) and the optimal linear-quadratic control (regulator) problem for observable Ito-Volterra systems with continuous states (see [2]). (Both papers [1, 2] contain the bibliography related to control and filtering problems for Ito-Volterra processes.) Based on the results obtained in [1, 2], the system of the optimal controller equations is first derived in the general case of Ito-Volterra state and observation equations, including a linear integral equation for the optimally controlled estimate and two integral Riccati equations for the estimate cross-correlation function and a constituent of the optimal regulator gain matrix. Those equations are then simplified in the case of a dynamic plant (the internal part of a state equation) governed by a differential equation, where the estimate cross-correlation function coincides with its variance (see [1]). In this situation, the estimate variance and the gain matrix constituent satisfying the Riccati equations depend on only one time variable, similarly to the variance in the filtering problem for a dynamic process over Ito-Volterra observations ([1]).

The optimal controller equations for Ito-Volterra systems with discontinuous states over discontinuous observations are obtained using the filtering procedure [6, 1] for deriving the filtering equations over discontinuous observations proceeding from the known filtering equations over continuous ones, which have already been obtained in the paper, and the dual results in the optimal control problem for Ito-Volterra systems [2]. In view of discontinuity of states and observations, the obtained optimal control law is discontinuous, and the optimal controller equations allow discontinuous solutions. Nevertheless, the obtained results enable one to compute jumps of the optimal filtering and control parameters (the optimally controlled state, the cross-correlation or variance, the gain matrix constituent, and the optimal control function) that can appear in points of discontinuity of states or observations.

Design of the optimal controller for Ito-Volterra state and observation equations and its simplification in the case of a dynamic plant (the internal part of a system) can serve as a background for subsequent design of the optimal controller for systems with delayed states and observations. The first obtained results (see [1]) have been based on the fact that a differential equation including even multiple time-varying delays presents a particular case of an integral (Ito)-Volterra equation. Further possible applications to controlling industrial processes, whose state and observation equations are subject to delays, are expected.

The paper is organized as follows. In Section 2, the optimal controller problem is stated and solved for unobservable continuous Ito-Volterra systems, using the separation principle for integral systems. Section 3 generalizes those results to discontinuous unobservable Ito-Volterra systems. The optimal controller equations are first obtained for Ito-Volterra state and observation equations and then simplified in the case of a dynamic plant (the internal part of a state equation).

# 2 Optimal Controller for Unobservable Continuous Ito-Volterra Systems

## 2.1 Problem statement

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be $F_t$-adapted Wiener processes. Let us consider the unobservable $F_t$-measurable random process $x(t)$ governed by the Ito-Volterra equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_0(t,s) + a(t,s)x(s) + b(t,s)u(t,s))ds + \int_{t_0}^{t} g(t,s)dW_1(s) \tag{1}$$

and the output (observation) process

$$y(t) = \int_0^t (A_0(t,s) + A(t,s)x(s))ds + \int_0^t B(t,s)dW_2(s). \tag{2}$$

Here, $x(t) \in R^n$ is the unobservable state vector, $u(t,s) \in R^p$ is the control variable, $y(t) \in R^m$ is the observation process, and the independent Wiener processes $W_1(t)$ and $W_2(t)$ represent random disturbances in state and observation equations, which are in turn independent of an initial Gaussian vector $x(t_0)$. Let $A(t,s)$ be a nonzero matrix and $B(t,s)B^T(t,s)$ be a positive definite matrix. In addition, the quadratic cost function $J$ to be minimized is defined as follows

$$J = E[\frac{1}{2}\,[x(T) - z_0]^T\,\Phi\,[x(T) - z_0] \tag{3}$$

$$+\frac{1}{2}\int_{t_0}^T u^T(t,s)K(s)u(t,s)ds + \frac{1}{2}\int_{t_0}^T x^T(s)L(s)x(s)ds],$$

where $z_0$ is a given vector, $\Phi$, $K$, $L$ are positive (nonnegative) definite symmetric matrices, $T > t_0$ is a certain time moment, the symbol $E[f(x)]$ means the expectation (mean) of a function $f$ of a random variable $x$, and $a^T$ denotes transpose to a vector (matrix) $a$.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1).

## 2.2 Separation principle in integral systems

As well as in linear stochastic systems governed by differential equations, the separation principle remains valid in linear integral stochastic systems governed by Ito-Volterra equations. Indeed, let us replace the unobservable system state $x(t)$ by its optimal estimate $m(t)$ given by the equation (see [1] for statement and derivation)

$$m(t) = m(t_0) + \int_0^t (a_0(t,s) + a(t,s)m(s) + b(t,s)u(t,s))ds+ \tag{4}$$

$$\int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))ds],$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$. Here, $m(t)$ is the best estimate for the Ito-Volterra process $x(t)$ at time $t$ based on the observation process $Y(t) = \{y(s), t_0 \le s \le t\}$, that is the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$. As shown in [1] and the previous papers [4, 7], it is impossible to obtain a closed system of filtering equations only for the optimal estimate $m(t)$ and its correlation function (variance) $S(t) =$

$E((x(t)-m(t)(x(t)-m(t))^T \mid F_t^Y)$, due to the Volterra nature of the equations (5) and (6). Designing a closed filter requires introducing the additional cross-correlation function $f(t,s)$ characterizing a deviation of the best estimate $m(t)$ from the real state $x(t)$:

$$f(t,s) = E((x_s^t - m_s^t)(x(s) - m(s))^T \mid F_{t,s}^Y), \tag{5}$$

where

$$x_s^t = \int_{t_0}^{s} (a_0(t,r) + a(t,r)x(r) + b(t,s)u(t,s))dr + \int_{t_0}^{s} g(t,r)dW_1(r),$$

$F_{t,s}^Y$ is the $\sigma$-algebra generated by the stochastic process $y_s^t$

$$y_s^t = \int_{t_0}^{s} (A_0(t,s) + A(t,s)x(s))ds + \int_{t_0}^{s} B(t,s)dW_2(s)$$

$$\text{and} \quad m_s^t = E(x_s^t \mid F_{t,s}^Y).$$

The equation for $f(t,s)$ takes the form (see [1] for derivation)

$$f(t,s) = f(t_0,t_0) + \int_{t_0}^{s} [a(s,r)f^T(t,r) + f(s,r)a^T(t,r) + \tag{6}$$

$$(1/2)(g(t,r)g^T(s,r) + g(s,r)g^T(t,r))]dr -$$

$$\int_0^s [f(t,r)A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f^T(s,r) +$$

$$f(s,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r) -$$

$$(1/2)f(t,r)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r) -$$

$$(1/2)f(s,r)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)]dr,$$

with the initial condition $f(t_0,t_0) = E((x(t_0) - m(t_0)(x(t_0) - m(t_0))^T \mid F_{t_0}^Y)$. Note that since $S(t) = f(t,t)$, the variance equation for $S(t)$ directly follows from (6):

$$S(t) = S(t_0) + \int_{t_0}^{t} [a(t,s)f^T(t,s) + f(t,s)a^T(t,s) +$$

$$g(t,s)g^T(t,s)]ds - \int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)f^T(t,s)ds,$$

where $S(t_0) = f(t_0,t_0)$. It is readily verified that the optimal control problem for the system state (1) and cost function (3) is equivalent to the optimal control problem for the optimal minmax estimate (4) and the cost function $J$ represented as

$$J = E\{\frac{1}{2}\,[m(T) - z_0]^T\,\Phi\,[m(T) - z_0]$$

$$+\frac{1}{2}\int_{t_0}^{T} u^T(t,s)K(s)u(t,s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds$$

$$+\frac{1}{2}\int_{t_0}^{T} tr[S(s)L(s)]ds + tr[S(T)\Phi]\}, \qquad (7)$$

where $tr[A]$ denotes trace of a matrix $A$. Since the latter part of $J$ is independent of control $u(t)$ or state $x(t)$, the reduced effective cost function $M$ to be minimized takes the form

$$M = E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0]$$

$$+\frac{1}{2}\int_{t_0}^{T} u^T(t,s)K(s)u(t,s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds\}. \qquad (8)$$

Thus, the solution for the optimal control problem specified by (1),(3) can be found solving the optimal control problem given by (4),(8). However, the minimal value of the criterion $J$ should be determined using (7). This conclusion presents the separation principle in integral Ito-Volterra systems.

## 2.3 Optimal control problem solution

Based on the solution of the optimal control problem obtained in [2] in the case of an observable system state, the following results are valid for the optimal control problem (4),(8), where the system state (the minmax estimate $m(t)$) is completely available and, therefore, observable.

The optimal control law is given by

$$u^*(t,s) = K^{-1}(s)b^T(t,s)q^T(t,s)m(s), \qquad (9)$$

where $q(t,s)$ is the solution of the integral Riccati equation

$$q(t,s) = q(t_0,t_0) + \int_{t_0}^{s}[-a^T(s,r)q^T(t,r) - q(s,r)a(t,r) + L(r)]dr - \qquad (10)$$

$$\int_{t_0}^{s}[q(t,r)b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r)+$$
$$q(s,r)b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r)-$$
$$(1/2)q(t,r)b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r)-$$
$$(1/2)q(s,r)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r)]dr,$$

with the terminal condition $q(T,T) = \Phi$.

Upon substituting the optimal control (9) into the equation (4) for the reconstructed system state $m(t)$, the following optimally controlled state estimate equation is obtained

$$m(t) = m(t_0) + \int_{t_0}^{t} A(t,s)m(s)ds + \int_{t_0}^{t} b(t,s)K^{-1}(s)b^T(t,s)q(t,s)m(s)ds$$

$$+\int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s)-(A_0(t,s)+A(t,s)m(s))ds]. \tag{11}$$

Thus, the optimally controlled state estimate equation (11), the gain matrix constituent equation (10), the optimal control law (9), and the cross-correlation equation (6) give the complete solution to the optimal controller problem for unobservable states of continuous integral systems governed by Ito-Volterra equations.

# 3 Optimal Controller for Unobservable Discontinuous Ito-Volterra Systems

## 3.1 Problem statement

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be $F_t$-adapted Wiener processes. Let us consider the unobservable $F_t$-measurable random process $x(t)$ governed by the Ito-Volterra equation

$$x(t) = x(t_0) + \int_{t_0}^t (a_0(t,s) + a(t,s)x(s))ds +$$

$$\int_{t_0}^t b(t,s)u(t,s)dv(s) + \int_{t_0}^t g(t,s)dW_1(s) \tag{12}$$

and the output (observation) process

$$y(t) = \int_0^t (A_0(t,s) + A(t,s)x(s))dw(s) + \int_0^t B(t,s)dW_2(w(s)), \tag{13}$$

where both state and observation equations are integral equations of the Volterra type with integration w.r.t. discontinuous measures, and the rest of the notation is the same as in Subsection 2.1.

The discontinuous measures in the state and observation equations are generated by scalar bounded variation functions $v(t)$ and $w(t)$, which can of course coincide or have discontinuities (jumps) at the same points. Therefore, the observation function $y(t)$ may be discontinuous due to discontinuity of the integral with discontinuous measure $dw(t)$ in the right-hand side of (13). This model of observations enables one to consider continuous and discrete observations in the common form: continuous observations correspond to the continuous component of a bounded variation function $w(t)$, and discrete observations correspond to its function of jumps.

The quadratic cost function $J$ to be minimized is defined as follows

$$J = E[\frac{1}{2}[x(T) - z_0]^T \Phi [x(T) - z_0] \tag{14}$$

$$+\frac{1}{2}\int_{t_0}^{T} u^T(t,s)K(s)u(t,s)dv(s) + \frac{1}{2}\int_{t_0}^{T} x^T(s)L(s)x(s)ds],$$

where $z_0$ is a given vector, $\Phi$, $K$, $L$ are positive (nonnegative) definite symmetric matrices, $T > t_0$ is a certain time moment.

The optimal control problem for the unobservable system state $x(t)$ is to find the *control* $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (12). The state trajectory $x(t)$ may also be discontinuous due to discontinuity of the integral with discontinuous function $v(t)$ in the right-hand side of (12). This model of system states enables one to consider sharp changes (jumps) in system position, as well as its gradual continuous movement. Modeling discontinuous unobservable system states of an Ito-Volterra system along with discontinuous observations of the Volterra type enables one to consider linear continuous, discrete, and delayed systems in the common form given by (12),(13), as it was done in [1].

## 3.2 Separation principle in discontinuous integral systems

The separation principle for discontinuous system states (12) and discontinuous observations (13) is based on the separation principle for continuous states and observations (5),(6). Actually, the corresponding filtering procedure was suggested [6] to obtain filtering equations over discontinuous observations proceeding from the known filtering equations over continuous ones. In the examined case, the following actions substantiated in [6] should be performed:

- assuming functions $v(t)$ and $w(t)$ in state and observation equations (12) and (13) to be absolutely continuous, write out the separation principle for continuous systems, obtained in Subsection 2.2, which yields the modified optimal control problem given by the state equation (4), effective criterion (8), cross-correlation function equation (6), and optimal value criterion (7);

- in thus obtained optimal control problem, assume the functions $v(t)$ and $w(t)$ to be arbitrary bounded variation ones again, keeping in mind that their derivative $\dot{v}(t)$ and $\dot{w}(t)$ can be generalized functions of zero singularity order (for example, $\delta$-functions), generating integration with the discontinuous measures $dv(t)$ and $dw(t)$.

As a result, the unobservable system state $x(t)$ of the system (12) is replaced by its optimal minmax estimate $\hat{x}(t)$ given by the equation (see [1] for statement and derivation)

$$m(t) = m(t_0) + \int_0^t (a_0(t,s) + a(t,s)m(s))ds + \int_0^t b(t,s)u(t,s))dv(s) + \quad (15)$$

$$\int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))dw(s)],$$

with the initial condition $m(t_0) = E(x(t_0) \mid F^Y_{t_0})$, where the cross-correlation function $f(t,s)$ satisfies the Riccati equation

$$f(t,s) = \int_0^s [a(s,r)f^T(t,r) + f(s,r)a^T(t,r)+ \tag{16}$$

$$(1/2)(g(t,r)g^T(s,r) + g(s,r)g^T(t,r))]dr-$$

$$\int_0^s [f(t,r)A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)+$$

$$f(s,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)-$$

$$(1/2)f(t,r)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)-$$

$$(1/2)f(s,r)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)]dw(r).$$

with the initial condition $f(t_0,t_0) = E((x(t_0) - m(t_0)(x(t_0) - m(t_0))^T \mid F^Y_{t_0})$.

Furthermore, the optimal control problem for the system state (12) and cost function (14) is equivalent to the optimal control problem for the optimal estimate (15) and the cost function $J$ represented as

$$J = E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0] + \frac{1}{2}\int_{t_0}^T u^T(t,s)K(s)u(t,s)dv(s)$$

$$+\frac{1}{2}\int_{t_0}^T m^T(s)L(s)m(s)ds + \frac{1}{2}\int_{t_0}^T tr[S(s)L(s)]ds + tr[S(T)\Phi]\}, \tag{17}$$

which can be reduced to the effective cost function $M$

$$M = E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0]$$

$$+\frac{1}{2}\int_{t_0}^T u^T(t,s)K(s)u(t,s)dv(s) + \frac{1}{2}\int_{t_0}^T m^T(s)L(s)m(s)ds\}. \tag{18}$$

Thus, the solution for the optimal control problem specified by (12),(14) can be found solving the optimal control problem given by (15),(18) and using (17) for the minimal value of the criterion $J$.

## 3.3 Optimal control problem solution for discontinuous systems

Based on the solution of the optimal control problem obtained in [2] in the case of an observable discontinuous system state, the following results are valid for the optimal control problem (15),(18), where the system state (the optimal estimate $m(t)$) is completely available and, therefore, observable.

The optimal control law is given by

$$u^*(t,s) = K^{-1}(s)b^T(t,s)q^T(t,s)m(s), \tag{19}$$

where $q(t,s)$ is the solution of the integral Riccati equation

$$q(t,s) = q(t_0,t_0) + \int_{t_0}^{s} [-a^T(s,r)q^T(t,r) - q(s,r)a(t,r) + L(r)]dr - \tag{20}$$
$$\int_{t_0}^{s} [q(t,r)b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r)+$$
$$q(s,r)b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r)-$$
$$(1/2)q(t,r)b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r)-$$
$$(1/2)q(s,r)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r)]dv(r).$$

with the terminal condition $q(T,T) = \Phi$.

Upon substituting the optimal control (19) into the equation (15) for the reconstructed system state $m(t)$, the following optimally controlled state estimate equation is obtained

$$m(t) = m(t_0) + \int_{t_0}^{t} (a_0(t,s) + a(t,s)m(s))ds$$
$$+ \int_{t_0}^{t} b(t,s)K^{-1}(s)b^T(t,s)q(t,s)m(s)dv(s)$$
$$+ \int_{0}^{t} f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))dw(s)], \tag{21}$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$.

The obtained equations (20)–(21), as well as the equation (16) for the cross-correlation function $f(t,s)$, are integral equations with integration w.r.t. discontinuous measures generated by bounded variation functions $v(t)$ and $w(t)$, which do not tell us how to compute jumps of the controller variables (the estimate $m(t)$, its cross-correlation function $f(t,s)$, and the gain matrix constituent $q(t,s)$) at the discontinuity points of the functions $v(t)$ and $w(t)$, corresponding to discontinuities in the system states $x(t)$ and the observation process $y(t)$. Nevertheless, the direct method for computing the jumps was given by Theorem 3 in [1], which yields the following jump expressions

$$\Delta m(t) = f(t,t-)[I + A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)f^T(t,t-)\Delta w(t)]^{-1}\times$$
$$A^T(t,t)(B(t,t)B^T(t,t))^{-1}[\Delta y(t) - (A_0(t,t) + A(t,t)m(t-))\Delta w(t)] +$$
$$b(t,t)K^{-1}(t)b^T(t,t)[I + A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)\times$$
$$f^T(t,t-)\Delta w(t)]^{-1}q(t,t-)m(t-)\Delta v(t),$$
$$\Delta f(t,s) = -[f(t,s-)[I + (A^T(s,s)(B(s,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)+$$
$$A^T(t,s)(B(t,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-$$

$$
\begin{aligned}
&(1/2)A^T(s,s)(B(s,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(t,s)(B(t,r)B^T(s,r))^{-1}A(s,s)f(s,s-))\Delta w(s)]^{-1}\times\\
&A^T(s,s)(B(s,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)+\\
&f(s,s-)[I+(A^T(s,s)(B(s,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)+\\
&A^T(t,s)(B(t,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(s,s)(B(s,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(t,s)(B(t,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-))\Delta w(s)]^{-1}\times\\
&A^T(t,s)(B(t,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)f(s,s-)[I+(A^T(s,s)(B(s,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)+\\
&A^T(t,s)(B(t,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(s,s)(B(s,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(t,s)(B(t,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-))\Delta w(s)]^{-1}\times\\
&A^T(s,s)(B(s,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)f(t,s-)[I+(A^T(s,s)(B(s,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)+\\
&A^T(t,s)(B(t,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(s,s)(B(s,r)B^T(t,r))^{-1}A(t,s)f^T(t,s-)-\\
&(1/2)A^T(t,s)(B(t,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-))\Delta w(s)]^{-1}\times\\
&A^T(t,s)(B(t,r)B^T(s,r))^{-1}A(s,s)f^T(s,s-)]\Delta w(s),
\end{aligned}
$$

where $I$ is the $n\times n$-dimensional identity matrix, and

$$
\begin{aligned}
\Delta q(t,s) = -[&q(t,s-)[I+(b(s,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)+\\
&b(t,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-\\
&(1/2)b(s,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-\\
&(1/2)b(t,s)(K(s))^{-1}b^T(s,s)q(s,s-))\Delta v(s)]^{-1}\times\\
&b(s,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)+\\
&q(s,s-)[I+(b(s,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)+\\
&b(t,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-\\
&(1/2)b(s,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-\\
&(1/2)b(t,s)(K(s))^{-1}b^T(s,s)q^T(s,s-))\Delta v(s)]^{-1}\times\\
&b(t,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-\\
&(1/2)q(s,s-)[I+(b(s,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)+\\
&b(t,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-
\end{aligned}
$$

$$(1/2)b(s,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-$$
$$(1/2)b(t,s)(K(s))^{-1}b^T(s,s)q^T(s,s-))\Delta v(s)]^{-1}\times$$
$$b(s,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-$$
$$(1/2)q(t,s-)[I+(b(s,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)+$$
$$b(t,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-$$
$$(1/2)b(s,s)(K(s))^{-1}b^T(t,s)q^T(t,s-)-$$
$$(1/2)b(t,s)(K(s))^{-1}b^T(s,s)q^T(s,s-))\Delta v(s)]^{-1}\times$$
$$b(t,s)(K(s))^{-1}b^T(s,s)q^T(s,s-)]\Delta v(s).$$

