where A_c is defined in Claim 1 with $\gamma_c = \tau\rho$. Define $V_\sigma = \sigma^T P_c \sigma$ then, we get that the difference equation $\Delta V_{\sigma_k} = V_{\sigma_{k+1}} - V_{\sigma_k}$ along the trajectories of $\sigma(k+1) = A_c\sigma(k)$ yields

$$\begin{aligned}\Delta V_{\sigma_k} &= V_{\sigma_{k+1}} - V_{\sigma_k} \\ &= \sigma^T(k)\left[A_c^T P_c A_c - P_c\right]\sigma(k).\end{aligned}$$

It is easy to see from Claim 1 with $\gamma_c = \tau\rho$, that

$$\begin{aligned}\Delta V_{\sigma_k} &= -\tau\rho\sigma^T(k)P_c\sigma(k) \\ &\quad -\tau\rho(1-\tau\rho)^n\sigma^T(k)F^TF\sigma(k) \\ \Delta V_{\sigma_k} &\leq -\tau\rho\,\|\sigma(k)\|_{P_c}^2\end{aligned}$$

Using this bound we now evaluate the difference equation $\Delta V_{\eta_k} = V_{\eta_{k-1}} - V_{\eta_k}$ where $V_{\eta_k} = \eta(k)^T P_c \eta(k)$ along the trajectories of (3.13) to obtain

$$\begin{aligned}\Delta V_{\eta_k} &= V_{\eta_{k-1}} - V_{\eta_k} \\ &\leq -\tau\rho\,\|\eta(k)\|_{P_c}^2 + \tau^2N^2\,\|\varepsilon(k)\|_{P_c}^2 \\ &\quad +2\tau N\,\|\varepsilon(k)\|_{P_c}\,\|\eta(k)\|_{P_c} \\ &\leq -\tau(\rho-1)\,\|\eta(k)\|_{P_c}^2 + \tau N^2\,\|\varepsilon(k)\|_{P_c}^2\end{aligned}$$

where we defined $N := \left\|\Omega_\rho\Delta_\theta^{-1}KC\Delta_\theta^{-1}\right\|$. Evaluating the sum from k_0 to ∞ on both sides of the inequality above, and using (3.12) we obtain that

$$\begin{aligned}\sum_{k=k_0}^{\infty}\Delta V_{\eta_k} &\geq \tau\sum_{k=k_0}^{\infty}\left\{(\rho-1)\,\|\eta(k)\|_{P_c}^2 - cN^2\,\|\varepsilon(k)\|^2\right\} \\ &\geq \tau(\rho-1)\sum_{k=k_0}^{\infty}\|\eta(k)\|_{P_c}^2 - cN^2\frac{\|\varepsilon(k_0)\|^2}{\delta}\end{aligned}$$

which implies that

$$\sum_{k=k_0}^{\infty}\|\eta(k)\|^2 \leq \frac{c}{\tau(\rho-1)}\left(\|\eta(k_0)\|^2 + \frac{N^2}{\delta}\,\|\varepsilon(k_0)\|^2\right)$$

hence, setting $\rho_{\min} > 1$ and since $\eta = \Omega_\rho z$, we finally obtain that

$$\left(\sum_{k=k_0}^{\infty}\|z(k)\|^2\right)^{1/2} \leq \frac{c}{\sqrt{\tau}}\,\|\xi(k_0)\| \tag{3.14}$$

where it is clear that c is independent of τ. To determine the last bound, we recall that

$$\Delta V_{\eta_k} \leq -\tau(\rho-1)\,V_{\eta_k} + \tau N^2\,\|\varepsilon(k)\|_{P_c}^2\,.$$

Then, using $\|\varepsilon(k)\|_{P_o} \leq \|\varepsilon(k_0)\|_{P_o}e^{-\delta\tau(k-k_0)}$, we obtain that

$$\Delta V_{\eta_k} \leq -\tau(\rho-1)\,V_{\eta_k} + \tau cN^2\,\|\varepsilon(k_0)\|^2\,e^{-2\delta\tau(k-k_0)}.$$

To show contradiction, assume that $V_{\eta_k} \to \infty$ as $k \to \infty$. From the above we see that there exists $k^* > 0$, such that $\Delta V_{\eta_k} \leq 0$, which implies that $\|\eta(k)\|^2 \leq c\,\|\eta(k_0)\|^2$ for all $k \geq k^*$. On the other hand, $\|\eta(k)\|^2 \leq c\,\|\eta(k_0)\|^2 + k^*N\tau_{\max}\,\|\varepsilon(k_0)\|^2$ for all $k < k^*$. Therefore, $\|\eta(k)\|^2 \leq c\,\|\eta(k_0)\|^2 + ck^*N\tau_{\max}\,\|\varepsilon(k_0)\|^2$ for all $k \geq k_0$. We conclude that there exists $c > 0$ independent of τ such that

$$\|z(k)\| \leq c\|\xi(k_0)\| \qquad \forall k \geq 0\,. \tag{3.15}$$

From the bounds (3.11), (3.12), (3.14), (3.15), and invoking Lemma 3 with $\nu = c$, $p = 2$ and $c_\tau := c\left(\max\{\frac{1}{\tau\delta}\,\frac{1}{\tau}\}\right)^{1/2}$ (which is obviously proportional to $\tau^{-1/2}$), we conclude that there exist $\kappa > 0$ and λ_τ, proportional to τ, such that (2.2) holds. ■

4. Application to a flexible-joint robot

We apply the results developed above to the control of the flexible-joint robot. The dynamic equations of a single link robot arm with a revolute elastic joint rotating in a vertical plane are given by

$$\begin{aligned}J_l\ddot{q}_1 + F_l\dot{q}_1 + k(q_1-q_2) + mgl\sin(q_1) &= 0 \\ J_m\ddot{q}_2 + F_m\dot{q}_2 - k(q_1-q_2) &= u \\ y &= q_1\end{aligned}$$

in which q_1 and q_2 are the link displacement and the rotor displacement, respectively. The link inertia J_l, the motor rotor inertia J_m, the elastic constant k, the link mass m, the gravity constant g, the center of mass l and the viscous friction coefficients F_l and F_m are positive constant parameters. The control u is the torque delivered by the motor. Assuming that only q_1 is measured, u is to be designed so that q_1 tracks a desired reference $q_{r1}(t)$ where the parameters are assumed to be known. Defining the state variables,

$$\xi_1 = q_1, \quad \xi_2 = \dot{q}_1, \quad \xi_3 = q_2, \quad \xi_4 = \dot{q}_2,$$

the model in state-space form is

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\frac{F_l}{J_l}\xi_2 - \frac{mgl}{J_l}\sin(\xi_1) - \frac{k}{J_l}(\xi_1-\xi_3) \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= -\frac{F_m}{J_m}\xi_4 - \frac{k}{J_m}(\xi_1-\xi_3) + \frac{1}{J_m}u.\end{aligned} \tag{4.1}$$

4.1. Control design

The system (4.1) is state-feedback linearizable by means of the change of coordinates (cf. [5])

$$x_1 = \xi_1,$$