where

$$\Psi_o^\tau(e,x,u) := [\alpha(e+x) - \alpha(x) + (\beta(e+x) - \beta(x))\, u].$$

In order to make a statement on the stability of the observer we need the following hypothesis.

Assumption A. The function Ψ_o along the trajectories of (2.1) and (3.7), driven by any admissible control input $u(k)$ satisfy

$$\|B\Psi_o^\tau(e(k), x(k), u(k))\| \le l_1 \|e(k)\|,$$
$$\forall\, k \ge k_0 \ge 0, \quad \forall \tau \in (0, \tau_{\max}).$$

Remark 1. Notice that this assumption holds for instance if, for each compact $\mathcal{X}$, and defining $\mathcal{U}_\tau := \{u \in R^n : u = \beta^{-1}(x)[v(x) - \alpha(x)], x \in \mathcal{X}\}$ there exists $l_1 > 0$ such that $\|B\Psi_o^\tau(e,x,u)\| \le l_1 \|e\|$, $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}_\tau$ for all $\tau \in (0, \tau_{\max})$ and all $k \ge k_0 \ge 0$.

Lemma 2. *Assume that the system (2.1) satisfies assumption* **A**. *Then, there exist $\tau_{\max} > 0$ sufficiently small and $\theta_{\min} > 0$ sufficiently large such that the estimation error dynamics (3.7) is uniformly globally exponentially stable with λ_τ proportional to $\tau \in (0, \tau_{\max})$, for all $\theta > \theta_{\min}$ such that $\theta_{\min}\tau_{\max} \in (0,1)$.*

The proof of this Lemma is based on the following claim which is the dual of Claim 1.

Claim 2 ([6]). *Let $A_o = I + \gamma_o(A - KC)$ where K is defined as in (3.6). Then for every $\gamma_o \in (0,1)$, the unique symmetric positive definite matrix P_o satisfying the algebraic equation,*

$$A_o^T P_o A_o - P_o = -\gamma_o P_o - \gamma_o(1-\gamma_o)^n C^T C,$$

is given by $P_o = M^T M$ where $M = \Lambda_o E_o$, $\Lambda_o = diag(1, (1-\gamma_o)^{\frac{1}{2}}, ..., (1-\gamma_o)^{\frac{n-1}{2}})$ and, letting i and j denote the rows and columns of E_o respectively, the elements of E_o, are $E_o(i,j) = (-1)^{i+j} C_{j-1}^{i-1}$ for $i \le j \le n$ and $E_o(i,j) = 0$ otherwise.

3.3. Main result

We can now establish the following result.

Theorem 1. *Consider the discretized nonlinear system*

$$\begin{aligned} x(k+1) &= A_\tau x(k) + \tau B\{\alpha(x(k)) + \beta(x(k))u(k)\} \\ y(k) &= Cz(k) \end{aligned}$$

under Assumption A. Then the observer-based output feedback control law,

$$\begin{aligned} z(k+1) &= A_\tau z(k) + \tau B[\alpha(z(k)) + \beta(z(k))u(k)] \\ &\quad + \tau\Delta_\theta^{-1} K[y(k) - \hat{y}(k)] \\ u(k) &= \beta^{-1}(z(k))[-F\Omega_\rho z(k) - \alpha(z(k))], \end{aligned}$$

renders the equilibrium $(x,z) = (0,0)$ of the closed-loop system (2.1), (3.1)-(3.3), (3.5)-(3.6) uniformly exponentially stable.

Proof: The result follows if and only if the origin of the estimation error and the observe dynamics, $(e,z) = (0,0)$, is exponentially stable. In view of Lemma 2, we only need to prove that the origin of the observer dynamics under the control action,

$$\begin{aligned} z(k+1) &= A_\tau z(k) + \tau B[\alpha(z(k)) + \beta(z(k))u(k)] \\ &\quad + \tau\Delta_\theta^{-1} K[y(k) - \hat{y}(k)], \end{aligned} \tag{3.8}$$

is uniformly globally exponentially stable.

To prove this, we will invoke the following result.

Lemma 3. If for a system $\xi(k+1) = f_\tau(k, \xi(k))$ there exist $p > 0$, $\tau_{\max} > 0$, $\nu > 0$ and c_τ proportional to $\tau^{-1/p}$ such that for all $k \ge k_o$, all $\xi(k_o) = \xi_o$ and all $T \in (0, \tau_{\max})$,

$$\max_{k \ge k_o} \|\xi(k)\| \le \nu\|\xi_o\| \tag{3.9}$$

$$\left(\sum_{k=k_o}^{\infty} \|\xi(k)\|^p\right)^{1/p} \le c_\tau \|\xi_o\| \tag{3.10}$$

then, there exist κ and $\lambda_\tau > 0$ proportional to τ such that (2.2) holds for all $\xi_0 \in R^m$. □

Hence, we proceed to compute the bounds (3.9), (3.10) with $\xi := \mathrm{col}[e,z]$. We start with the bounds for $\|e(k)\|$. From Lemma 2, it follows that $\|\varepsilon(k)\|_{P_o} \le \|\varepsilon(k_0)\|_{P_o} e^{-\delta\tau(k-k_0)}$, *i.e.* $\|\varepsilon(k)\|_{P_o} \le \|\varepsilon(k_0)\|_{P_o}$ and therefore, there exists $c > 0$ such that

$$\|e(k)\| \le c\|e(k_0)\| \qquad \forall\, k \ge k_0. \tag{3.11}$$

Also from Lemma 2, we obtain $\Delta V_{\varepsilon_k} \le -\tau 2\delta V_{\varepsilon_k}$, then evaluating the sum from k_0 to ∞ on both sides of $\Delta V_{\varepsilon_k} \le -\tau 2\delta V_{\varepsilon_k}$, it follows that

$$V_{\varepsilon_{k_0}} \ge -\sum_{k=k_0}^{\infty} \Delta V_{\varepsilon_k} \ge \sum_{k=k_0}^{\infty} \tau\delta \|\varepsilon(k)\|_{P_o}^2$$

so using the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{P_o}$ we conclude that there exists $c > 0$ such that

$$\left(\sum_{k=k_0}^{\infty} \|e(k)\|^2\right)^{1/2} \le \frac{c}{\sqrt{\tau\delta}} \|e(k_0)\|. \tag{3.12}$$

Next, we proceed to compute similar bounds for $z(k)$. To this end, reconsider the observer dynamics under the control action, and under the coordinate transformation $\eta = \Omega_\rho z$, *i.e.*,

$$\begin{aligned} \eta(k+1) &= (I_n + \tau\rho(A - BF))\eta(k) \\ &\quad + \tau\Omega_\rho\Delta_\theta^{-1} KC\Delta_\theta^{-1}\varepsilon(k) \\ &= A_c\eta(k) + \tau\Omega_\rho\Delta_\theta^{-1} KC\Delta_\theta^{-1}\varepsilon(k) \end{aligned} \tag{3.13}$$