Following [1], the obtained jump expressions can be incorporated into the controller and filtering equations (20), (21), (16) using the equivalent form of integral equations with integration w.r.t. a discontinuous measure

$$m(t)=m_0+\int_{t_0}^{t}(a_0(t,s)+a(t,s)m(s))ds+\int_{t_0}^{t}b(t,s)u(t,s)ds$$
$$+\int_{t_0}^{t}b(t,s)K^{-1}(s)b^T(t,s)\{I+A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)$$
$$\times f(t,s-)\Delta w(s)\}^{-1}q(t,s-)m(s-)dv(s)$$
$$+\int_{t_0}^{t}f(t,s-)\left\{I+A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)f(t,s-)\Delta w(s)\right\}^{-1}$$
$$\times A^T(t,s)(B(t,s)B^T(t,s))^{-1}\left[dy(s)-A(t,s)m(s-)dw(s)\right], \tag{22}$$

with the initial condition $m(t_0)=E(x(t_0)\mid F_{t_0}^Y)$,

$$f(t,s)=f(t_0,t_0)+\int_{t_0}^{s}[a(s,r)f^T(t,r)+f(s,r)a^T(t,r)+ \tag{23}$$
$$(1/2)(g(t,r)g^T(s,r)+g(s,r)g^T(t,r))]dr-$$
$$\int_{t_0}^{s}[f(t,r-)[I+(A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)+$$
$$A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta w(r)]^{-1}\times$$
$$A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f^T(s,r-)+$$
$$f(s,r-)[I+(A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)+$$
$$A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$

$$(1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta w(r)]^{-1}\times$$
$$A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r-)-$$
$$(1/2)f(s,r-)[I+(A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)+$$
$$A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta w(r)]^{-1}\times$$
$$A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r-)-$$
$$(1/2)f(t,r-)[I+(A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)+$$
$$A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)-$$
$$(1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta w(r)]^{-1}\times$$
$$A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r-)]dw(r).$$

with the initial condition $f(t_0,t_0)=E((x(t_0)-m(t_0)(x(t_0)-m(t_0))^T \mid F^Y_{t_0})$, the function $f(t,s)$ is continuous in $t$, and

$$q(t,s)=q(t_0,t_0)+\int_{t_0}^{s}[-a^T(s,r)q^T(t,r)-q(s,r)a(t,r)+L(r)]dr- \quad (24)$$
$$\int_{t_0}^{s}[q(t,r-)[I+(b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)+$$
$$b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(t,r)(K(r))^{-1}b^T(s,r)q(s,r-))\Delta v(r)]^{-1}\times$$
$$b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)+$$
$$q(s,r-)[I+(b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)+$$
$$b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r-))\Delta v(r)]^{-1}\times$$
$$b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)q(s,r-)[I+(b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)+$$
$$b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r-))\Delta v(r)]^{-1}\times$$

$$\cdot\, b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)q(t,r-)[I+(b(s,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)+$$
$$b(t,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(s,r)(K(r))^{-1}b^T(t,r)q^T(t,r-)-$$
$$(1/2)b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r-))\Delta v(r)]^{-1}\times$$
$$b(t,r)(K(r))^{-1}b^T(s,r)q^T(s,r-)]\Delta v(r).$$

with the terminal condition $q(T,T)=\Phi$, the function $q(t,s)$ is continuous in $t$. Here $\Delta w(t)$, $\Delta v(t)$, and $\Delta y(t)$ are the jumps of the bounded variation functions $w(t)$, $v(t)$, and the observation process $y(t)$ at a point $t$, respectively, and $m(t-)$, $f(t,s-)$, and $q(t,s-)$ are the values of the discontinuous controller and filtering parameters (the estimate $m(t)$, its cross-correlation function $f(t,s)$, and the gain matrix constituent $q(t,s)$) at points $t$ and $(t,s)$ from the left.

The optimally controlled state estimate equation (22), the gain matrix constituent equation (24), the cross-correlation equation (23), and the optimal control law (19) give the complete solution to the optimal controller problem for unobservable states of discontinuous integral systems governed by Ito-Volterra equations, including analytic expressions for jumps of the controller and filtering variables at the discontinuity points of the real system state $x(t)$ and the observation process $y(t)$.

## 3.4 Optimal controller for dynamic plant

As shown in this section, the equations of Subsection 3.3 can be significantly simplified in the case of a dynamic system, if the state equation (12) has an internal differential part, i.e., is given by

$$x(t)=x(t_0)+\int_{t_0}^{t}(a_0(s)+a(s)x(s))ds+ \tag{25}$$
$$\int_{t_0}^{t}b(t,s)u(t,s))dv(s)+\int_{t_0}^{t}g(t,s)dW_1(s),$$

and the observation process $y(t)$ (13) and the quadratic cost function $J$ (14) are the same.

As was proved in [1], in the case of a dynamic plant equation (25), it is possible to obtain a closed system of the optimal filtering equations with respect to only two variables, the optimal estimate $m(t)$ and its variance $S(t)$, without introducing the cross-correlation $f(t,s)$. Those filtering equations take the form [1]

$$m(t)=m(t_0)+\int_{t_0}^{t}(a_0(s)+a(s)m(s))ds+\int_{0}^{t}b(t,s)u(t,s))dv(s)+ \tag{26}$$

$$\int_0^t S(s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))dw(s)],$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$, where the variance function $S(t)$ satisfies the Riccati equation (which is given in the equivalent form and follows from (23) using $S(t) = f(t,t)$)

$$S(t) = S(t_0) + \int_{t_0}^{s} [a(s)S(s) + S(s)a^T(s) + g(t,s)g^T(t,s)]ds- \tag{27}$$

$$\int_{t_0}^{t} [S(s-)\{I + A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)S(s-)\Delta w(s)\}^{-1}$$

$$A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)S(s-)]dw(s).$$

with the initial condition $S(t_0) = f(t_0,t_0) = E((x(t_0) - m(t_0)(x(t_0) - m(t_0))^T \mid F_{t_0}^Y)$.

Furthermore, the optimal control problem for the system state (25) and cost function (14) is equivalent to the optimal control problem for the optimal estimate (26) and the cost function (17), which can be reduced to the effective cost function (18). Thus, the solution for the optimal control problem specified by (25),(14) can be found solving the optimal control problem given by (26),(18) and using (17) for the criterion minimal value.

Based on the results obtained in [2] for the optimal control problem in an Ito-Volterra system with a dynamic internal part, the optimal control law is given by

$$u^*(t,s) = K^{-1}(s)b^T(t,s)P(s)m(s), \tag{28}$$

where $P(t)$ is the solution of the integral Riccati equation (which is given in the equivalent form and follows from (24) using $P(t) = q(t,t)$)

$$P(t) = P(t_0) + \int_{t_0}^{t} [-a^T(s)P(s) - P(s)a(s) + L(s)]ds- \tag{29}$$

$$\int_{t_0}^{t} [P(s-)\{I + A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)P(s-)\Delta w(s)\}^{-1}\times$$

$$b(t,s)(K(s))^{-1}b^T(t,s)P(s-)]dv(s),$$

with the terminal condition $P(T) = q(T,T) = \Phi$.

Upon substituting the optimal control (27) into the equation (26) for the reconstructed system state $m(t)$, the following optimally controlled state estimate equation is obtained (which is given in the equivalent form and follows from (22) using $S(t) = f(t,t)$ and $P(t) = q(t,t)$)

$$m(t) = m_0 + \int_{t_0}^{t} (a_0(s) + a(s)m(s))ds + \int_{t_0}^{t} b(t,s)u(t,s)ds$$

$$+\int_{t_0}^{t} b(t,s)K^{-1}(s)b^T(t,s)\{I + A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)$$

$$\times S(s-)\Delta w(s)\}^{-1}P(s-)m(s-)dv(s)$$

$$+\int_{t_0}^{t} S(s-)\{I + A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)S(s-)\Delta w(s)\}^{-1}$$

$$\times A^T(t,s)(B(t,s)B^T(t,s))^{-1}\left[dy(s) - A(t,s)m(s-)dw(s)\right], \qquad (30)$$

with the initial condition $m(t_0) = E(x(t_0) \mid F^Y_{t_0})$.

Thus, in the case of an Ito-Volterra system with a dynamic internal plant, the optimal controller problem solution is completely given by the optimal controller equation (30), the variance equation (27), the gain matrix constituent equation (29), and the optimal control law (28). Obviously, the case of continuous state and observation equations in Ito-Volterra system with a dynamic internal part (considered in Section 2) is recovered assuming $v(t) = t$ and $w(t) = t$ in (25)–(30).

The jumps of all the optimal controller variables in the discontinuous case also take the simplified form

$$\Delta m(t) = S(t-)[I + A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)S(t-)\Delta w(t)]^{-1}\times$$

$$A^T(t,t)(B(t,t)B^T(t,t))^{-1}\left[\Delta y(t) - (A_0(t,t) + A(t,t)m(t-))\Delta w(t)\right]+$$

$$b(t,t)K^{-1}(t)b^T(t,t)[I + A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)\times$$

$$S(t-)\Delta w(t)]^{-1}P(t-)m(t-)\Delta v(t),$$

$$\Delta S(t) = -[S(t-)[I + (A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)S(t-))\Delta w(t)]^{-1}\times$$

$$A^T(t,t)(B(t,t)B^T(t,t))^{-1}A(t,t)S(t-)]\Delta w(t).$$

and

$$\Delta P(t) = -[P(t-)[I + (b(t,t)(K(t))^{-1}b^T(t,t)P(t-))\Delta v(t)]^{-1}\times$$

$$b(t,t)(K(t))^{-1}b^T(t,t)P(t-)]\Delta v(s).$$

Let us finally note that the results in design of the optimal controller for a system with an internal dynamic part can readily be applied to solution of the optimal controller problem of launching a missile with continuous and impulsive jet motors and unobservable velocity to the maximal possible altitude with the minimal fuel consumption (see [3] for its initial continuous statement), as it has been done for the corresponding optimal regulator problem in [2].

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# OPTIMAL POLYNOMIAL FILTER AND REGULATOR FOR THIRD DEGREE POLYNOMIAL SYSTEMS

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This paper presents the optimal nonlinear filter for a stochastic system state given by a polynomial equation of degree 3 or 4 and linear observations confused with white Gaussian noises. The obtained filtering equations are applied to solution of the state estimation problem for a nonlinear automotive system. Simulation results are compared for the optimal polynomial filter given in this paper and the linear Kalman-Bucy filter applied to the linearized system. Using the duality principle, the optimal regulator is then designed for a polynomial system of degree 3 with linear control input and quadratic cost criterion, applied to the nonlinear automotive system, and compared to the optimal linear regulator dual to the Kalman-Bucy filter.

## 1. Introduction

Although the general optimal solution of the filtering problem for nonlinear state and observation equations confused with white Gaussian noises is given by the Kushner equation for the conditional density of an unobservable state with respect to observations [6], there are a very few known examples of nonlinear systems where the Kushner equation can be reduced to a finite-dimensional closed system of filtering equations for a certain number of lower conditional moments. The most famous result, the Kalman-Bucy filter [5], is related to the case of linear state and observation equations, where only two moments, the estimate itself and its variance, form a closed system of filtering equations. However, the optimal nonlinear finite-dimensional filter can be obtained in some other cases, if, for example, the state vector can take only a finite number of admissible states [13] or if the observation equation is linear and the drift term in the state equation satisfies the Riccati equation $df/dx + f^2 = x^2$ (see [3]). The complete classification of the "general situation" cases (this means that there are no special assumptions on the structure of state and observation equations), where the optimal nonlinear finite-dimensional filter exists, is given in [14].

This paper presents the optimal nonlinear filter for a stochastic system state given by a polynomial equation of degree 3 or 4 and linear observations confused with white Gaussian noises. This relatively simple case seems to be important for practical applications, since a nonlinear state equation can usually be well approximated by a polynomial of degree 3 or 4 and the observations are frequently direct, that is linear. Moreover, the filtering problem for a polynomial state equation of lower degree is significant itself, because many, for example, chemical processes are described by quadratic equations (see [10]). As shown in the paper, the polynomial filter of lower degree represents a particular case of the polynomial filter of any superior degree, so the quadratic filter is a particular case of the polynomial filter of degree 3, 3 of 4, etc.

The obtained optimal filter for a polynomial state equation of degree 3 is applied to solution of the state estimation problem for a nonlinear automotive system [9] whose state equation for car orientation angle is nonlinear (contains tangent). Along with the original state equation, its expansion to Taylor polynomial up to degree 3 is also considered. For both state equations and linear observations, the optimal filtering equations for a polynomial state of third degree are written and then compared to the linear Kalman-Bucy filter applied to the linearized system. Simulations are conducted for both original and approximate systems and also compared to the linear Kalman-Bucy filter applied to the linearized system. The simulation results given in the paper show significant advantage of the optimal polynomial filter in comparison to the Kalman-Bucy one, especially for the original nonlinear state equation.

Although the optimal control (regulator) problem for linear system states was solved, as well as the filtering one, in 1960s [7, 4], the optimal control function for nonlinear systems has to be determined using the general principles of maximum [11] or dynamic programming [2] which do not provide an explicit form for the optimal control in most cases. However, taking into account that the optimal control problem can be solved in the linear case applying the duality principle to the solution of the optimal filtering problem, this paper exploits the same idea for designing the optimal control in a polynomial system with linear control input, using the optimal filter for polynomial system states over linear observations. Based on the obtained polynomial filter of the third degree, the optimal regulator for a polynomial system of degree 3 with linear control input and quadratic cost criterion is obtained in a closed form, finding the optimal regulator gain matrix as dual transpose to the optimal filter gain one and constructing the optimal regulator gain equation as dual to the variance equation in the optimal filter. The results obtained by virtue of the duality principle could be rigorously verified through the general equations of [11] or [2] applied to a specific polynomial case, although the physical duality seems obvious: if the optimal filter exists in a closed from, the optimal closed-form regulator should also exist, and vice versa. Finally, the obtained optimal control for a polynomial system of the third degree is applied for regulation of the same automotive system [9], with the objective of increasing values of the state variables and consuming the minimum control energy.

The paper is organized as follows. The Kalman-Bucy filter for linear state and observation equations is reminded in Section 2. Section 3 presents an intermediate result: the Kushner equation in the case of polynomial state and linear observation equations. The optimal nonlinear filter for a polynomial state equation of degree 3 and linear observations is derived in Section 4. This result is generalized to a polynomial state equation of degree 4 in Section 5. The optimal control problem for a polynomial system is stated, the duality principle is briefly reminded, and the optimal control problem for a polynomial system state of degree 3 is solved in Section 6. In Section 7, the obtained results are applied to the filtering problem for a nonlinear automotive system with two state variables, orientation and steering angles, over direct linear observations confused with white Gaussian noises. Graphic simulation results are also obtained and compared with those for the linear Kalman-Bucy filter applied to the linearized system. Section 8 presents application of the optimal polynomial regulator to the same automotive system, with the objective to increase values of the state variables and consume the minimum control energy. Graphic simulation results are conducted for polynomial control of degree 3 and compared with those for lineal control.

## 2. Optimal filtering for linear state and observation equations

For reference purposes, this section briefly describes the Kalman-Bucy filter [5] for linear state and observation equations. Let an unobservable random process x(t) satisfy a linear equation

$$dx(t) = (a_0(t) + a(t)x(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0,$$

and linear observations are given by:

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t),$$

where $W_1(t)$ y $W_2(t)$ are Wiener processes, whose weak derivatives are Gaussian white noises and which are assumed independent of each other and of the Gaussian initial value $x_0$.

The filtering problem is to find dynamical equations for the best estimate for the real process x(t) at time t, based in the observations $Y(t) = \{y(s) / t_0 \le s \le t\}$, that is the conditional expectation $m(t) = E[x(t) / Y(t)]$ of the real process x(t) with respect to the observations Y(t). Let $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T / Y(t)]$ be the estimate variance (correlation function).

The solution to this problem is given by the following system of filtering equations, which is closed with respect to the introduced variables, *m(t)* and *P(t)*:

$$dm(t) = (a_0(t) + a(t)m(t))dt + P(t)A^T(t)(B(t)B^T(t))^{-1}[dy - (A_0(t) + A(t)m(t))]dt,$$
$$m(t_0) = E[x(t_0) / y(t_0)],$$
$$dP(t) = (a(t)P(t) + P(t)a^T(t) + b(t)b^T(t))dt - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt,$$
$$P(t_0) = E[(x(t_0) - m(t_0))(x(t_0) - m(t_0))^T / y(t_0)].$$

## 3. Kushner equation for nonlinear state and linear observations

In the case of a nonlinear state equation, the problem is more complicated. Let an unobservable random process x(t) satisfy a nonlinear equation

(1) $$dx(t) = (f(x(t)))dt + b(t)dW_1(t), \ x(t_0) = x_0,$$

and linear observations are given by

(2) $$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t),$$

where $W_1(t)$ and $W_2(t)$ are Wiener processes independent of each other and of the Gaussian initial value $x_0$. The desirable best estimate is the conditional expectation $m(t) = E[x(t) / Y(t)]$ of the real process x(t) with respect to the observations Y(t), and $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T / Y(t)]$ is its variance (correlation function).

Since the observation equation is linear, the innovations process $\upsilon(t) = y(t) -$

$\int_{t0}^{t} (A_0(t) + A(t)m(t))\,dt = \int_{t0}^{t} (A_0(t) + A(t)x(t))\,dt + B(t)dW_2(t) - (A_0(t) + A(t)m(t))\,dt = \int_{t0}^{t} (A(t)(x(t) - m(t)))\,dt + B(t)dW_2(t)$ is a Wiener process in the case of Gaussian disturbances (see [8]), and $\int_{t0}^{t} B(t)dW_2(t)$ is also a Wiener process, then the random variable $A(t)(x(t) - m(t))$ is Gaussian with respect to observations for each fixed t [12]. If the matrix $A^{-1}$ exists, then the random vector $(x(t) - m(t))$ is also Gaussian [12].

Moreover, in this case, the Kushner equation for the optimal estimate $m(t) = E[x(t) / Y(t)]$ takes the form which readily follows from the general form of the Kushner equation (see [8]) and the observation equation (2)):

(3) $dm(t) = E[f(x(t)) / Y(t)]\,dt + P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))dt]$,
$m(t_0) = E[x(t_0) / y(t_0)]$.

Let us note [1] that if the function $f(x(t)) = a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t) + \ldots$ is a polynomial, it should be possible to compute a finite-dimensional filter in a closed form for variables m(t) and P(t), using the fact that the random variable $(x(t)-m(t))$ is Gaussian. This implies that all conditional odd central moments of this Gaussian variable $\mu_1 = E[(x(t) - m(t)) / Y(t)]$, $\mu_3 = E[(x(t) - m(t))^3 / Y(t)]$, $\mu_5 = E[(x(t) - m(t))^5 / Y(t)]$,... are equal to 0, and all even central moments $\mu_2 = E[(x(t) - m(t))^2 / Y(t)]$, $\mu_4 = E[(x(t) - m(t))^4 / Y(t)]$, .... can be represented as functions of the variance P(t). For example: $\mu_2 = P$, $\mu_4 = 3P^2$, $\mu_6 = 15P^3$,... etc. Thus, all higher moments of x(t) with respect to Y(t) can be expressed using P(t), and this yields additional relations for representing each higher initial moment of x(t) with respect to Y(t) and, finally, the possibility to obtain the optimal filter in a closed form, i.e., the optimal finite-dimensional filter should exist in the polynomial-linear case.

## 4. Optimal filter for polynomial state equation of degree 3 and linear observations

In this section, the outlined procedure is applied to deriving the optimal filter for the case of a polynomial state equation of degree 3, obtained from (1) if $f(x) = a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t)$:

(4) $dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + b(t)dW_1(t)$, $x(t_0) = x_0$,

and the linear observations (2), where $x \in R^n$, $x^2(t) = (x_1^2(t), x_2^2(t), x_3^2(t), \ldots, x_n^2(t))$, $x^3(t) = (x_1^3(t), x_2^3(t), x_3^3(t), \ldots, x_n^3(t))$.

Since all odd central moments for $(x(t) - m(t))$ are equal to 0 and all even central moments can be represented as functions of P(t), the higher initial moments of the state x(t) with respect to the observations Y(t) can also be expressed as functions of m(t) and P(t), as it is done below.

Let $m(t) \in R^n$ be the best estimate vector, $m(t) = (m_1(t), m_2(t), \ldots, m_n(t))$; $P(t) \in R^{n\times n}$ be the covariance matrix; $p(t) \in R^n$ be the vector whose components are the variances of the components of x(t), i.e., the diagonal elements of P(t); $m^2(t)$ be defined as the vector of squares of the components of m(t): $m^2(t) = (m_1^2(t), m_2^2(t), \ldots, m_n^2(t))$; P(t)m(t) be the conventional product of a matrix P(t) by a vector m(t); and p(t)*m(t) be the product of two vectors by components: $p(t)*m(t) = [p_1(t)m_1(t), p_2(t)m_2(t), \ldots, p_n(t)m_n(t)]$.

Using the introduced notation, the expression for the second initial moment is given by

(5) $E[x^2(t)/Y(t)] = p(t) + m^2(t)$.

Since $E[(x(t) - m(t))^3 / Y(t)] = 0$, then $E[(x(t) - m(t))^3 / Y(t)] = E[x^3(t) /Y(t)] -$

$3m(t) * E[x^2(t) / Y(t)] + 3m^2(t) * E[x(t) / Y(t)] - m^3(t) = 0.$

Substituting (5) into the last equation, the third initial moment expression is obtained

(6) $$E[x^3(t) / Y(t)] = 3p(t)*m(t) + m^3(t).$$

Taking into account $E[(x(t) - m(t))^4 /Y(t)] = 3p^2(t)$, the following equality is valid: $E[(x(t) - m(t))^4/Y(t)] = E[x^4(t) /Y(t)] - 4m(t)*E[x^3(t) /Y(t)] + 6m^2(t)*E[x^2(t) / Y(t)] - 4m^3(t)*E[x(t) /Y(t)] + m^4(t) = 3p^2(t)$, where $m^3(t) = (m_1^3(t), m_2^3(t),...m_n^3(t))$ and $m^4(t) = (m_1^4(t), m_2^4(t),...m_n^4(t))$. Substituting (5) and (6) and making the corresponding algebraic transformations, the fourth initial moment expression follows

(7) $$E[x^4(t) / Y(t)] = 3p^2(t) +6p(t)*m^2(t) + m^4(t).$$

The fifth initial moment representation can be obtained analogously using the equality $E[x^5(t) / Y(t)] = 0$ and substituting the previously obtained expressions (5)-(7):

(8) $$E[x^5(t) / Y(t)] = 15m(t)*p^2(t) + 10p(t)*m^3(t) + m^5(t),$$

where $m^5(t) = (m_1^5(t), m_2^5(t),...m_n^5(t))$. Thus, in the case of a polynomial state equation of degree 3, $f(x(t)) = a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t)$, the Kushner equation (3) for the optimal estimate $m(t) = E[x(t) / Y(t)]$ can be reduced to

$$dm(t) = (E[a_0(t) /Y(t)] + E[a_1(t)x(t) / Y(t)] + E [a_2(t)x^2(t) / Y(t)] + E[a_3(t)x^3(t) / Y(t)])dt + P(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) +A(t)m(t))dt).$$

Using the representations (5) and (6) for the second and third moments, the optimal estimate equation takes the form

(9) $$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + a_3(t)(3p(t)*m(t) + m^3(t)))dt + P(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt), \quad m(t_0) = E[x(t_0) / y(t_0)].$$

The next step is to obtain the equation for the covariance matrix

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T / Y(t)].$$

Upon differentiating the last equality in t

$$dP(t) = dE[(x(t) - m(t))(x(t) - m(t))^T / Y(t)] = E[d\{x(t)(x(t) - m(t))^T + m(t)(x(t) - m(t))^T\}/ Y(t)] = E[\{ (dx(t))(x(t) - m(t))^T + x(t)(d(x(t) - m(t))^T)\} / Y(t)] = E[\{(dx(t))(x(t) - m(t))^T + x(t)(dx(t) - dm(t))^T\}/ Y(t)] = E[(dx(t))(x(t) - m(t))^T / Y(t)] + E[x(t)(dx(t) - dm(t))^T / Y(t)]$$

and substituting the expressions for dx(t) and dm(t) given by (4) and (9), the following equation follows

$$dP(t) = E[(((a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t)) dt + b(t)dW_1(t))(x(t) - m(t))^T + x(t)((a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + b(t)dW_1(t) + (- a_0(t) -a_1(t)m(t)-a_2(t)p(t) - a_2(t)m^2(t) + 3a_3(t)p(t)*m(t) - a_3(t)m^3(t))dt +K d\upsilon)^T / Y(t)],$$

where

$$K(t) = P^T(t)A^T(t)(B(t)B^T(t))^{-1}$$

and $\upsilon(t)$ is the innovations process,

$$d\upsilon(t) = dy(t) - (A_0(t) + A_1(t)m(t))dt.$$

The latter equation can be transformed into

$$dP(t) = (a_0(t)E[(x(t) - m(t))^T / Y(t)] + a_1(t)E[(x(t)(x(t) - m(t))^T / Y(t)] + a_2(t)E[x^2(t)(x(t) - m(t))^T / Y(t)] + a_3(t)E[(x^3(t)(x(t) -m(t))^T / Y(t)] + E[x(t)(a_1(t)x (t))^T / Y(t)] + E[x(t)(a_2(t)x^2(t))^T / Y(t)] + E[x (t)a_3(t)x^3(t))^T / Y(t)] - E[x(t)(a_1(t)m(t))^T/ Y(t)] - E[x(t)(a_2(t)p(t))^T / Y(t)] - E[x(t)(a_2(t)m^2(t))^T / Y(t)\} - E[x(t)(a_3(t)p(t)*m(t))^T / Y(t)] - E(x(t)(a_3(t)m^3(t))^T / Y(t)) + b(t)b^T(t) - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt.$$

Finally, upon substituting (5)-(7) and making the corresponding algebraic transformations, the variance equation takes the form

(10)
$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)*P(t) + 2(P(t)*m^T(t))a_2^T(t) +$$
$$3a_3(t)(p(t)*P(t)) + 3(p(t)*P(t))^T a_3^T(t) + 3a_3(t)(m^2(t)*P(t)) +$$
$$3(P(t)*(m^2(t))^T)a_3^T(t) + b(t)b^T(t) - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt,$$
$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T / y(t_0)),$$

where the product $m(t)*P(t)$ between a vector $m(t)$ and a matrix $P(t)$ is defined as the matrix whose rows are equal to rows of $P(t)$ multiplied by the same corresponding element of $m(t)$:

$$\begin{array}{l} m_1(t) \\ m_2(t) \\ \cdot \\ \cdot \\ m_n(t) \end{array} \quad \begin{array}{llll} P_{11}(t) & P_{12}(t) & \ldots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \ldots & P_{2n}(t) \\ \cdot & \cdot & \ldots & \\ \cdot & \cdot & \ldots & \\ P_{n1}(t) & P_{n2}(t) & \ldots & P_{nn}(t) \end{array} \qquad \begin{array}{llll} m_1(t)P_{11}(t) & m_1(t)P_{12}(t) & \ldots & m_1(t)P_{1n}(t) \\ m_2(t)P_{21}(t) & m_2(t)P_{22}(t) & \ldots & m_2(t)P_{2n}(t) \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ m_n(t)P_{n1}(t) & m_n(t)P_{n2}(t) & \ldots & m_n(t)P_{nn}(t) \end{array}$$

The transposed matrix $P(t)*m^T(t) = (m(t)*P(t))^T$ is defined as the matrix whose columns are equal to columns of $P(t)$ multiplied by the same corresponding element of $m(t)$:

$$\begin{array}{llll} [m1(t) & m2(t) & \ldots & mn(t)] \\ P_{11}(t) & P_{12}(t) & \ldots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \ldots & P_{2n}(t) \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ P_{n1}(t) & P_{n2}(t) & \ldots & P_{nn}(t) \end{array} \qquad \begin{array}{llll} & & & \\ m_1(t)P_{11}(t) & m_2(t)P_{12}(t) & \ldots & m_n(t)P_{1n}(t) \\ m_1(t)P_{21}(t) & m_2(t)P_{22}(t) & \ldots & m_n(t)P_{2n}(t) \\ \cdot & \cdot & \ldots & \\ \cdot & \cdot & \ldots & \\ m_1(t)P_{n1}(t) & m_2(t)P_{n2}(t) & \ldots & m_n(t)P_{nn}(t) \end{array}$$

Thus, the equation (9) for the optimal estimate $m(t)$ and the equation (10) for its covariance matrix $P(t)$ form a closed system of filtering equations in the case of a polynomial state equation of degree 3 and linear observations given by the equations (4) and (2), respectively.

## 5. Optimal filter for polynomial state equation of degree 4 and linear observations

Generalizing the result of the previous section, the outlined procedure is now applied to deriving the optimal filter for the case of a polynomial state equation of degree 4, obtained from (1) if $f(x) = a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t) + a_4(t)x^4(t)$:

(11)
$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t) + a_4(t)x^4(t))dt + b(t)dW_1(t),$$
$$x(t_0) = x_0,$$

and the linear observations (2), where $x \in R^n$, $x^4(t) = (x_1^4(t), x_2^4(t), x_3^4(t), \ldots, x_n^4(t))$.

Following the scheme of the previous section and substituting the expressions (5)-(8) for the conditional initial moments of $x(t)$ in the Kushner equation (3), the following equation is obtained for the optimal estimate in the case of a polynomial state equation of degree 4 and linear observations

(12)
$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + 3a_3(t)p(t)*m(t) +$$
$$a_3(t)m^3(t) + 3a_4(t)\,p^2(t) + 6a_4(t)p(t)*m^2(t) + a_4(t)m^4(t))dt +$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt),$$
$$m(t_0) = E[x(t_0) / y(t_0)].$$

Correspondingly, the variance equation takes the form

(13)
$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)(m(t)*P(t)) + 2(P(t)*m^T(t))a_2^T(t) +$$
$$3a_3(t)(p(t)*P(t)) + 3(p(t)*P(t))^T a_3^T(t) + 3a_3(t)(m^2(t)*P(t)) +$$

$3(P(t)*(m^2(t))^T)a_3^T(t) + 12a_4(t)((m(t)*p(t))*P(t)) +$
$12(P(t)*(m(t)*p(t))^T)(a_4(t))^T + 4a_4(t)(m^3(t)*P(t)) +$
$4(P(t)*(m^3(t))^T)(a_4(t))^T + b(t)b^T(t) - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt,$
$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T / y(t_0)).$

*Remark.* If we continue obtaining the filters for polynomial state equations of degrees 5, 6, etc., the corresponding equations for the estimate m(t) and the variance P(t) would contain the terms of the preceding lower degrees, complemented with new terms. In other words, the filtering equations for the quadratic state contain all terms of the linear filtering equations, plus new quadratic terms, the filtering equations for the cubic state contain all terms of the linear and quadratic filtering equations, plus new cubic terms, and so on.

# 6. Optimal control for polynomial state degree 3 with linear control input

Consider the polynomial system

(14) $dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + G(t)u(t)dt, \; x(t_0) = x_0,$

where $x(t) \in R^n$ is the system state, $x^2(t) = (x_1^2(t), x_2^2(t), x_3^2(t), \ldots, x_n^2(t))$, $x^3(t) = (x_1^3(t), x_2^3(t), x_3^3(t), \ldots, x_n^3(t))$, and u(t) is the control variable. The quadratic cost function to be minimized is defined as follows:

(15) $J = \frac{1}{2}[x(T)-x_1]^T\Psi[x(T) - x_1] + \frac{1}{2}\int_{t0}^{T} u^T(s)R(s)u(s)\,ds +$
$\frac{1}{2}\int_{t0}^{T} x^T(s)L(s)x(s)\,ds,$

where $x_1$ is a given vector, $\Psi$, R, L are positive (nonnegative) definite symmetric matrices, and $T > t_0$ is a certain time moment.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0,T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (14). To find the solution to this optimal control problem, the duality principle [7] could be used. For linear systems, if the optimal control exists in the optimal control problem for a linear system with the quadratic cost function J, the optimal filter exists for the dual linear system with Gaussian disturbances and can be found from the optimal control problem solution, using simple algebraic transformations (duality between the gain matrices and the matrix and variance equations), and vice versa. Taking into account the physical duality of the filtering and control problems, the last conjecture should be valid for all cases where the optimal control (or, vice versa, the optimal filter) exists in a closed finite-dimensional form. This proposition is now applied to a third order polynomial system, for which the optimal filter has already been obtained (see Section 4).

Let us return to the optimal control problem for the polynomial state (14) with linear control input and the cost function (15). This problem is dual to the filtering problem for the polynomial state (4) of degree 3 and the linear observations (2). Since the optimal polynomial filter gain matrix in (9) is equal to

$$K_f = P(t)A^T(t)(B(t)B^T(t))^{-1},$$

the gain matrix in the optimal control problem takes the form of its dual transpose

$$K_c = (R(t))^{-1}G^T(t)Q(t),$$

and the optimal control law is given by

(16) $u^*(t) = K_c x = (R(t))^{-1}G^T(t)Q(t)x(t),$

where the matrix function is the solution of the following equation dual to the variance equation (10)

(17) $dQ(t) = (-a_1^T(t)Q(t) - Q(t)a_1(t) - 2a_2^T(t)Q(t)*x^T(t) - 2x(t)*Q(t)a_2(t) - 3a_3^T(t)Q(t)*q^T(t) - 3q(t)*Q(t)a_3(t) - 3a_3^T(t)Q(t)*((x^2)^T(t)) - 3(x^2(t)*Q(t))a_3(t) + L(t) - Q(t)G(t)R^{-1}(t)G^T(t)Q(t))dt,$

with the terminal condition $Q(T)=\Psi$. The binary operation * has been introduced in Section 4, and $q(t) = (q_1(t), q_2(t), \ldots, q_n(t))$ denotes the vector consisting of the diagonal elements of $Q(t)$.

Upon substituting the optimal control (16) into the state equation (14), the optimally controlled state equation is obtained

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + G(t)(R(t))^{-1}G^T(t)Q(t)x(t)dt, \; x(t_0) = x_0,$$

Note that if the real state vector x(t) is unknown (unobservable), the optimal controller uniting the obtained optimal filter and regulator equations, can be constructed using the separation principle [7] for polynomial systems, which should also be valid if solutions of the optimal filtering and control problems exist in a closed finite-dimensional form.

The results obtained in this section by virtue of the duality principle could be rigorously verified through the general equations of the Pontryagin maximum principle [11] or Bellman dynamic programming [2].

# 7. Application of optimal polynomial filter to automotive system

This section presents application of the obtained filter for a polynomial state of degree 3 over linear observations to estimation of the state variables, orientation and steering angles, in a nonlinear kinematical model of car movement [9] satisfying the following equations:

(18)
$$dx(t) = \upsilon \cos\varphi(t)\, dt,$$
$$dy(t) = \upsilon \sin\varphi(t)\, dt,$$
$$d\varphi(t) = (\upsilon/l) \tan\delta(t)\, dt,$$
$$d\delta(t) = u(t)\, dt.$$

Here, x(t) and y(t) are Cartesian coordinates of the mass center of the car, $\varphi(t)$ is the orientation angle, $\upsilon$ is the velocity, l is the longitude between the two axes of the car, $\delta(t)$ is the steering wheel angle, and u(t) is the control variable (steering angular velocity). The zero initial conditions for all variables are assumed.

The problem is to find the optimal estimates for the variables $\varphi(t)$ and $\delta(t)$, using direct linear observations confused with independent and identically distributed disturbances modeled as white Gaussian noises. The corresponding observation equations are

(19)
$$dz_\varphi(t) = \varphi(t)dt + \phi_1(t)dt,$$
$$dz_\delta(t) = \delta(t)dt + \phi_2(t)dt,$$

where $z_\varphi(t)$ is the observation variable for $\varphi(t)$, $z_\delta(t)$ is the observation variable for $\delta(t)$, and $\phi_1(t)$ and $\phi_2(t)$ are white Gaussian noises independent of each other.

To apply the obtained filtering algorithms to the nonlinear system (18) and linear observations (19), let us make the Taylor expansion of the two last equations in (18) at the origin up to degree 3 (the fourth degree does not appear in the Taylor series for tangent)

(20)
$$d\varphi(t) = ((\upsilon/l)\delta(t) + (\upsilon/l)(\delta^3(t))/3)dt$$
$$d\delta(t) = u(t)dt$$

The filtering equations (9) and (10) for the third degree polynomial state (20) over linear observations (19) take the form

(21) $$dm_\varphi = ((\upsilon/l)m_\delta + (\upsilon/3l)(3p_\delta + m^3{}_\delta) + p_{\varphi\varphi}(z_\varphi - m_\varphi) + p_{\varphi\delta}(z_\delta - m_\delta))dt,$$
$$dm_\delta = (u(t) + p_{\delta\varphi}(z_\varphi - m_\varphi) + p_{\delta\delta}(z_\delta - m_\delta))dt,$$
$$dp_{\varphi\varphi} = ((2\upsilon/l)p_{\delta\varphi}p_{\delta\delta} + (2\upsilon/l)p_{\delta\varphi} + (2\upsilon/l)m^2{}_\varphi p_{\delta\varphi} - p^2{}_{\varphi\varphi} - p^2{}_{\varphi\delta})dt,$$
$$dp_{\varphi\delta} = ((\upsilon/l)p_{\delta\delta} + (\upsilon/l)m^2{}_\delta p_{\delta\delta} - p_{\varphi\varphi}p_{\varphi\delta} - p_{\varphi\delta}p_{\delta\delta})dt,$$
$$dp_{\delta\delta} = (-p^2{}_{\delta\varphi} - p^2{}_{\delta\delta})dt,$$

where $m_\varphi$ and $m_\delta$ are the estimates for variables $\varphi$ and $\delta$, and $p_{\varphi\varphi}$, $p_{\varphi\delta}$, $p_{\delta\delta}$ are elements of the symmetric covariance matrix P.

The estimates obtained upon solving the equations (21) are compared to the conventional Kalman-Bucy estimates satisfying the following Kalman-Bucy filtering equations for a state of the linearized system (18) (only the linear term is present in the Taylor expansion for tangent) over linear observations (19)

(22) $$dm_\varphi = ((\upsilon/l)m_\delta + p_{\varphi\varphi}(z_\varphi - m_\varphi) + p_{\varphi\delta}(z_\delta - m_\delta))dt,$$
$$dm_\delta = (u(t) + p_{\delta\varphi}(z_\varphi - m_\varphi) + p_{\delta\delta}(z_\delta - m_\delta))dt,$$
$$dp_{\varphi\varphi} = ((2\upsilon/l)p_{\delta\varphi} - p^2{}_{\varphi\varphi} - p^2{}_{\varphi\delta})dt,$$
$$dp_{\varphi\delta} = ((\upsilon/l)p_{\delta\delta} - p_{\varphi\varphi}p_{\varphi\delta} - p_{\varphi\delta}p_{\delta\delta})dt,$$
$$dp_{\delta\delta} = (-p^2{}_{\delta\varphi} - p^2{}_{\delta\delta})dt.$$

Numerical simulation results are obtained solving the systems of filtering equations (21) and (22). The obtained values of the estimates $m_\varphi$ and $m_\delta$ are compared, in both cases, to the real values of the variables $\varphi$ and $\delta$ in the original system (18) and its polynomial approximation (20).

Thus, two sets of graphs are obtained.

1) Graphs of variables $\varphi$ and $\delta$ for the polynomial approximation system (20); graphs of the Kalman-Bucy filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (22); and graphs of the optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (21) (Figs. 1 and 2).
2) Graphs of variables $\varphi$ and $\delta$ for the original system (18); graphs of the Kalman-Bucy filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (22); and graphs of the optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (21) (Figs. 3 and 4).

For each of the four filters and two reference systems involved in simulation, the following values of the input variables and initial values are assigned: $\upsilon = 1$, $l = 1$, $u(t) = 0.05$, $m_\varphi(0) = 10$, $m_\delta(0) = 0.1$, $\varphi(0) = \delta(0) = 0$, $P_{\varphi\varphi}(0) = 100$, $P_{\varphi\delta}(0) = 10$, $P_{\delta\delta}(0) = 1$. Gaussian disturbances $\phi_1(t)$ and $\phi_2(t)$ in (21) are realized as sinusoidal signals: $\phi_1(t) = \phi_2(t) = \sin t$.

The obtained values of the reference variables $\varphi$ and $\delta$ satisfying the polynomial approximation system (20) are compared to the Kalman-Bucy filter and optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ at the terminal time T=20 in the following table (corresponding to Figs. 1 and 2).

| Kalman-Bucy filter | Third degree polynomial filter |
|---|---|
| $\varphi(20) = 5$ | $\varphi(20) = 5$ |
| $\delta(20) = 1$ | $\delta(20) = 1$ |
| $m_\varphi(20) = 3.35$ | $m_\varphi(20) = 5.2$ |
| $m_\delta(20) = 0.48$ | $m_\delta(20) = 0.73$ |

The obtained values of the reference variables $\varphi$ and $\delta$ satisfying the original system (18) are compared to the Kalman-Bucy filter and optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ at the terminal time T=20 in the following table (corresponding to Figs. 3 and 4)

| Kalman-Bucy filter | Third degree polynomial filter |
|---|---|
| $\varphi(20) = 12.3$ | $\varphi(20) = 12.3$ |
| $\delta(20) = 1$ | $\delta(20) = 1$ |
| $m_\varphi(20) = 7.35$ | $m_\varphi(20) = 11.83$ |
| $m_\delta(20) = 0.61$ | $m_\delta(20) = 0.905$ |

The simulation results show that the estimates obtained by using the optimal third degree polynomial filter are closer to the real values of the reference variables than those obtained by using the conventional Kalman-Bucy linear filter, especially for the original nonlinear system (18). Although this conclusion follows from the developed theory, the numerical comparison serves as a convincing illustration.

## 8. Application of optimal polynomial regulator to automotive system

This section presents application of the obtained optimal regulator for a polynomial system of degree 3 with linear control input and quadratic criterion to controlling the state variables, orientation and steering angles, in the nonlinear kinematical model of car movement [9] given by the nonlinear equations (18). The optimal control problem is to maximize the orientation angle $\varphi$ using the minimum energy of control u.

The corresponding criterion J to be minimized takes the form

$$J = ½ [\varphi(T) - \varphi^*]^2 + ½ \int_0^T u^2(s)\, ds,$$

where T = 0.3, and $\varphi^* = 1$ is a large value of $\varphi(t)$ unreachable for time T. Since R = 1 and $G^T = [0\ 1]$, the optimal control law (16) $u^*(t) = (R(t))^{-1}G^T(t)Q(t)x(t)$ takes the form $u^*(t) = q_{21}(t)\varphi(t) + q_{22}(t)\delta(t)$, where the elements $q_{11}(t)$, $q_{21}(t)$, $q_{22}(t)$ of the symmetric matrix Q(t) satisfy the equations

(23) $$dq_{11}(t) = -(q_{21}(t))^2\, dt,$$
$$dq_{12}(t) = (-3(\upsilon/l)\,(q_{11}(t))^2 - q_{11}(t)q_{12}(t) - (\upsilon/l)q_{11}(t) - 3(\upsilon/l)\varphi^2(t)q_{11}(t))dt,$$
$$dq_{22}(t) = (-2(\upsilon/l)q_{12}(t) - 2(\upsilon/l)\, q_{12}(t)q_{22}(t) - 2(\upsilon/l)\delta^2(t)q_{12}(t) - (q_{22}(t))^2)dt.$$

The system composed of the two last equations of (18) and the equations (20) should be solved with initial conditions $\varphi(0) = 0.1$, $\delta(0) = 0.1$ and terminal conditions $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$. This boundary problem is solved numerically using the iterative method of direct and reverse passing as follows. The first initial conditions for q's are guessed, and the system is solved in direct time with the initial conditions at t = 0, thus obtaining certain values for $\varphi$ and $\delta$ at the terminal point T = 0.3. Then, the system is solved in reverse time, taking the obtained terminal values for $\varphi$ and $\delta$ in direct time as the initial values in reverse time, thus obtaining certain values for q's at the initial point t = 0, which are taken as the initial values for the passing in direct time, and so on. The given initial conditions $\varphi(0) = 0.1$, $\delta(0) = 0.1$ are kept fixed for any direct passing, and the given terminal conditions $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ are used as the fixed initial conditions for any reverse passing. The algorithm stops when the system arrives at values $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ after direct passing and at values $\varphi(0) = 0.1$, $\delta(0) = 0.1$ after reverse passing. The obtained simulation graphs for $\varphi$, $\delta$, and the criterion J are given in Fig. 5. These results for polynomial regulator of degree 3 are then compared to the results obtained using the optimal linear regulator, whose matrix Q(t) elements satisfy the Riccati equations

(24) $$dq_{11}(t) = -(q_{21}(t))^2\, dt$$
$$dq_{12}(t) = (-q_{11}(t)q_{12}(t) - (\upsilon/l)q_{11}(t))\, dt,$$
$$dq_{22}(t) = (-2(\upsilon/l)q_{12}(t) - (q_{22}(t))^2)\, dt,$$

with terminal conditions $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$. Note that in the linear case the only reverse passing for $q$'s is necessary, because the system (24) does not depend on $\varphi$ and $\delta$, and the initial values for q's at t = 0 are obtained after single reverse passing. The simulation graphs for the linear case are given in Fig. 6.

Thus, two sets of graphs are obtained.

1) Graphs of variables $\varphi$ and $\delta$ satisfying the original system (18) and controlled using the optimal linear regulator defined by (24); graphs of the corresponding values of the criterion J (Fig. 5).
2) Graphs of variables $\varphi$ and $\delta$ satisfying the original system (18) and controlled using the optimal third order polynomial regulator defined by (23); graphs of the corresponding values of the criterion J (Fig. 6).

The obtained values of the controlled variables $\varphi$ and $\delta$ and the criterion J are compared for the optimal third order polynomial and linear regulators at the terminal time T = 0.3 in the following table (corresponding to Figs. 5 and 6).

| Linear regulator | Third degree polynomial regulator |
|---|---|
| $\varphi(0.3) = 0.132$ | $\varphi(0.3) = 0.138$ |
| $\delta(0.3) = 0.1045$ | $\delta(0.3) = 0.1278$ |
| $J(0.3) = 0.759$ | $J(0.3) = 0.75$ |

The simulation results show that the values of the controlled variables $\varphi$ and $\delta$ at the terminal point T = 0.3 are greater for the third order regulator than for the linear one (although only the variable $\varphi$ is maximized) and the criterion value at the terminal point is less for the third order regulator also. Thus, the third order polynomial regulator controls the system variables better than the linear one from both points of view, thus illustrating, as well as for the filtering problem, the theoretical conclusion.

# 9. Conclusions

The optimal nonlinear filter for a stochastic system state given by a polynomial equation of degree 3 or 4 and linear observations confused with white Gaussian noises has been obtained. The optimal polynomial filter of degree 3 has been then applied to solution of the estimation problem for state variables, orientation and steering angles, of a nonlinear automotive system describing kinematics of car movement. The estimates obtained by using the optimal third degree polynomial filter have been compared to the conventional Kalman-Bucy linear filter estimates. The numerical simulation results have demonstrated that the values of the polynomial filter estimates are closer to the real values of reference variables than the Kalman-Bucy ones, showing that the polynomial filter yields better estimation results for nonlinear systems. Using the duality principle, the optimal regulator has been designed for a polynomial system of degree 3 with linear control input and quadratic cost criterion. Application of the obtained regulator to the nonlinear automotive system have yielded lesser values of the criterion and greater values of the controlled variables in comparison to the optimal linear regulator. In both cases, the numerical simulation results confirm the theoretical conclusions.

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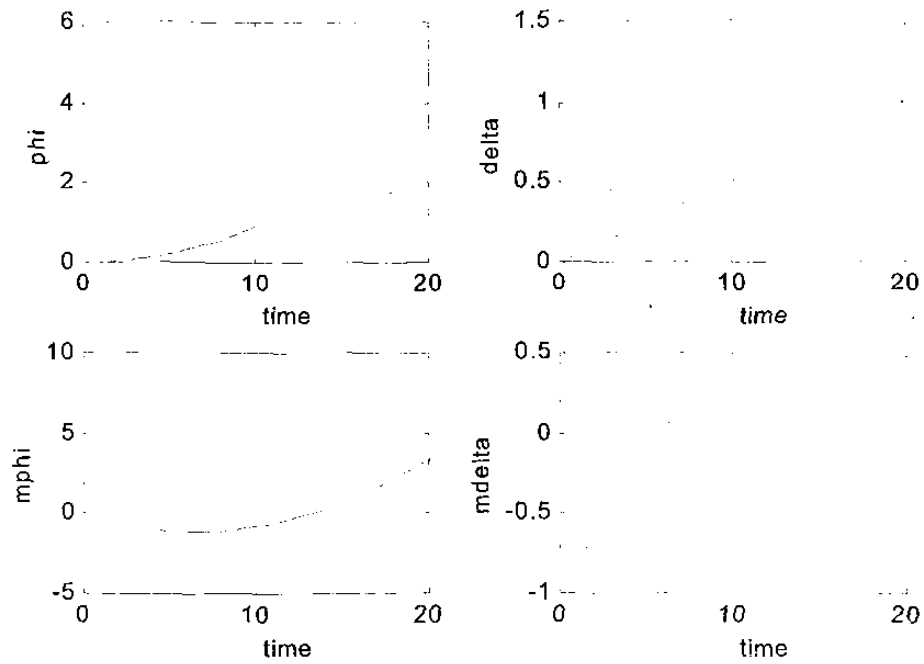


**Fig. 1.** Graphs of variables φ and δ for the polynomial approximation system *(20)*; Graphs of the Kalman-Bucy filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (22).

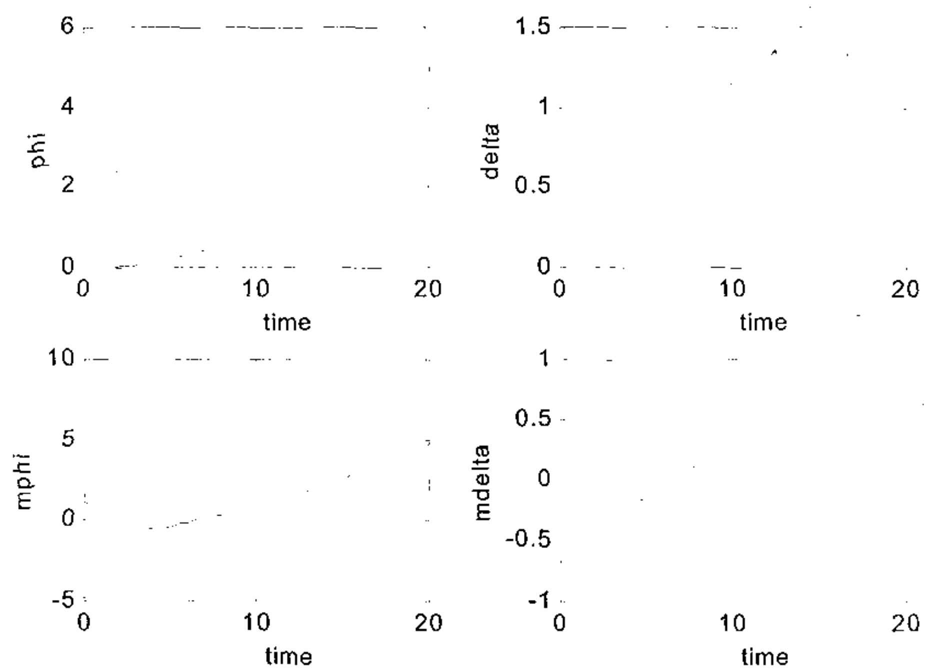


**Fig. 2.** Graphs of variables φ and δ for the polynomial approximation system (20); Graphs of the optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (21).

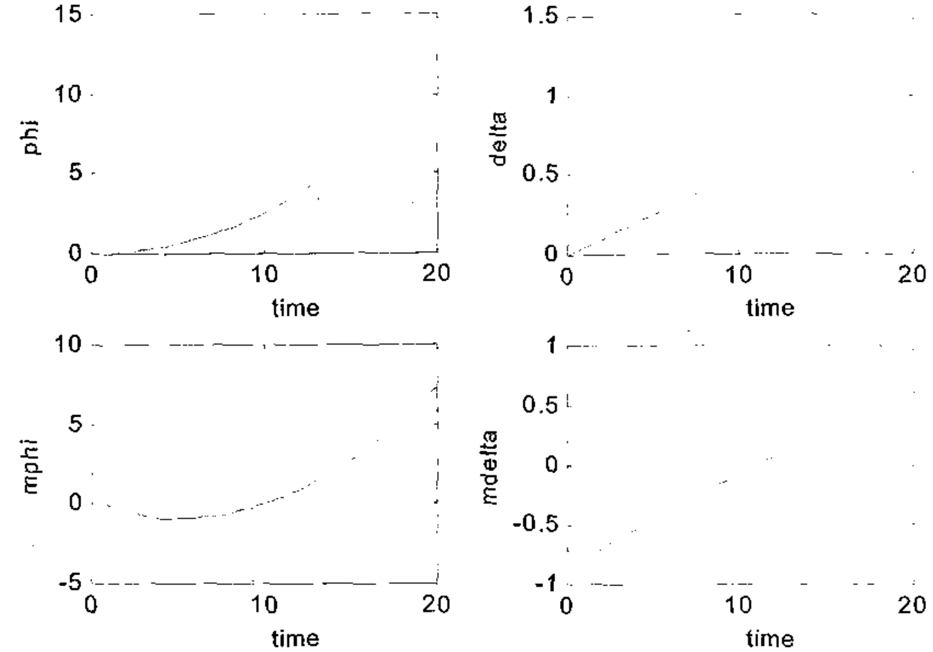


**Fig. 3.** Graphs of variables φ and δ for the original system (*18*); *Graphs of the* Kalman-Bucy filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (22).

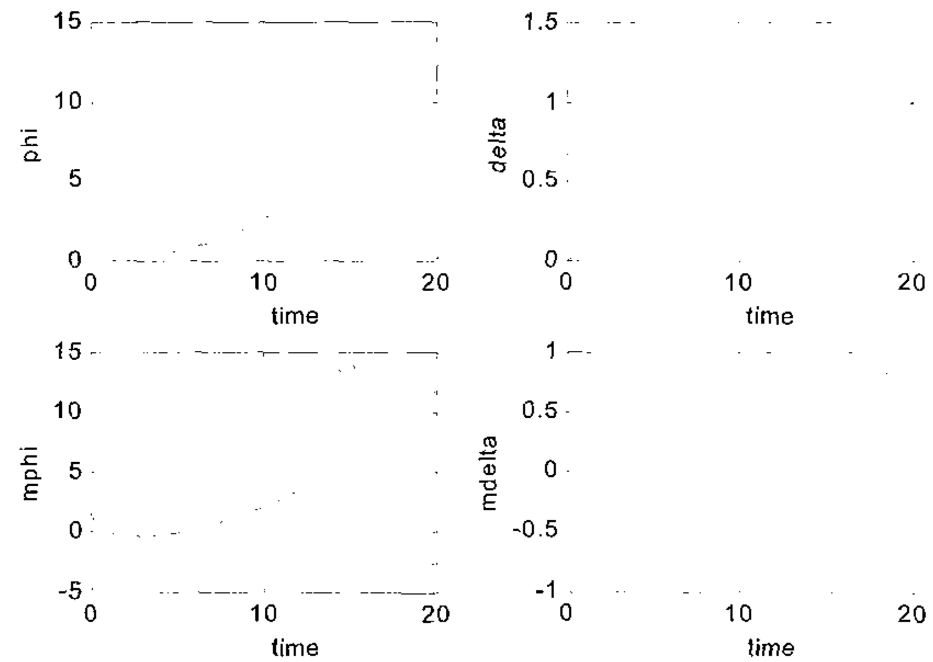


**Fig. 4.** *Graphs of variables* φ *and* δ for the original system (18); Graphs of the optimal third degree polynomial filter estimates $m_\varphi$ and $m_\delta$ satisfying the equations (21).

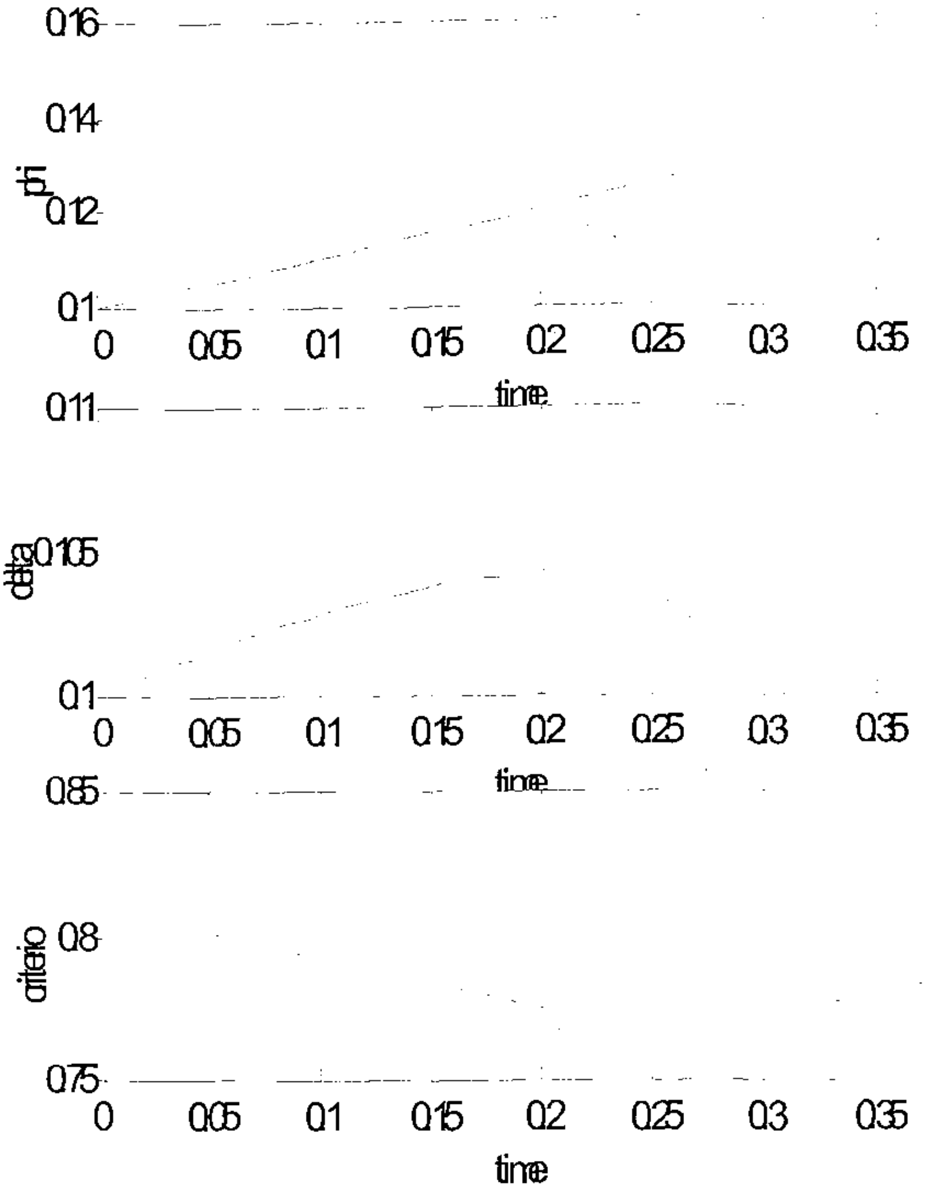


**Fig. 5.** Graphs of variables φ and δ satisfying the original system (18) and controlled using the optimal linear regulator defined by (24); graphs of the corresponding values of the criterion J.

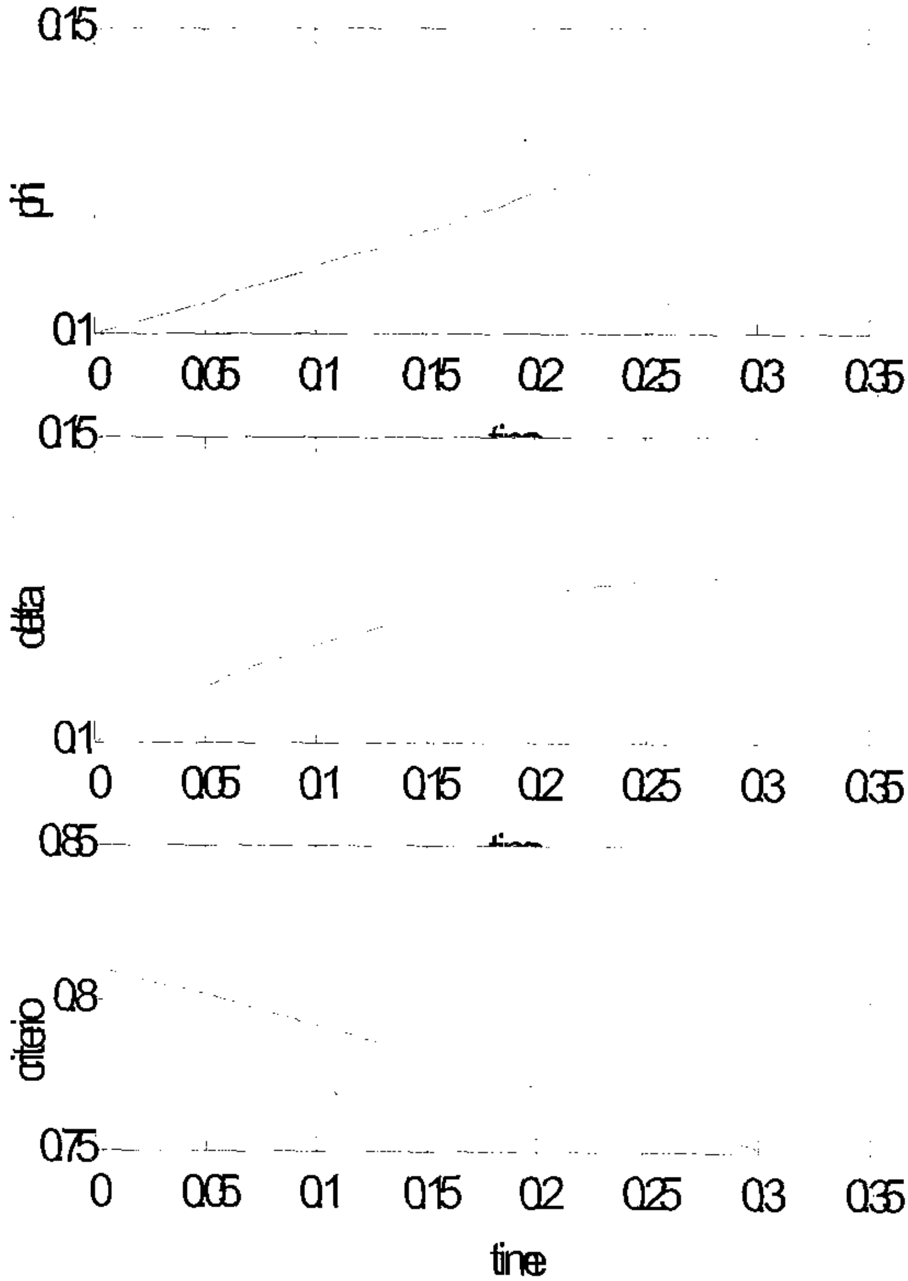


**Fig. 6.** Graphs of variables φ and δ satisfying the original system (18) and controlled using the optimal third order polynomial regulator defined by (23); graphs of the corresponding values of the criterion J.

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# Optimal Filtering for Bilinear System States and Its Application to Polymerization Process Identification

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**Abstract.** The paper presents the optimal nonlinear filter for bilinear state and linear observation equations confused with white Gaussian disturbances. The general scheme for obtaining the optimal filter in case of polynomial state and linear observation equations is announced. The obtained bilinear filter is applied to solution of the identification problem for the bilinear terpolymerization process and compared to the optimal linear filter available for the linearized model and to the mixed filter designed as a combination of those filters.

## 1 Introduction

It is virtually the common opinion that the optimal nonlinear finite-dimensional filter exists and can be obtained in a closed form only in the case of linear state and observation equations. This famous construction is called the linear Kalman-Bucy filter [1], referring to the scientists who derived it in 1960s. However, it is much less known that the optimal nonlinear finite-dimensional filter can be obtained in many other cases, if, for example, the state vector can take only a finite number of admissible states [2] or if the observation equation is linear and the drift term in the state equation satisfies the Riccati equation $\frac{df}{dx} + f^2 = x^2$ (see [3]). Moreover, the complete classification of the "general situation" cases (this means that there are no special assumptions on the structure of state and observation equations) when the optimal nonlinear finite-dimensional filter exists is given in [4].

This paper would like to attract attention to relatively simple (but important in practical applications, see [5]) cases when the optimal nonlinear finite-dimensional filter can be obtained in a closed form. Indeed, if the observation equation is linear and the observation matrix is invertible, then, as shown below in the paper, it is possible to obtain the optimal finite-dimensional filter for a polynomial state equation, provided that the system coefficients depend on time only. In the case of a bilinear state equation, the corresponding filtering equations are derived in the paper directly. The possibility to derive similar results for an arbitrary polynomial state equation is underlined.

The paper is organized as follows. Section 2 briefly reminds the linear Kalman-Bucy filter for reference purposes, considers the case of nonlinear state and linear observation equations, establishes the procedure to obtain a closed system of the filtering equations for polynomial state and linear observation equations, and gives the optimal filter for bilinear system states and linear observations in the explicit form. In Section 3, the obtained bilinear filter is applied to solution of the identification problem for the bilinear terpolymerization process and compared to the optimal linear filter available for the linearized model and to the mixed filter designed as a combination of those filters. The simulation results show an advantage of the optimal bilinear filter in comparison to the other filters.

# 2 Optimal filtering for polynomial state equation

## 2.1 Linear Kalman-Bucy filter

It is well known that the linear optimal filter [1] can be designed in a closed form, if the state and observation equations of a dynamic system are linear. Let an unobservable random process $x(t)$ satisfy a linear equation

$$dx(t) = (a_0(t) + a(t)x(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{1}$$

and linear observations are given by

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t). \tag{2}$$

Here, $W_1(t)$ and $dW_2(t)$ are Wiener processes, whose weak derivatives are Gaussian noises and which are assumed independent of each other and of the initial value $x_0$. The last equation can also be written in the algebraic form:

$$\dot{y}(t) = A_0(t) + A(t)x(t) + B(t)\psi(t). \tag{3}$$

where $\psi(t)$ is a white Gaussian noise (a weak derivative of $W_2(t)$).

The estimation problem is to find the best estimate for the real process $x(t)$ at time $t$ based on the observations $Y(t) = \{y(s), t_0 \leq s \leq t\}$, that is the conditional expectation $m(t) = E(x(t) \mid Y(t))$ of the real process $x(t)$ with respect to the observations $Y(t)$. Let $P(t) = E((x(t)-m(t)(x(t)-m(t))^T \mid Y(t))$ be the estimate variance (correlation function).

The solution to this problem is given by the following system of filtering equations, which is closed with respect to the introduced variables, $m(t)$ and $P(t)$:

$$dm(t) = (a_0(t) + a(t)m(t))dt + P(t)A^T(t)(B(t)B^T(t))^{-1}\times$$

$$[dy(t) - (A_0(t) + A(t)m(t))dt], \tag{4}$$

$$m(t_0) = E(x(t_0) \mid Y(t_0))$$

$$dP(t) = (a(t)P(t) + P(t)a^T(t) + b(t)b^T(t))dt- \tag{5}$$

$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt,$$

$$P(t_0) = E((x(t_0) - m(t_0)(x(t_0) - m(t_0))^T \mid Y(t_0).$$

The advantages of the Kalman-Bucy filter are very well known: the equations are simple, the variance equation is independent of the observations $y(t)$ and can be solved off-line, the estimate equation is linear and the variance one is quadratic of the Riccati type.

## 2.2 Nonlinear filtering equation

In the case of nonlinear state and observation equations, the problem is more complicated. Let an unobservable random process $x(t)$ satisfy a nonlinear equation

$$dx(t) = f(x(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{6}$$

and nonlinear observations are given by

$$dy(t) = h(x(t))dt + B(t)dW_2(t). \tag{7}$$

There exist two principal results related to this case [6]. First, as in the previous linear case, the innovations process $\vartheta(t) = y(t) - \int_{t_0}^{t} E(h(x(s)) \mid Y(s))ds$ is a Wiener process and, second, contains the same new information as the observation process $y(t)$ itself. The first result means that for every fixed $t$, the random variable $\vartheta(t)$ is Gaussian and the second one implies that for every function $\varphi(x)$ depending on the real unobservable vector $x(t)$, the expectations with respect to the observation and innovations processes are the same: $E(\varphi(x(t)) \mid Y(t)) = E(\varphi(x(t)) \mid \{\vartheta(s), t_0 \le s \le t\})$, in particular, if $\varphi(x) = x$, then $m(t) = E(x(t) \mid Y(t)) = E(x(t) \mid \{\vartheta(s), t_0 \le s \le t\})$.

Using these basic properties, it is possible to obtain the equation for the optimal estimate $m(t) = E(x(t) \mid Y(t))$, the so-called nonlinear filtering equation, first derived by Kushner [7], in the form

$$\begin{gathered} dm(t) = E(f(x(t)) \mid Y(t))dt+ \\ [E(h(x(t))x^T(t) \mid Y(t)) - E(h(x(t)) \mid Y(t))m^T(t)]^T \times \\ (B(t)B^T(t))^{-1}[dy(t) - E(h(x(s)) \mid Y(t))dt], \\ m(t_0) = E(x(t_0) \mid Y(t_0)). \end{gathered} \tag{8}$$

However, the computation of $m(t)$ requires computing the functions in the right-hand side of this equation, which, in turn, requires computing the quantities: $E(f(x(t)) \mid Y(t))$, $E(h(x(t))x(t) \mid Y(t))$, and $E(h(x(t)) \mid Y(t))$. Each of them is a nonlinear function of $x$ and, as a consequence, a non-Gaussian random variable. Thus, one has to solve a nonlinear stochastic differential equation for each of these variables, which involves higher moments of these variables in its right-hand side. Hence, an infinite-dimensional system of nonlinear stochastic equations should be obtained as the optimal filter. In other words, the optimal filter cannot be obtained in a closed form, i.e., with respect to a finite number of filtering variables (there are two, $m(t)$ and $P(t)$, in the linear Kalman-Bucy filter), or one can say that the optimal finite-dimensional filter does not exist. Actually, there are only a few number of examples where the optimal finite-dimensional filter exists for a nonlinear model of state and observation processes [2–4] in the "general situation."

## 2.3 Polynomial state and linear observation equations

Nonetheless, it should be possible to obtain the optimal finite-dimensional filter in a closed form in the following case. Let a unobserved random process $x(t)$ satisfy a nonlinear equation

$$dx(t) = f(x(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{9}$$

and linear observations are given by

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t), \tag{10}$$

where the function $f(x(t)) = a_0(t) + a_1(t)x + a_2(t)x^2 + ...$ is a polynomial and the observation matrix $A(t)$ is invertible, i.e., the inverse matrix $A^{-1}(t)$ exists.

Since the observation equation is linear, the first result of nonlinear filtering implies that the innovations process $\vartheta(t) = y(t) - \int_{t_0}^{t}(A_0(s) + A(s)m(s))ds = \int_{t_0}^{t}(A_0(s) + A(s)x(s))ds + \int_{t_0}^{t} B(s)dW_2(s) - \int_{t_0}^{t}(A_0(s) + A(s)m(s))ds = \int_{t_0}^{t} A(s)(x(s) - m(s))ds + \int_{t_0}^{t} B(s)dW_2(s)$ is a Wiener process, and, since $\int_{t_0}^{t} B(s)dW_2(s)$ is also a Wiener process, the random variable $A(t)(x(t) - m(t))$ is Gaussian for every fixed $t$. If the inverse matrix $A^{-1}(t)$ exists, then the random vector $(x(t) - m(t))$ is also Gaussian [8].

Moreover, in this case, the second term in the nonlinear filtering equation is equal to

$$[E(h(x(t))x^T(t) \mid Y(t)) - E(h(x(t)) \mid Y(t))m^T(t)]^T\times$$
$$(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt] =$$
$$[E(x(t)x^T(t)A^T \mid Y(t)) - m(t)E(x^T(t)A^T(t) \mid Y(t))]\times$$
$$(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt] =$$
$$[E(x(t)x^T(t) \mid Y(t))A^T(t) - m(t)E(x^T(t) \mid Y(t))A^T(t)]\times$$
$$(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt] =$$
$$[E(x(t)x^T(t) \mid Y(t)) - m(t)m^T(t)]A^T(t)\times$$
$$(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt] =$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt].$$

Hence, the nonlinear filtering equation for the optimal estimate $m(t)$ takes the form:

$$dm(t) = E(f(x(t)) \mid Y(t))dt+$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt],$$
$$m(t_0) = E(x(t_0) \mid Y(t_0)). \tag{11}$$

Let us note now that if the function $f(x(t)) = a_0(t) + a_1(t)x + a_2(t)x^2 + ...$ is a polynomial, it should be possible to compute a finite-dimensional filter in a closed form for variables $m(t)$ and $P(t)$, using the fact that the random variable $(x(t) - m(t))$ is Gaussian. Since all the system coefficients in (9), (10) do not depend on state $x(t)$ and observations $y(t)$, the conditional moments of $(x(t) - m(t))$ with respect to observations $y(t)$ coincide with the unconditional ones. This implies that all odd central conditional moments of this Gaussian variable $\mu_1 = E((x(t) - m(t)) \mid Y(t))$, $\mu_3 = E((x(t) - m(t))^3 \mid Y(t))$, $\mu_5 = E((x(t) - m(t))^5 \mid Y(t)), ...$ are equal to 0, and all even central conditional moments $\mu_2 = E((x(t) - m(t))^2 \mid Y(t))$, $\mu_4 = E((x(t) - m(t))^4 \mid Y(t))$, $\mu_6 = E((x(t) - m(t))^6 \mid Y(t)), ...$ can be represented as functions of the variance $P(t)$. For example, $\mu_2 = P$, $\mu_4 = 3P^2$, $\mu_6 = 15P^3, ....$ Thus, all higher moments of $(x(t) - m(t))$ can be expressed using $P(t)$, and this yields additional relations for representing every higher initial moment of $x(t)$ and, finally, the possibility to obtain the optimal filter in a closed form, i.e., the optimal finite-dimensional filter should exist in the polynomial-linear case.

For example, if the function

$$f(x) = a_0(t) + a_1(t)x + a_2(t)xx^T \tag{12}$$

is a bilinear polynomial, where $x$ is now an $n$-dimensional vector, $a_1$ is an $n \times n$ - matrix, and $a_2$ is a 3D tensor of dimension $n \times n \times n$, the system of filtering equations is as follows

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t))dt +$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt],$$
$$m(t_0) = E(x(t_0) \mid Y(t_0)), \tag{13}$$
$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) +$$
$$2a_2(t)m(t)P(t) + 2P(t)m^T(t)a_2^T(t) +$$
$$b(t)b^T(t))dt - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt,$$
$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid Y(t_0)), \tag{14}$$

since the third central moment $\mu_3$ is equal to 0, and the third initial moment of $x(t)$ can be expressed using its second and first moments, i.e., $P(t)$ and $m(t)$. In this bilinear-linear case, the variance equation is also independent of the observations $y(t)$, but has the bilinear terms $m(t)P(t)$ in its right-hand side and depends on $m(t)$, thus making both the equations interconnected. The estimate equation is bilinear with respect to $m$, as expected.

# 3 Application

The obtained optimal filter for bilinear system states and linear observations is applied to solution of the terpolymerization process identification problem in the presence of direct linear observations. The mathematical model of terpolymerization process given by Ogunnaike [5] is reduced to ten equations for the concentrations of input reagents, the zeroth live moments of the product molecular weight distribution (MWD), and its first bulk moments. These equations are intrinsically nonlinear (bilinear), so their linearization leads to large deviations from the real system dynamics, as it could be seen from the simulation results. Of course, the assumption that the MWD moments can be measured in the real time is artificial, since this can be done only with large time delays, however, at this step, the objective is to verify the performance the obtained nonlinear filtering algorithm for a nonlinear system and compare it with other filtering algorithms based on the linearized model. Taking into account delays in some of the observation components would be the subject of subsequent papers.

Let us rewrite the bilinear state equations (9),(12) and the linear observation equations (10) in the component form using index summations

$$dx_k(t)/dt = a_{0k}(t) + \sum_i a_{1ki}(t)x_i(t) + \tag{15}$$

$$\sum_{ij} a_{2kij}(t)x_i(t)x_j(t) + \sum_i b_{ki}(t)\psi_{1i}(t), \quad k = 1, n,$$

$$y_k(t) = \sum_i A_{ki}(t) x_i(t) + \sum_i B_{ki}(t)\psi_{2k}(t),$$

where $\psi_1(t)$ and $\psi_2(t)$ are white Gaussian noises. Then, the filtering equations (13),(14) can be rewritten in the component form as follows:

$$dm_k(t)/dt = (a_{0k}(t) + \tag{16}$$
$$\sum_i a_{1ki}(t) m_i(t) + \sum_{ij} a_{2kij}(t) m_i(t) m_j(t) +$$
$$\sum_{ij} a_{2kij}(t) P_{ij}(t))dt + \sum_{ijlps} P_{kj}(t) A^T_{jl}(t)(B_{lp}(t) B_{ps}(t)))^{-1}[dy_s - \sum_r A_{sr}(t) m_r(t) dt]$$

with

$$m_k(t_0) = E[x_k(t_0) \mid Y(t_0)],$$

$$dP_{ij}(t) = \sum_k a_{1ik}(t) P_{kj}(t) + \sum_j P_{kj}(t) a_{1jk}(t) + \tag{17}$$
$$2\sum_{kl} a_{2ikl}(t) m_l(t) P_{kj} + 2\sum_{kl} a_{2jkl}(t) m_l(t) P_{ki}(t) +$$
$$\sum_k b_{ik}(t) b_{kj}(t) - \sum_{klpsr} P_{ik}(t) A^T_{kl}(t)(B_{lp}(t) B_{ps}(t)))^{-1} A_{sr}(t) P_{rj}(t),$$

with

$$P_{ij}(t_0) = E[(x_i(t_0) - m_i(t_0))(x_j(t_0) - m_j(t_0))^T \mid Y(t_0)].$$

The terpolymerization process model reduced to 10 bilinear equations selected from [5] is given by

$$dC_{m1}/dt = [(1/V) d\Delta_{m1}/dt - ((1/\theta) + K_{L1}C^* + K_{11}\mu^o_P + K_{21}\mu^o_Q + K_{31}\mu^o_R)C_{m1}; \tag{18}$$
$$dC_{m2}/dt = (1/V) d\Delta_{m2}/dt - ((1/\theta) + K_{L2}C^* + K_{12}\mu^o_P + K_{22}\mu^o_Q)C_{m2};$$
$$dC_{m3}/dt = (1/V) d\Delta_{m3}/dt - ((1/\theta) + K_{13}\mu^o_P)C_{m3};$$
$$dC_{m4}/dt = (1/V) d\Delta_{m^*}/dt - ((1/\theta) + K_d + K_{L1}C_{m1} + K_{L2}C_{m2})C^*;$$
$$d\mu^o_P/dt = (-1/\theta - K_{t1})\mu^o_P + K_{L1}C_{m1}C^* - (K_{12}C_{m2} + K_{13}C_{m3})\mu^o_P +$$
$$K_{21}C_{m1}\mu^o_Q + K_{31}C_{m1}\mu^o_R;$$
$$d\mu^o_Q/dt = (-1/\theta)\mu^o_Q + K_{L2}C_{m2}C^* - (K_{21}C_{m1} + K_{t2})\mu^o_Q + K_{12}C_{m2}\mu^o_P;$$
$$d\mu^o_R/dt = (-1/\theta)\mu^o_R - (K_{31}C_{m1} + K_{t3})\mu^o_R + K_{13}C_{m3}\mu^o_P;$$
$$d\lambda^{100}_1/dt = (-1/\theta)\lambda^{100}_1 + K_{L1}C_{m1}C^* + K_{L2}C_{m2}C^* + K_{11}C_{m1}\mu^o_P +$$
$$K_{21}C_{m1}\mu^o_Q + K_{31}C_{m1}\mu^o_R;$$
$$d\lambda^{010}_1/dt = (-1/\theta)\lambda^{010}_1 + K_{L1}C_{m1}C^* + K_{L2}C_{m2}C^* +$$
$$K_{12}C_{m2}\mu^o_P + K_{22}C_{m2}\mu^o_Q;$$
$$d\lambda^{001}_1/dt = (-1/\theta)\lambda^{001}_1 + (K_{L1}C_{m1} + K_{L2}C_{m2})C^* + K_{13}C_{m3}\mu^o_P;$$

Here, the state variables are: $C_{m1}$, $C_{m2}$, and $C_{m3}$ are the reagent (monomer) concentrations, $C^*$ is the active catalyst concentration; $\mu_P^o$, $\mu_Q^o$, and $\mu_R^o$ are the zeroth live moments of the product MWD. and $\lambda_1^{100}$, $\lambda_1^{010}$, and $\lambda_1^{001}$ are its first bulk moments. The reactor volume $V$ and residence time $\theta$, as well as all coefficients $K$'s, are known parameters, and $\Delta_{m1}, \Delta_{m2}, \Delta_{m3}, \Delta_{m^*}$ stand for net molar flows of the reagents and active catalyst into the reactor.

The identification (filtering) problem is to find the optimal estimate for the unobservable states (18) assuming that the direct observations $Y_i$ mixed with Gaussian noises $\psi_2$'s are provided for each of the ten state components $x_i$

$$y_i = x_i + \psi_{2i}.$$

Here, $x_1$ denotes $C_{m1}$, $x_2$ denotes $C_{m2}$, and so on up $x_{10}$. In this situation, the bilinear filtering equations (16) for the vector of the optimal estimates $m(t)$ take the form

$$dm_1(t)/dt = (1/V)d\Delta_{m1}/dt - ((1/\theta) + K_{L1}m_4(t) + K_{11}m_5(t)+ \tag{19}$$

$$K21m_6(t) + K_{31}m_7(t))m_1(t) - K_{L1}P_{14}(t) - K_{11}P_{15}(t)-$$

$$K_{21}P_{16}(t) - K_{31}P_{17}(t) + \sum_j P_{1j}[dy_j/dt - m_j]$$

$$dm_2(t)/dt = (1/V)d\Delta_{m2}/dt - ((1/\theta) + K_{L2}m_4(t) + K_{12}m_5(t)+$$

$$K_{22}m_6(t))m_2(t) - K_{L2}P_{24}(t) - K_{12}P_{25}(t) - K_{22}P_{26}(t)+$$

$$\sum_j P_{2j}[dy_j/dt - m_j]$$

$$dm_3(t)/dt = (1/V)d\Delta_{m3}/dt - ((1/\theta) + K_{13}m_5(t))m_3(t)-$$

$$K_{13}P_{35}(t) + \sum_j P_{3j}[dy_j/dt - m_j]$$

$$dm_4(t)/dt = (1/V)d\Delta_{m^*}/dt - ((1/\theta) + K_d + K_{L1}m_1(t)+$$

$$K_{12}m_2(t))m_4(t) - K_{L1}P_{14}(t) - K_{12}P_{24}(t)+$$

$$\sum_j P_{4j}[dy_j/dt - m_j]$$

$$dm_5(t)/dt = (-1/\theta - K_{t1})m_5(t) + K_{L1}m_4(t)m_1(t)-$$

$$K_{12}m_2(t)m_5(t) + K_{21}m_6(t)m_1(t)+$$

$$K_{31}m_7(t)m_1(t) - K_{13}m_5(t)m_3(t)+$$

$$K_{L1}P_{14}(t) + K_{21}P_{16}(t) + K_{31}P_{17}(t) - K_{12}P_{25}(t)-$$

$$K_{13}P_{35}(t) + \sum_j P_{5j}[dy_j/dt - m_j]$$

$$dm_6(t)/dt = (-1/\theta - K_{t2} - K_{21}m_1(t))m_6(t)+$$

$$K_{L2}m_4(t)m_2(t) + K_{12}m_5(t)m_2(t)$$

$$-K_{21}P_{16}(t) + K_{L2}P_{24}(t) + K_{12}P_{25}(t)+$$

$$\sum_j P_{6j}[dy_j/dt - m_j]$$

$$dm_7(t)/dt = (-1/\theta - K_{t3} - K_{31}m_1(t))m_7(t) + K_{13}m_5(t)m_3(t) -$$
$$K_{31}P_{17}(t) + K_{13}P_{35}(t) + \sum_j P_{7j}[dy_j/dt - m_j]$$
$$dm_8(t)/dt = (-1/\theta)m_8(t) + (K_{L1}m_4(t) + K_{11}m_5(t) +$$
$$K_{21}m_6(t) + K_{31}m_7(t))m_1(t) + K_{L2}m_4(t)m_2(t) +$$
$$K_{L1}P_{14}(t) + K_{11}P_{15}(t) + K_{21}P_{16}(t) + K_{31}P_{17}(t) +$$
$$K_{L2}P_{24}(t) + \sum_j P_{8j}[dy_j/dt - m_j]$$
$$dm_9(t)/dt = (-1/\theta)m_9(t) + K_{L1}m_4(t)m_1(t) + K_{L2}m_4(t)m_2(t) +$$
$$K_{12}m_5(t)m_2(t) + K_{22}m_6(t)m_2(t) + K_{L1}P_{14}(t) +$$
$$K_{L2}P_{24}(t)K_{12}P_{25}(t) + K_{22}P_{26}(t) + \sum_j P_{9j}[dy_j/dt - m_j];$$
$$dm_{10}(t)/dt = (-1/\theta)m_{10}(t) + K_{L1}m_4(t)m_1(t) + K_{L2}m_4(t) \times$$
$$m_2(t) + K_{13}m_5(t)m_3(t) + K_{L1}P_{14}(t) + K_{L2}P_{24}(t) +$$
$$K_{13}P_{35}(t) + \sum_j P_{10j}[dy_j/dt - m_j].$$

Here, $m_1(t)$ is the optimal estimate for $C_{m1}$, $m_2(t)$ for $C_{m2}$, and so on up to $m_{10}(t)$. The fifty-five variance component equations are similarly generated by the equations (17), however are not given here due to place shortage.

In the simulation process, the initial conditions at $t = 0$ are equal to zero for the state variables $C_{m1}, ..., \lambda_1^{001}$, to 0.5 for the estimates $m_1(t), ..., m_{10}(t)$, to 1 for the diagonal entries of the variance matrix, and to zero for its other entries. The system parameter values are all set to 1: $V = 1; d\Delta_{m1}/dt = 1; K_{L1} = 1; K_{11} = 1; K_{21} = 1; K_{31} = 1; K_{32} = 1; d\Delta_{m2}/dt = 1; d\Delta_{m3}/dt = 1; d\Delta_{m^*}/dt = 1;$ $K_{L2} = 1; K_{L3} = 1; K_{12} = 1; K_{13} = 1; K_{22} = 1; K_d = 1; K_{t1} = 1; K_{t2} = 1; K_{t3} = 1; \theta = 1$. The white Gaussian noises in the equations (19) are realized as sinusoidal signals: $\psi_i = \sin t$ for $i = 1, 10$.

In Figure 1, the obtained values of the state variables $C_{m1}, ..., \lambda_1^{001}$ are given in the blue, and the values of the bilinear optimal filter estimates $m_1(t), ..., m_{10}(t)$ are depicted in the red.

The performance of the optimal bilinear filter (16),(17) is compared to the performance of the optimal linear Kalman-Bucy filter available for the linearized system. This linear filter consists of only the linear terms and innovations processes in the equations (16) (or (19)) for the optimal estimates and the Riccati equations for the variance matrix components corresponding to the equations (17):

$$dm_k(t)/dt = (a_{0k}(t) + \sum_i a_{1ki}(t)m_i(t) + \tag{20}$$
$$\sum_{jlps} P_{kj}(t)A_{jl}^T(t)(B_{lp}B_{ps}))^{-1}(t)[dy_s - \sum_r A_{sr}(t)m_r(t)dt]$$

with

$$m_k(t_0) = E[x_k(t_0) \mid Y(t_0)];$$
$$dP_{ij}(t)/dt = \sum_k a_{1ik}(t)P_{kj}(t) + \sum_k P_{ki}(t)a_{1jk}(t) + \tag{21}$$

$$\sum_k b_{ik}(t)b_{kj}(t) - \sum_{klpsr} P_{ik}(t)A^T_{kl}(t)(B_{lp}B_{ps}))^{-1}A_{sr}P_{rj}(t).$$

with

$$P_{ij}(t_0) = E[(x_i(t_0) - m_i(t_0))(x_j(t_0) - m_j(t_0))^T \mid Y(t_0)].$$

The graphs of the estimates obtained using this linear Kalman-Bucy filter are shown in Figure 1 in the green.

Finally, the performance of the optimal bilinear filter (16),(17) is compared to the performance of the mixed filter designed as follows. The estimate equations in this filter coincide with the equations (16) (or (19)) from the optimal bilinear filter, and the variance equations coincide with the equations (21) from the linear Kalman-Bucy filter. The graphs of the estimates obtained using this mixed filter are shown in Figure 1 in the black. The initial conditions and white Gaussian noise realizations remain the same for all the filters involved in the simulation.

Upon comparing all simulation results given in Figure 1, it can be concluded that the optimal bilinear filter gives the best estimate in comparison to two other filters. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

## 4 References


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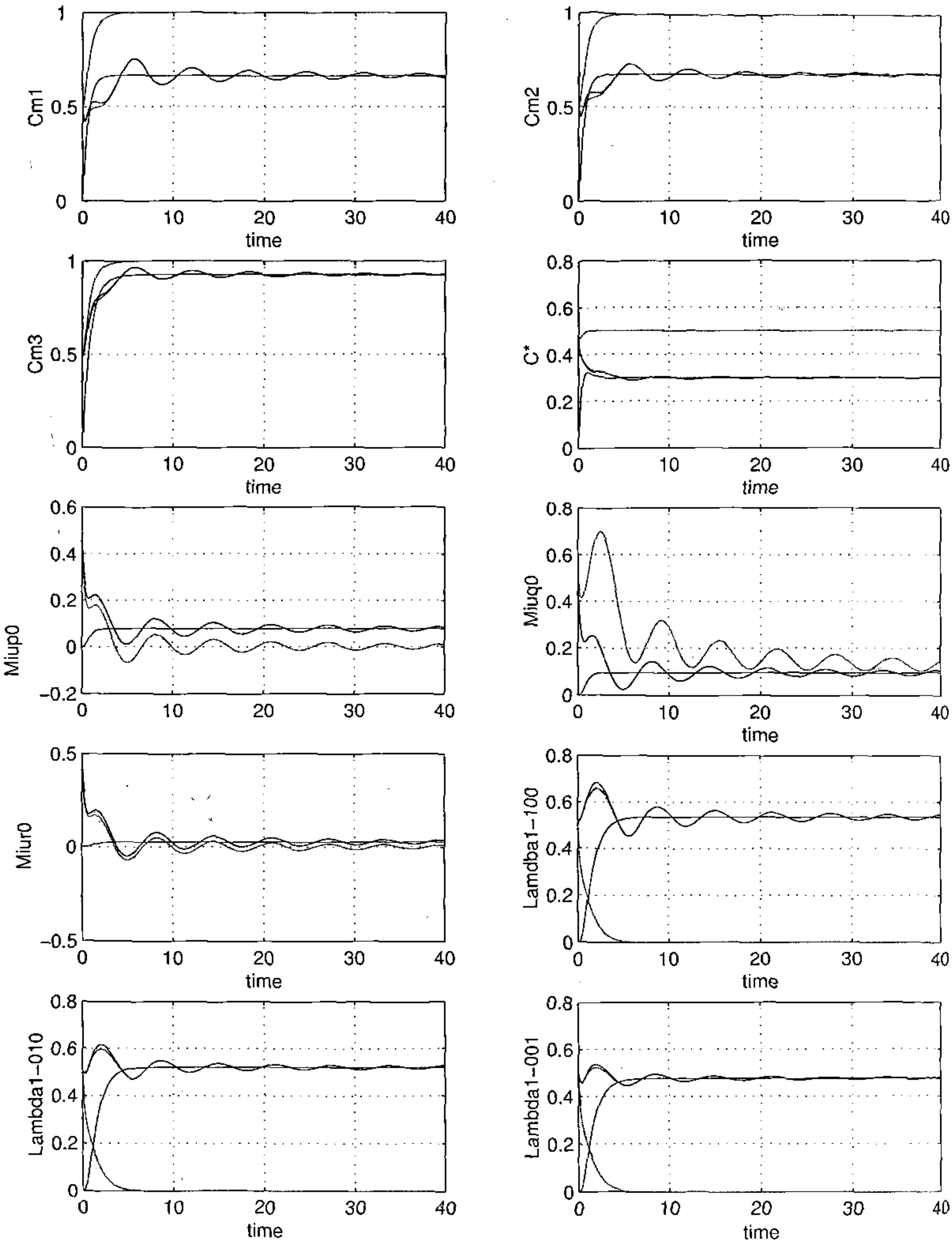


Figure 1: Graphs of the ten state variables (18) (blue), the estimates given by the optimal bilinear filter (16),(17) (red), the estimate given by the linear Kalman-Bucy filter (20),(21) (green), the estimates

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# Optimal Control for Third Degree Polynomial Systems *

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**Abstract**

This paper presents the optimal regulator for a nonlinear system state given by a polynomial equation of degree 3 with linear control input and quadratic cost criterion. The optimal regulator equations are obtained using the duality principle, which is applied to the optimal filter for a polynomial system state of degree 3 over linear observations. The obtained results are applied to solution of the optimal control problem for a nonlinear automotive system. Simulation results are compared for the optimal polynomial regulator given in this paper and the linear optimal regulator.

## 1 Introduction

Although the optimal control (regulator) problem as well as the filtering one were solved in the 1960s [4, 3], the optimal control function for nonlinear systems has to be determined by using the general principles of maximum [7] or dynamic programming [2] which do not provide an explicit form for the optimal control in most cases. However, taking into account that the optimal control problem can be solved in the linear case by applying the duality principle to the solution of the optimal filtering problem, this paper exploits the same idea for designing the optimal control in a polynomial system with linear control input, using the optimal filter for polynomial system states over linear observations. Based on the obtained polynomial filter of the third degree [1], the optimal regulator for a polynomial system of degree 3 with linear control input and quadratic cost criterion is obtained in a closed form, finding the optimal regulator gain matrix as dual transpose to the optimal filter gain one and constructing the optimal regulator gain equation as dual to the variance equation in the optimal filter. The results obtained by virtue of the duality principle could be rigorously verified through the general equations of [7] or [2] applied to a specific polynomial case, although the physical duality seems obvious: if the optimal filter exists in a closed from, the optimal closed-form regulator should also exist, and vice versa. Finally, the obtained optimal control for a polynomial system of the third degree is applied to regulation

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of a nonlinear automotive system [5] whose state equation for car orientation angle is nonlinear (contains tangent). To apply the polynomial regulator, the nonlinear equation is expanded into its Taylor polynomial up to degree 3. The optimal regulator equations for a polynomial state of third degree are written and then compared to the optimal linear regulator for the linearized system. Simulations are conducted for both polynomial and linear regulators applied to the original nonlinear system. The simulation results show significant advantage of the polynomial regulator in comparison to the linear one, five times in the values of the controlled variable and ten times in the criterion performance.

This relatively simple case treated in the paper seems to be important for practical applications, since a nonlinear state equation can usually be well approximated by a polynomial of degree 3 and the control input is, as a rule, linear. Moreover, the optimal control problem for a polynomial state equation of lower degree is significant itself, because many, for example, chemical processes are described by quadratic equations (see [6]). The quadratic state equation is, of course, a particular case of the third degree one, as well the cubic state equation is a particular case of that of fourth degree, etc.

The paper is organized as follows. Section 2 states the optimal control problem for a polynomial system of degree 3 and the duality principle for a closed-form situation. For reference purposes, the optimal filtering equations for a polynomial state equation of degree 3 and linear observations are briefly recalled in Section 3. The optimal control problem for a polynomial system state of degree 3 is solved in Section 4. Section 5 presents application of the optimal polynomial regulator to a nonlinear automotive system with two state variables, orientation and steering angles, with the objective to increase the value of the orientation angle and consume the minimum control energy. Graphic simulation results are conducted for polynomial control of degree 3 and compared with those for linear control.

## 2 Optimal Control Problem

Consider the polynomial system

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + G(t)u(t)dt, \; x(t_0) = x_0, \quad (1)$$

where $x(t) \in R^n$ is the system state, $x^2(t) = (x_1^2(t), ..., x_n^2(t))$, $x^3(t) = (x_1^3(t), ..., x_n^3(t))$, and $u(t)$ is the control variable. The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}[x(T) - x_1]^T \psi [x(T) - x_1] + \frac{1}{2}\int_{t_0}^{T} \left(u^T(s)R(s)u(s) + x^T(s)L(s)x(s)\right) ds, \quad (2)$$

where $x_1$ is a given vector, $\psi$, $R$, $L$ are positive (nonnegative) definite symmetric matrices, and $T > t_0$ is a certain time moment. We remark that the transpose of a vector $x$ is also denoted by $x^T$, which, however, should not cause any confusion.

The optimal control problem is to find the control $u(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1). To find the solution to this optimal control problem, the duality principle [4] could be used. For linear systems, if the optimal control exists in the optimal control problem for a linear system with the quadratic cost function $J$, the optimal filter exists for the dual linear system with Gaussian disturbances and can be found from the optimal control problem solution, using simple algebraic transformations (duality between the gain matrices and between the gain matrix and variance equations), and vice versa. Taking into account the physical duality of the filtering and control problems, the last conjecture should be valid for all cases where the optimal control (or, vice versa, the optimal filter) exists in a closed finite-dimensional form. This proposition is now applied to a third order polynomial system, for which the optimal filter has already been obtained (see [1]).

## 3 Optimal Filter

In this section, the optimal filtering equations for a polynomial state equation of degree 3 over linear observations (obtained in [1]) are briefly recalled for reference purposes. Let an unobservable random process $x(t)$ satisfy a polynomial equation of third degree

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + b(t)dW_1(t), \; x(t_0) = x_0, \quad (3)$$

and linear observations are given by:

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t),$$

where $x \in R^n$, $x^2(t) = (x_1^2(t), ..., x_n^2(t))$, $x^3(t) = (x_1^3(t), ..., x_n^3(t))$. $W_1(t)$ and $W_2(t)$ are Wiener processes, whose weak derivatives are Gaussian white noises and which are assumed independent of each other and of the Gaussian initial value $x_0$.

The filtering problem is to find dynamical equations for the best estimate for the real process $x(t)$ at time $t$, based on the observations $Y(t) = [y(s) \mid t_0 \le s \le t]$, that is the conditional expectation $m(t) = E[x(t) \mid Y(t)]$ of the real process $x(t)$ with respect to the observations $Y(t)$. Let $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid Y(t)]$ be the estimate covariance (correlation function).

The following notations are used. Let $m(t) = (m_1(t), ..., m_n(t)) \in R^n$ be the best estimate vector; $P(t) \in R^{n\times n}$ be the covariance matrix; $p(t) \in R^n$ be the vector whose components are the variances of the components of $x(t)$, i.e., the diagonal elements of $P(t)$; $m^2(t) = (m_1^2(t), ..., m_n^2(t))$; $m^3(t) = (m_1^3(t), ..., m_n^3(t))$; $P(t)m(t)$ be the conventional product of a matrix $P(t)$ by a vector $m(t)$; and $p(t) * m(t)$ be the product of two vectors defined componentwise: $p(t) * m(t) = [p_1(t)m_1(t), ..., p_n(t)m_n(t)]$.

The solution to the stated problem is given by the following system of filtering equations, which is closed with respect to the introduced variables, $m(t)$ and $P(t)$:

$$\begin{aligned} & dm(t) \\ = \; & (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + a_3(t)(3p(t) * m(t) + m^3(t)))dt \\ & + P(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt), \end{aligned} \quad (4)$$

$$m(t_0) = E[x(t_0)/y(t_0)].$$

$$\begin{aligned} dP(t) &= (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t) * P(t) \\ &+2(P(t) * m^T(t))a_2^T(t) + 3a_3(t)(p(t) * P(t)) + 3(p(t) * P(t))^T a_3^T(t) \\ &+3a_3(t)(m^2(t) * P(t)) + 3(P(t) * (m^2(t))^T)a_3^T(t) + b(t)b^T(t) \\ &-P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt, \end{aligned} \tag{5}$$

$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T / y(t_0)),$$

where the product $m(t) * P(t)$ between a vector $m(t)$ and a matrix $P(t)$ is defined as the matrix whose rows are equal to rows of $P(t)$ multiplied by the same corresponding element of $m(t)$:

$$\begin{bmatrix} m_1(t) & P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ m_2(t) & P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_n(t) & P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{bmatrix}$$

$$= \begin{bmatrix} m_1(t)P_{11}(t) & m_1(t)P_{12}(t) & \cdots & m_1(t)P_{1n}(t) \\ m_2(t)P_{21}(t) & m_2(t)P_{22}(t) & \cdots & m_2(t)P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ m_n(t)P_{n1}(t) & m_n(t)P_{n2}(t) & \cdots & m_n(t)P_{nn}(t) \end{bmatrix}.$$

The transposed matrix $P(t) * m^T(t) = (m(t) * P(t))^T$ is defined as the matrix whose columns are equal to columns of $P(t)$ multiplied by the same corresponding element of $m(t)$:

$$\begin{bmatrix} m_1(t) & m_2(t) & \cdots & m_n(t) \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{bmatrix}$$

$$= \begin{bmatrix} m_1(t)P_{11}(t) & m_2(t)P_{12}(t) & \cdots & m_n(t)P_{1n}(t) \\ m_1(t)P_{21}(t) & m_2(t)P_{22}(t) & \cdots & m_n(t)P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(t)P_{n1}(t) & m_2(t)P_{n2}(t) & \cdots & m_n(t)P_{nn}(t) \end{bmatrix}.$$

Thus, the equation (4) for the optimal estimate $m(t)$ and the equation (5) for its covariance matrix $P(t)$ form a closed system of filtering equations in the case of a polynomial state equation of degree 3 and linear observations.

## 4 Optimal Solution

Let us return to the optimal control problem for the polynomial state (1) with linear control input and the cost function (2). This problem is dual to the filtering problem for the polynomial state (3) of degree 3 and linear observations. Since the optimal polynomial filter gain matrix in (4) is equal to

$$K_f = P(t)A^T(t)(B(t)B^T(t))^{-1},$$

the gain matrix in the optimal control problem takes the form of its dual transpose

$$K_c = (R(t))^{-1}G^T(t)Q(t),$$

and the optimal control law is given by

$$u^*(t) = K_c x = (R(t))^{-1}G^T(t)Q(t)x(t), \tag{6}$$

where the matrix function $Q(t)$ is the solution of the following equation dual to the variance equation (5)

$$\begin{aligned} dQ(t) \; = \; & (-a_1^T(t)Q(t) - Q(t)a_1(t) - 2a_2^T(t)Q(t) * x^T(t) - 2x(t) * Q(t)a_2(t) \\ & -3a_3^T(t)Q(t) * q^T(t) - 3q(t) * Q(t)a_3(t) - 3a_3^T(t)Q(t) * ((x^2(t))^T) \\ & -3(x^2(t) * Q(t))a_3(t) + L(t) - Q(t)G(t)R^{-1}(t)G^T(t)Q(t))dt, \end{aligned} \tag{7}$$

with the terminal condition $Q(T) = \psi$. The binary operation $*$ has been introduced in Section 3, and $q(t) = (q_1(t), q_2(t), \dots, q_n(t))$ denotes the vector consisting of the diagonal elements of $Q(t)$.

Upon substituting the optimal control (6) into the state equation (1), the optimally controlled state equation is obtained

$$\begin{aligned} dx(t) \; &= \; (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt \\ & \quad + G(t)(R(t))^{-1}G^T(t)Q(t)x(t)dt, \\ x(t_0) \; &= \; x_0, \end{aligned}$$

Note that if the real state vector $x(t)$ is unknown (unobservable), the optimal controller uniting the obtained optimal filter and regulator equations, can be constructed using the separation principle [4] for polynomial systems, which should also be valid if solutions of the optimal filtering and control problems exist in a closed finite-dimensional form.

The results obtained in this section by virtue of the duality principle could be rigorously verified through the general equations of the Pontryagin maximum principle [7] or Bellman dynamic programming [2].

## 5 Application to Automotive System

This section presents application of the obtained optimal regulator for a polynomial system of degree 3 with linear control input and quadratic criterion to controlling the state variables, orientation and steering angles, in the nonlinear kinematical model of car movement [5] given by the following nonlinear equations

$$\begin{aligned} dx(t) &= v\cos(\phi(t))dt \\ dy(t) &= v\sin(\phi(t))dt \\ d\phi(t) &= (v/l)\tan(\delta(t))dt \\ d\delta(t) &= u(t)dt \end{aligned} \tag{8}$$

Here, $x(t)$ and $y(t)$ are Cartesian coordinates of the mass center of the car, $\phi(t)$ is the orientation angle, $v$ is the velocity, $l$ is the longitude between the two axes of the car, $\delta(t)$ is the steering wheel angle, and $u(t)$ is the control variable (steering angular velocity).

The optimal control problem is to maximize the orientation angle $\phi$ using the minimum energy of control $u$. The examined values of the velocity and longitude are $v = 17$, $l = 2$, and the motion time is $T = 0.1$, which correspond to the idle engine mode of a full-size car in the time interval of 6 seconds. The initial conditions for the angles are $\phi(0) = 0.1$ and $\delta(0) = 0.1$. In other words, the problem is to make the maximum turn of the running wheels from their initial position, using the minimum steering energy.

The corresponding criterion $J$ to be minimized takes the form

$$J = \frac{1}{2}[\phi(T) - \phi^*]^2 + \frac{1}{2}\int_0^T u^2(t)dt \tag{9}$$

where $T = 0.1$, and $\phi^* = 1$ is a large value of $\phi(t)$ *a priori* unreachable for time $T$. To apply the obtained optimal control algorithms to the nonlinear system (8), let us make the Taylor expansion of the two last equations in (8) at the origin up to degree 3 (the fourth degree does not appear in the Taylor series for tangent)

$$\begin{aligned} d\phi(t) &= \left(\tfrac{v}{l}\right)\delta(t)dt + \left(\tfrac{v}{l}\right)\left(\tfrac{\delta^3(t)}{3}\right)dt \\ d\delta(t) &= u(t)dt \end{aligned} \tag{10}$$

Now, since $R = 1$ and $G^T = [0, 1]$ in (9), the optimal control law (6) takes the form $u^*(t) = q_{21}(t)\phi(t) + q_{22}(t)\delta(t)$, where the elements $q_{11}(t)$, $q_{21}(t)$, $q_{22}(t)$ of the symmetric matrix $Q(t)$ satisfy the equations

$$\begin{aligned} dq_{11}(t) &= -q_{21}^2(t) \\ dq_{12}(t) &= -\tfrac{3v}{l}q_{11}^2(t) - q_{12}(t)q_{22}(t) - \tfrac{v}{l}q_{11}(t) - \tfrac{3v}{l}\phi^2 q_{11}(t) \\ dq_{22}(t) &= -\tfrac{2v}{l}q_{12}(t) - \tfrac{6v}{l}q_{12}(t)q_{22}(t) - \tfrac{6v}{l}\delta^2 q_{12}(t) - q_{22}^2(t) \end{aligned} \tag{11}$$

The system composed of the two last equations of (8) and the equations (10) should be solved with initial conditions $\phi(0) = 0.1$, $\delta(0) = 0.1$ and terminal conditions $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$. This boundary problem is solved numerically using the iterative method of direct and reverse passing as follows. The first initial conditions for $q$'s are guessed, and the system is solved in direct time with the initial conditions at $t = 0$, thus obtaining certain values for $\phi$ and $\delta$ at the terminal point $T = 0.1$. Then, the system is solved in reverse time, taking the obtained terminal values for $\phi$ and $\delta$ in direct time as the initial values in reverse time, thus obtaining certain values for $q$'s at the initial point $t = 0$, which are taken as the initial values for the passing in direct time, and so on. The given initial conditions $\phi(0) = 0.1$, $\delta(0) = 0.1$ are kept fixed for any direct passing, and the given terminal conditions $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ are used as the fixed initial conditions for any reverse passing. The algorithm stops when the system arrives at values $q_{11}(T) = 1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ after direct passing and at values $\phi(0) = 0.1$, $\delta(0) = 0.1$ after reverse passing. The initial conditions for $q$'s in the final direct iteration are $q_{11}(0) = 1.32$, $q_{12}(0) = 16$, $q_{22}(0) = 1640$. The obtained simulation graphs for $\phi$ and the criterion $J$ are given in Fig. 2. These results for polynomial regulator of degree 3 are then compared to the results obtained using the optimal linear regulator, whose matrix $Q(t)$ elements satisfy the Riccati equations

$$\begin{aligned} dq_{11}(t) &= -q_{12}^2(t) \\ dq_{12}(t) &= -q_{12}q_{22} - \tfrac{v}{l}q_{11} \\ dq_{22}(t) &= -\tfrac{2v}{l}q_{12} - q_{22}^2 \end{aligned} \tag{12}$$

with terminal conditions $q_{11}(T) = 1, q_{12}(T) = 0, q_{22}(T) = 0$. Note that in the linear case the only reverse passing for $q$'s is necessary, because the system (12) does not depend on $\phi$ and $\delta$, and the initial values for $q$'s at $t = 0$ are obtained after single reverse passing. The initial conditions for $q$'s in the direct iteration are $q_{11}(0) = 1.025$, $q_{12}(0) = 0.87$, $q_{22}(0) = 0.74$. The simulation graphs for the linear case are given in Figure 1, which consists of the graph of the variable $\phi$ satisfying the original system (8) and controlled using the optimal linear regulator defined by (12) and the graph of the corresponding values of the criterion $J$.

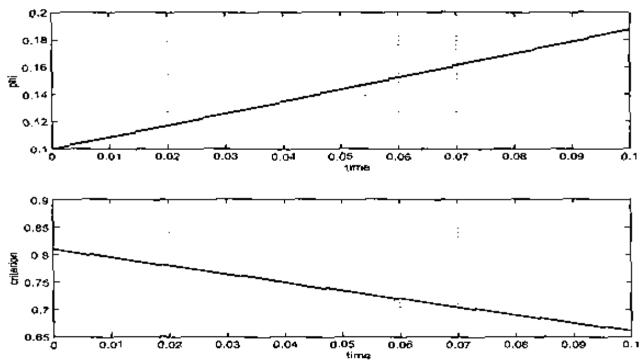


Figure 1.

In Figure 2, we show the graph of the variable $\phi$ satisfying the original system (8) and controlled using the optimal third order polynomial regulator defined by (11) and the graph of the corresponding values of the criterion $J$.

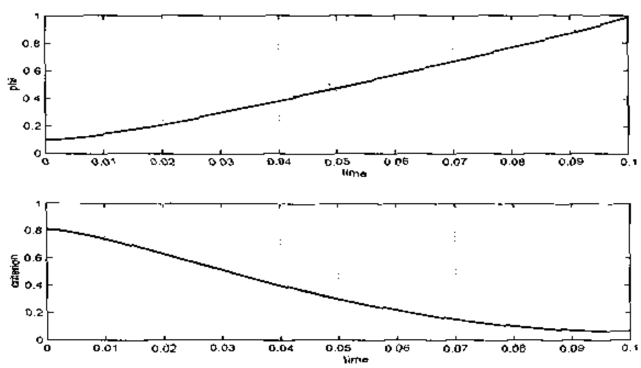


Figure 2.

The obtained values of the controlled variable $\phi$ and the criterion $J$ are compared for the optimal third order polynomial and linear regulators at the terminal time $T = 0.1$ in the following table (corresponding to Figs. 1 and 2).

| Linear regulator | Third degree polynomial regulator |
|---|---|
| $\phi(0.1) = 0.1875$ | $\phi(0.1) = 0.989$ |
| $J = 0.661$ | $J = 0.065$ |

Graphs of control functions $u^*(t)$ corresponding to the optimal linear regulator and the optimal third order polynomial regulator are given in Figs. 3 and 4, respectively.

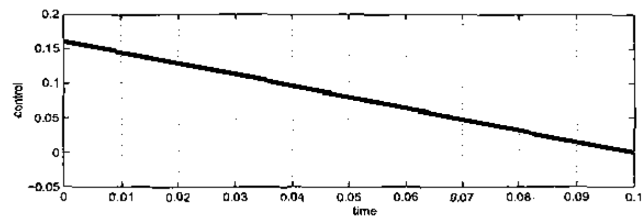


Figure 3.

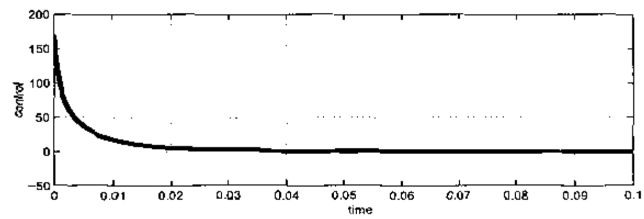


Figure 4.

# 6 Conclusions

The simulation results show that the values of the controlled variable $\phi$ at the terminal point $T = 0.1$ are five times greater for the third order regulator than for the linear one and the criterion value at the terminal point is ten times less for the third order regulator. Thus, the third order polynomial regulator controls the system variables significantly better than the linear one from both points of view. The obtained results show that the best gain matrix based on the linearized model could still be too far from achieving the optimal performance. The considered example validates design and implementation of the regulators based on polynomial approximations of nonlinear systems.

Finally note that the better performance of the cost function and controlled variable has been achieved without changing the system dynamics (in both simulation cases, the designed control algorithms are applied to the original nonlinear system (8)), but by assigning a better regulator gain matrix ($Q(t)$ satisfies (11) instead of (12)). Thus, the principal result in the considered application consists in designing a better regulator and not in using more accurate system dynamics, as it could seem after the first glance.

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# OPTIMAL CONTROLLER FOR THIRD DEGREE POLYNOMIAL SYSTEM

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**Abstract.** This paper presents the solution to the optimal controller problem for a stochastic system given by a polynomial equation of third degree, linear observations confused with white Gaussian noises, and a quadratic cost function. The obtained controller equations are applied to solution of the state controlling problem for a nonlinear automotive system. Simulation results are compared for the optimal polynomial controller given in this paper and the best linear controller available for the linearized system.

## 1 Introduction

Although the optimal controller problem for linear system states was solved in 1960s, based on the solutions to the optimal filtering [3] and regulator [4, 2] problems, solution to the optimal controller problem for nonlinear (in particular, polynomial) systems has been impossible due to the absence of the solution to the corresponding filtering and control problems for nonlinear systems. This paper presents solution to the optimal controller problem for unobservable third degree polynomial system states over linear observations and quadratic criterion. Due to the separation principle for polynomial systems with linear observations and quadratic criterion, which is stated and substantiated in the paper analogously to that for linear ones (see [4]), the original controller problem is split into the optimal filtering problem for third degree polynomial system states over linear observations and the optimal control (regulator) problem for observable third degree polynomial system states with quadratic criterion. (The statements and solutions of both those problems can be found in [1]).

The relatively simple case of third degree polynomial systems considered in this paper seems to be important for practical applications, since a nonlinear state equation can usually be well approximated by a polynomial of degree 3, the observations are frequently direct, that is linear, and the cost function in the controlling problems, where the desired value of the controlled variable should be maintained or maximized using the minimum control energy, is intrinsically quadratic. Moreover, the controlling problem for a polynomial state equation of lower degree is significant itself, because many, for example, chemical processes are described by quadratic equations (see [7]).

The obtained optimal controller for a polynomial state equation of degree 3 is applied to solution of the state controlling problem for a nonlinear automotive system [6] whose state equation for car orientation angle is nonlinear (contains tangent), with the objective of increasing values of the state variables and consuming the minimum control energy. To apply the developed polynomial technique, the original state equation is expanded as a Taylor polynomial, up to degree 3. The optimal controller equations for a polynomial state of third degree are written and then compared to the best linear controller available for the linearized system. Numerical simulations are conducted for the optimal polynomial controller and also compared to those for the linear one applied to the linearized system. The simulation results given in the paper show a significant, more than one and half times, advantage of the optimal polynomial controller performance in comparison to the linear one.

The paper is organized as follows. In Section 2, the optimal controller problem is stated and solved for unobservable third degree polynomial system states, using the separation principle for polynomial systems with linear observations and quadratic criterion. Section 3 presents application of the obtained results to the controlling problem for a nonlinear automotive system with two state variables, orientation and steering angles, over direct linear observations confused with white Gaussian noises, with the objective of increasing values of the state variables and consuming the minimum control energy. Graphic simulation results are obtained and compared to those for the best linear controller available for the linearized system.

# 2 Optimal Controller Problem

## 2.1 Problem statement

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be $F_t$-adapted Wiener processes. Let us consider the unobservable $F_t$-measurable random process $x(t)$ governed by the third degree polynomial state equation

$$dx(t) = (a_0(t)+a_1(t)x(t)+a_2(t)x^2(t)+a_3(t)x^3(t))dt+G(t)u(t)dt+b(t)dW_1(t),$$

$$x(t_0) = x_0, \tag{1}$$

and the linear output (observation) process

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t). \tag{2}$$

Here, $x(t) \in R^n$ is the unobservable state vector, whose second and third degrees are defined in the componentwise sense $x^2(t) = (x_1^2(t), x_2^2(t), x_3^2(t), \ldots, x_n^2(t))$, $x^3(t) = (x_1^3(t), x_2^3(t), x_3^3(t), \ldots, x_n^3(t))$, $u(t) \in R^p$ is the control variable, $y(t) \in R^m$ is the observation process, and the independent Wiener processes $W_1(t)$ and $W_2(t)$ represent random disturbances in state and observation equations, which are also independent of an initial Gaussian vector $x_0$. Let $A(t)$ be a nonzero matrix and $B(t)B^T(t)$ be a positive definite matrix. In addition, the quadratic cost function $J$ to be minimized is defined as follows

$$J = E[\frac{1}{2}[x(T) - z_0]^T \Phi [x(T) - z_0] + \tag{3}$$

$$\frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} x^T(s)L(s)x(s)ds],$$

where $z_0$ is a given vector, $K$ is positive definite and $\Phi$, $L$ are nonnegative definite symmetric matrices, $T > t_0$ is a certain time moment, the symbol $E[f(x)]$ means the expectation (mean) of a function $f$ of a random variable $x$, and $a^T$ denotes transpose to a vector (matrix) $a$.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion $J$ along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1).

## 2.2 Separation principle for polynomial systems

As well as for a linear stochastic system, the separation principle remains valid for a stochastic system given by a third order polynomial equation, linear observations, and a quadratic criterion. Indeed, let us replace the unobservable system state $x(t)$ by its optimal estimate $m(t)$ given by the equation (see [1] for statement and derivation)

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + \tag{4}$$

$$a_3(t)(3p(t) * m(t) + m^3(t))dt + G(t)u(t) +$$

$$P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt),$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$. Here, $m(t)$ is the best estimate for the unobservable process $x(t)$ at time $t$ based on the observation process $Y(t) = \{y(s), t_0 \le s \le t\}$, that is the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$, $m(t) = m_1(t), m_2(t), \ldots, m_n(t))$; $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid Y(t)] \in \mathbb{R}^n$ is the error covariance matrix; $p(t) \in \mathbb{R}^n$ is the vector whose components are the variances of the components of $x(t) - m(t)$, i.e., the diagonal elements of $P(t)$; $m^2(t)$ and $m^3(t)$ are defined as the vectors of squares and cubes of the components of $m(t)$: $m^2(t) = (m_1^2(t), m_2^2(t), \ldots, m_n^2(t)$, $m^3(t) = (m_1^3(t), m_2^3(t), \ldots, m_n^3(t)$; $P(t)m(t)$ is the conventional product of a matrix $P(t)$ by a vector $m(t)$; and $p(t)*m(t)$ is the product of two vectors by components: $p(t)*m(t)=[p_1(t)m_1(t), p_2(t)m_2(t), \ldots, p_n(t)m_n(t)]$. The best estimate $m(t)$ minimizes the criterion

$$H = E[(x(t)-m(t))^T(x(t)-m(t))], \tag{5}$$

with respect to selection of the estimate $m$ as a function of observations $y(t)$, at every time moment $t$ ([5]).

The complementary equation for the covariance matrix $P(t)$ takes the form (see [1] for derivation)

$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)*P(t)+ \tag{6}$$
$$2(P(t)*m^T(t))a_2^T(t) + 3a_3(t)(p(t)*P(t))+$$
$$3(p(t)*P(t))^T a_3^T(t) + 3a_3(t)(m^2(t)*P(t))+$$
$$3(P(t)*(m^2(t))^T)a_3^T(t) + (b(t)b^T(t))-$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt,$$

with the initial condition $P(t_0)=E((x(t_0)-m(t_0))(x(t_0)-m(t_0))^T \mid y(t_0))$, where the product $m(t)*P(t)$ between a vector $m(t)$ and a matrix $P(t)$ is defined as the matrix whose rows are equal to rows of $P(t)$ multiplied by the same corresponding element of $m(t)$:

$$\begin{bmatrix} m_1(t) & P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ m_2(t) & P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_n(t) & P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{bmatrix} =$$
$$\begin{bmatrix} m_1(t)P_{11}(t) & m_1(t)P_{12}(t) & \cdots & m_1(t)P_{1n}(t) \\ m_2(t)P_{21}(t) & m_2(t)P_{22}(t) & \cdots & m_2(t)P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ m_n(t)P_{n1}(t) & m_n(t)P_{n2}(t) & \cdots & m_n(t)P_{nn}(t) \end{bmatrix}.$$

It is readily verified (see [4]) that the optimal control problem for the system state (1) and cost function (3) is equivalent to the optimal control problem for the estimate (4) and the cost function $J$ represented as

$$J = E\{\frac{1}{2}[m(T)-z_0]^T \Phi [m(T)-z_0] + \tag{7}$$
$$\frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds$$
$$+\frac{1}{2}\int_{t_0}^{T} tr[P(s)L(s)]ds + tr[P(T)\Phi]\},$$

where $tr[A]$ denotes trace of a matrix $A$. Since the latter part of $J$ is independent of control $u(t)$ or state $x(t)$, the reduced effective cost function $M$ to be minimized takes the form

$$M = E\{\frac{1}{2}[m(T) - z_0]^T \Phi [m(T) - z_0] + \tag{8}$$

$$\frac{1}{2}\int_{t_0}^{T} u^T(s)K(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds\}.$$

Thus, the solution for the optimal control problem specified by (1),(3) can be found solving the optimal control problem given by (4),(8). However, the minimal value of the criterion $J$ should be determined using (7). This conclusion presents the separation principle in third order polynomial systems.

## 2.3 Optimal control problem solution

Based on the solution of the optimal control problem obtained in [1] in the case of an observable system state governed by a third order polynomial equation, the following results are valid for the optimal control problem (4),(8), where the system state (the estimate $m(t)$) is completely available and, therefore, observable.

The optimal control law is given by

$$u^*(t) = K^{-1}(t)G^T(t)Q(t)m(t), \tag{9}$$

where the matrix function is the solution of the following equation dual to the variance equation

$$dQ(t) = (-a_1^T(t)Q(t) - Q(t)a_1^T(t) - 2a_2^T(t)Q(t) * m^T(t) -$$
$$2m(t) * Q(t)a_2(t) - 3a_3^T(t)Q(t) * q^T(t) - \tag{10}$$
$$3q(t) * Q(t)a_3(t) - 3a_3^T(t)Q(t) * ((m^2)^T(t)) -$$
$$3(m^2(t) * Q(t))a_3(t) + L(t) - Q(t)G(t)K^{-1}(t)G^T(t)Q(t))dt,$$

with the terminal condition $Q(T) = \Phi$. The binary operation $*$ has been introduced in Subsection 2.2, and $q(t) = (q_1(t), q_2(t), ..., q_n(t))$ denotes the vector consisting of the diagonal elements of $Q(t)$. In the process of derivation of the equation (10), it has been taken into account that the last term in the equation (4), $P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt)$, is a Gaussian white noise.

Upon substituting the optimal control (9) into the equation (4) for the reconstructed system state $m(t)$, the following optimally controlled state estimate equation is obtained

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + \tag{11}$$

$$a_3(t)(3p(t) * m(t) + m^3(t))dt + G(t)(K(t))^{-1}G^T(t)Q(t)m(t)dt+$$
$$P^T(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt),$$
$$m(t_0) = E(x(t_0) \mid F_{t_0}^Y).$$

Thus, the optimally controlled state estimate equation (11), the gain matrix constituent equation (10), the optimal control law (9), and the variance equation (6) give the complete solution to the optimal controller problem for unobservable states of third degree polynomial systems.

# 3 Application of optimal polynomial controller to automotive system

This section presents application of the obtained controller for a polynomial state of degree 3 over linear observations and a quadratic cost function to controlling the unobservable state variables, orientation and steering angles, in a nonlinear kinematical model of car movement [6] satisfying the following equations:

$$dx(t) = v\cos\phi(t)dt, \tag{12}$$
$$dy(t) = v\sin\phi(t)dt,$$
$$d\phi(t) = (v/l)\tan\delta(t)dt,$$
$$d\delta(t) = u(t)dt.$$

Here, $x(t)$ and $y(t)$ are Cartesian coordinates of the mass center of the car, $\phi(t)$ is the orientation angle, $u$ is the velocity, $l$ is the longitude between the two axes of the car, $\delta(t)$ is the steering wheel angle, and $u(t)$ is the control variable (steering angular velocity). The zero initial conditions for all variables are assumed.

The observation process for the unobservable variables $\phi(t)$ and $\delta(t)$ is given by direct linear observations confused with independent and identically distributed disturbances modelled as white Gaussian noises. The corresponding observations equations are

$$dz_\phi(t) = \phi(t)dt + w_1(t)dt, \tag{13}$$
$$dz_\delta(t) = \delta(t)dt + w_2(t)dt,$$

where $z_\phi(t)$ is the observation variable for $\phi(t)$, $z_\delta(t)$ is the observation variable for $\delta(t)$, and $w_1(t)$ and $w_2(t)$ are white Gaussian noises independent of each other.

The examined values of the velocity and longitude are $v = 17$, $l = 2$, which correspond to the idle engine mode of a full-size car. In other words, the problem is to make the maximum turn of the running wheels from their initial position, using the minimum steering energy. For the reasons of economizing the fuel and reducing air pollution, the weight of the control term in the criterion is assigned ten times more than the weight of the state terminal term. The corresponding criterion $J$ to be minimized takes the form

$$J = [\phi(t) - \phi^*]^2 + 10 \int_0^T u^2(t)dt, \tag{14}$$

where $T = 0.3$, and $\phi^* = 10$ is a large value of $\phi(t)$ unreachable for time $T$.

To apply the obtained optimal controller algorithms to the nonlinear system (12), linear observations (13), and the quadratic criterion (14), let us make the Taylor expansion of the two last equations in (12) at the origin up to degree 3 (the fourth degree does not appear in the Taylor series for tangent)

$$d\phi(t) = ((\frac{v}{l})\delta(t) + (\frac{v}{l})(\frac{\delta^3(t)}{3}))dt, \tag{15}$$
$$d\delta(t) = u(t)dt.$$

The solution to the stated optimal controller problem is given as follows. Since $K = 1$ and $G^T = [0, 1]$ in (14) and (15), the optimal control $u^*(t) = (K(t))^{-1}G^T(t)Q(t)m(t)$ takes the form

$$u^*(t) = q_{21}(t)m_\phi(t) + q_{22}(t)m_\delta(t), \tag{16}$$

and the following optimal controller equations (9)–(11) and (6) for the third degree polynomial state (15) over the linear observations (13) and the quadratic criterion (14) are obtained

$$dm_\phi = ((\frac{v}{l})m_\delta + (\frac{v}{3l})(3p_\delta + m_\delta^3) + p_{\phi\phi}(z_\phi - m_\phi) + p_{\phi\delta}(z_\delta - m_\delta))dt. \tag{17}$$

$$dm_\delta = (u^*(t) + p_{\delta\phi}(z_\phi - m_\phi) + p_{\delta\delta}(z_\delta - m_\delta))dt,$$

$$dp_{\phi\phi} = ((2v/l)p_{\delta\phi}p_{\delta\delta} + \frac{2v}{l}p_{\delta\phi} + \frac{2v}{l}m_\phi^2 p_{\delta\phi} - p_{\phi\phi}^2 - p_{\phi\delta}^2)dt.$$

$$dp_{\phi\delta} = (\frac{v}{l}p_{\delta\delta} + \frac{v}{l}m_\delta^2 p_{\delta\delta} - p_{\phi\phi}p_{\phi\delta} - p_{\phi\delta}p_{\delta\delta})dt,$$

$$dp_{\delta\delta} = (-p_{\delta\phi}^2 - p_{\delta\delta}^2)dt,$$

$$dq_{11}(t) = (-q_{21}^2(t))dt,$$

$$dq_{12}(t) = (-\frac{v}{l}q_{11}^2 - q_{12}q_{22} - \frac{v}{l}q_{11} - \frac{v}{l}m_\phi^2 q_{11})dt,$$

$$dq_{22}(t) = (-\frac{2v}{l}q_{12} - \frac{2v}{l}q_{12}q_{22} - \frac{2v}{l}m_\delta^2 q_{12} - q_{22}^2)dt.$$

Here, $m_\phi$ and $m_\delta$ are the estimates for variables $\phi$ and $\delta$; $p_{\phi\phi}, p_{\phi\delta}, p_{\delta\delta}$ are elements of the symmetric covariance matrix $P$; and $q_{11}(t), q_{21}(t), q_{22}(t)$ are elements of the symmetric matrix $Q(t)$ forming the optimal control (16).

The following values of the input variables and initial values are assigned: $v = 17, l = 2, m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$. Gaussian disturbances $w_1(t)$ and $w_2(t)$ in (13) are realized as sinusoidal signals: $w_1(t) = w_2(t) = \sin t$. The terminal conditions for the matrix $Q$ elements are: $q_{11}(T) = 0.1, q_{12}(T) = 0, q_{22}(T) = 0$, where the final time $T = 0.3$.

Thus, the system composed of the two last equations of (12) and the equations (17) should be solved with initial conditions $m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$ and terminal conditions $q_{11}(T) = 0.1, q_{12}(T) = 0, q_{22}(T) = 0$. This boundary problem is solved numerically using the iterative method of direct and reverse passing as follows. The first initial conditions for $q$'s are guessed, and the system is solved in direct time with the initial conditions at $t = 0$, thus obtaining certain values for the other listed variables at the terminal point $T = 0.3$. Then, the system is solved in reverse time, taking the obtained terminal values for the other variables in direct time as their initial values in reverse time, thus obtaining certain values for $q$'s at the initial point $t = 0$, which are taken as their initial values for the passing in direct time, and so on. The given initial conditions $m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$ are kept fixed for any direct passing, and the given terminal conditions $q_{11}(T) = 0.1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ are used as the fixed initial conditions for any reverse passing. The algorithm stops when the system arrives at values $q_{11}(T) = 0.1$, $q_{12}(T) = 0$, $q_{22}(T) = 0$ after direct passing and at values $m_\phi(0) = 1, m_\delta(0) = 0.1, \phi(0) = \delta(0) = 0, P_{\phi\phi}(0) = 10, P_{\phi\delta}(0) = 1, P_{\delta\delta}(0) = 1$ after reverse passing. The obtained simulation graphs for $\phi$, $\delta$, $m_\phi$, $m_\delta$, the criterion $J$, and the optimal control $u^*(t)$ are given in Fig. 1. These results for the polynomial controller of degree 3 are then compared to the results obtained using the best linear controller available for the linearized model (only the linear term is present in the Taylor expansion for tangent). The optimal control law in this linear controller is the same as in (16) and the optimal linear controller equations are given by

$$dm_\phi = (\frac{v}{l}m_\delta + p_{\phi\phi}(z_\phi - m_\phi) + p_{\phi\delta}(z_\delta - m_\delta))dt, \qquad (18)$$

$$dm_\delta = (u^*(t) + p_{\delta\phi}(z_\phi - m_\phi) + p_{\delta\delta}(z_\delta - m_\delta))dt,$$

$$dp_{\phi\phi} = (\frac{2v}{l}p_{\delta\phi} - p_{\phi\phi}^2 - p_{\phi\delta}^2)dt,$$

$$dp_{\phi\delta} = (\frac{v}{l}p_{\delta\delta} - p_{\phi\phi}p_{\phi\delta} - p_{\phi\delta}p_{\delta\delta})dt,$$

$$dp_{\delta\delta} = (-p_{\delta\phi}^2 - p_{\delta\delta}^2)dt,$$

$$dq_{11}(t) = (-q_{21}^2(t))dt,$$

$$dq_{12}(t) = (-q_{12}q_{22} - \frac{v}{l}q_{11})dt,$$

$$dq_{22}(t) = (-\frac{2v}{l}q_{12} - q_{22}^2)dt.$$

Note that in the linear case the only reverse passing for $q$'s is necessary, because the the equations for $q$'s in (18) do not depend of $\phi$, $\delta$, $m_\phi$, or $m_\delta$, and the initial values for $q$'s at $t = 0$ can be obtained after single reverse passing. The simulation graphs for the linear case are given in Fig. 2.

Thus, two sets of graphs are obtained.

1. Graphs of the variables $\phi$ and $\delta$ satisfying the polynomial system (15) and controlled using the optimal linear regulator defined by (16), (18); graphs of the estimates $m_\phi$ and $m_\delta$ satisfying the system (18) and controlled using the optimal linear regulator defined by (16), (18); graphs of the corresponding values of the criterion $J$; graphs of the corresponding values of the optimal control $u^*$ (Fig. 1).

2. Graphs of the variables $\phi$ and $\delta$ satisfying the polynomial system (15) and controlled using the optimal third order polynomial controller defined by (16), (17); graphs of the estimates $m_\phi$ and $m_\delta$ satisfying the system (17) and controlled using the optimal third order polynomial controller defined by (16), (17); graphs of the corresponding values of the criterion $J$; graphs of the corresponding values of the optimal control $u^*$ (Fig. 2).

The obtained values of the controlled variable $\phi$ and the criterion $J$ are compared for the optimal third order polynomial and linear controllers at the terminal time $T = 0.3$ in the following table (corresponding to Figs. 1 and 2).

| Lineal controller | Third degree polynomial controller |
|---|---|
| $\phi(0.3) = 0.0545$ | $\phi(0.3) = 0.0876$ |
| $J = 98.9625$ | $J = 98.3884$ |

The simulation results show that the value of the controlled variable $\phi$ at the terminal point $T = 0.3$ is more than one and half times greater for the third order polynomial controller than for the linear one, and the difference between the initial and final criterion values is more than one and half times greater for the third order polynomial controller as well. Thus, the third order polynomial controller regulates the system variables better than the linear one from both points of view, thus illustrating the theoretical conclusion.

# 4 Conclusions

The optimal nonlinear controller for a stochastic system state given by a polynomial equation of degree 3, linear observations confused with white Gaussian noises, and a quadratic criterion has been obtained. The optimal polynomial controller of degree 3 has been then applied to solution of the controlling problem for state variables, orientation and steering angles, of a nonlinear automotive system describing kinematics of car movement. Application of the obtained controller to the nonlinear automotive system has yielded more than one and half times better values of the criterion and greater values of the controlled variable in comparison to the best linear controller available for the linearized model. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

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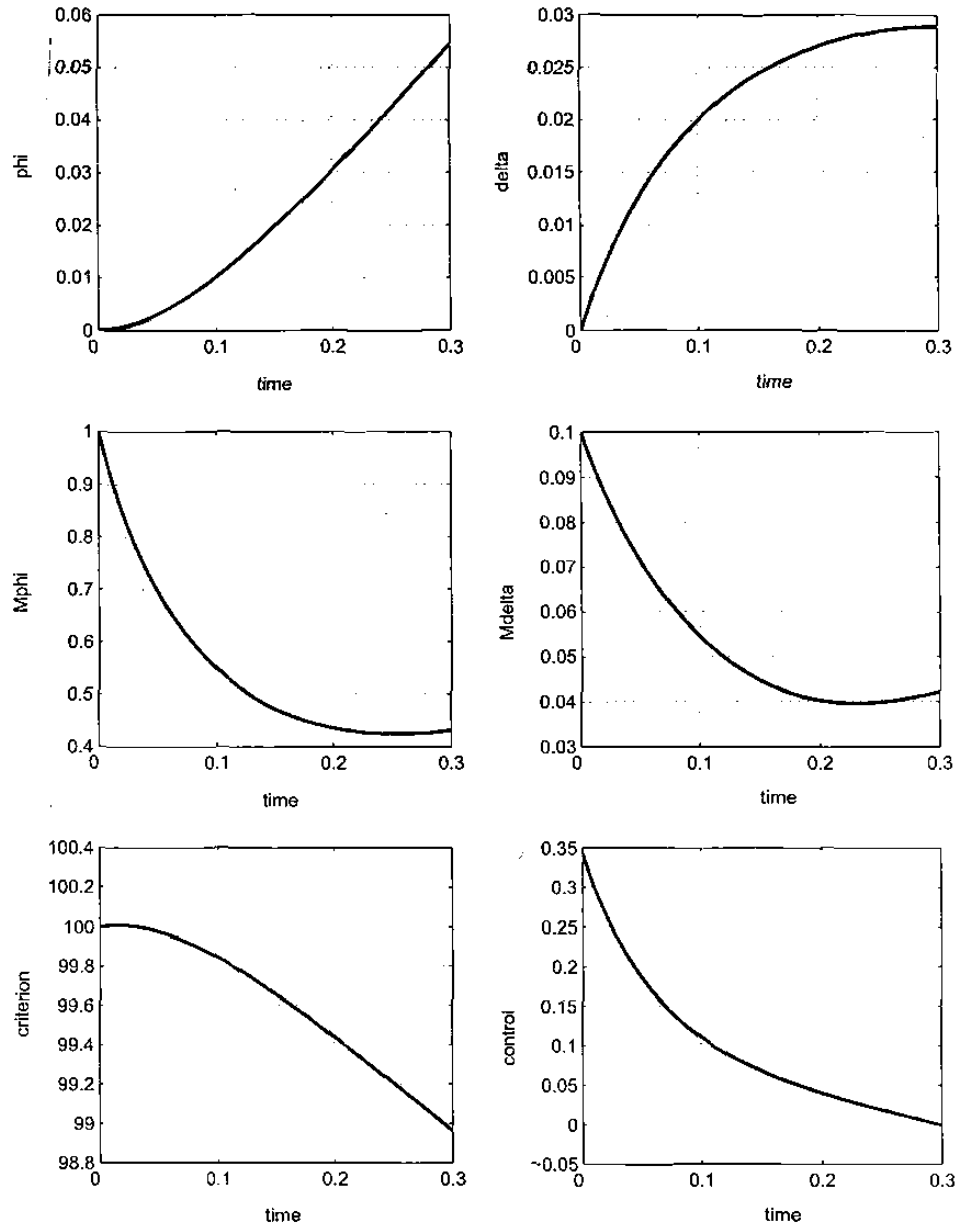


Figure 1: Graphs of the variables $\phi$ and $\delta$ satisfying the polynomial system (15) and controlled using the optimal linear regulator defined by (16), (18); graphs of the estimates $m_\phi$ and $m_\delta$ satisfying the system (18) and controlled using the optimal linear regulator defined by (16), (18); graphs of the corresponding values of the criterion $J$; graphs of the corresponding values of the optimal control $u^*$.

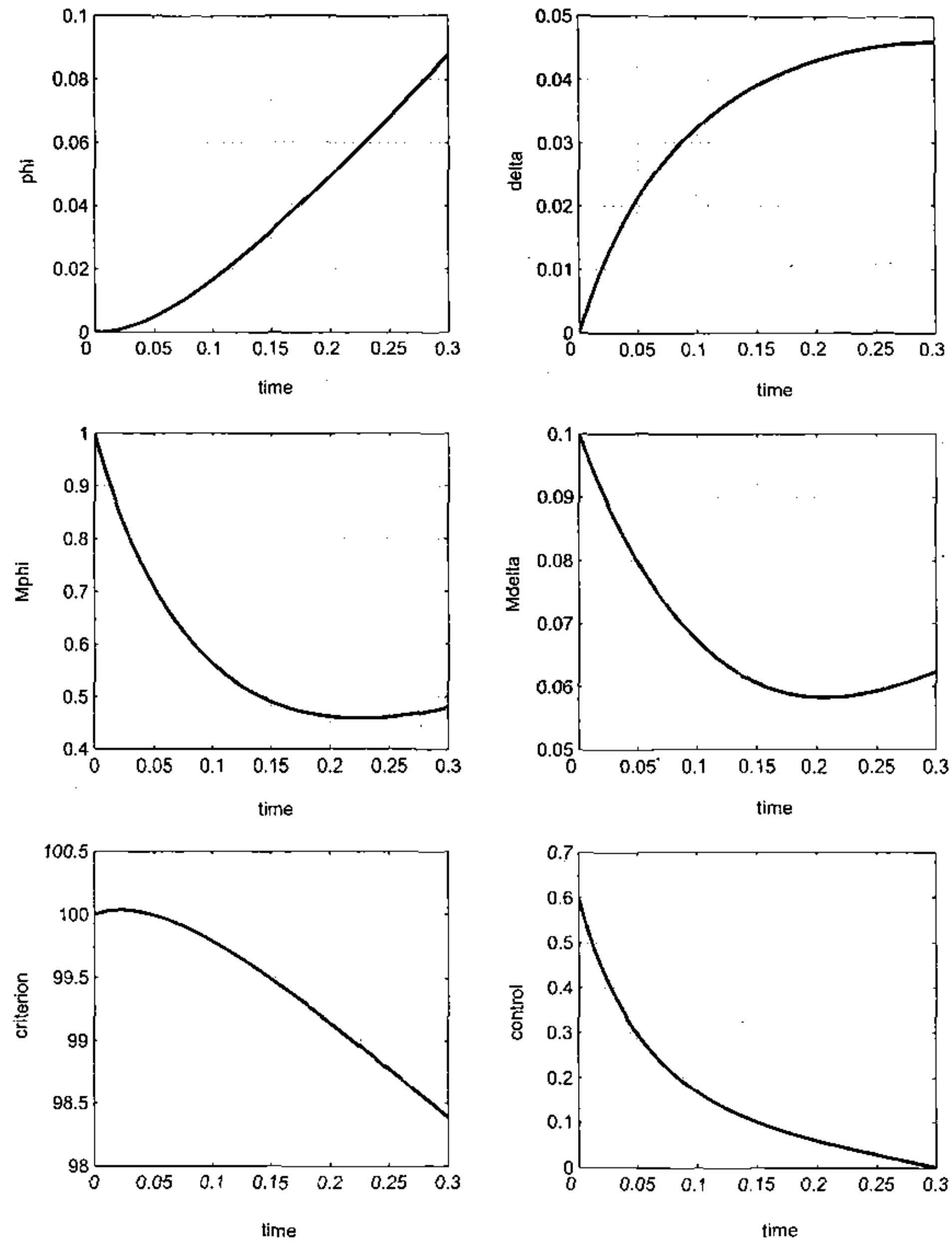


Figure 2: Graphs of the variables $\phi$ and $\delta$ satisfying the polynomial system (15) and controlled using the optimal third order polynomial controller defined by (16), (17); graphs of the estimates $m_{\phi}$ and $m_{\delta}$ satisfying the system (17) and controlled using the optimal third order polynomial controller defined by (16), (17); graphs of the corresponding values of the criterion $J$; graphs of the corresponding values of the optimal control $u^*$.