$$
\begin{aligned}
x_2 &= \xi_2 \\
x_3 &= -\frac{F_l}{J_l}\xi_2 - \frac{mgl}{J_l}\sin(\xi_1) - \frac{k}{J_l}(\xi_1 - \xi_3) \\
x_4 &= \frac{F_l^2}{J_l^2}\xi_2 + \frac{F_l mgl}{J_l^2}\sin(\xi_1) \\
&\quad + \frac{F_l k}{J_l^2}(\xi_1 - \xi_3) - \frac{k}{J_l}(\xi_2 - \xi_4) \\
&\quad - \frac{mgl}{J_l\tau}\sin(\xi_1) + \frac{mgl}{J_l\tau}\sin\xi_1
\end{aligned}
$$

and feedback

$$u = \beta^{-1}\left[v(x) - \alpha(x)\right]$$

where $\beta(x) = \tau\frac{k}{J_l J_m}$ and

$$
\begin{aligned}
\alpha(x) &= \left(\frac{mgl}{J_l}\sin x_1 + \frac{F_1 mgl}{J_l^2}\cos x_1 + \frac{kF_l{}^2}{J_l}\right)x_2 \\
&\quad + \frac{mgl}{J_l}\cos x_1 + \left(\frac{k}{J_l} - \frac{F_l^2}{J_l^2}\right) \times \\
&\quad \times \left[\frac{F_l}{J_l}x_2 + \frac{mgl}{J_l}\sin x_1 + \frac{x}{J_l}(x_1 - x_3)\right] \\
&\quad - \frac{kF_l}{J_l^2}x_4 + \frac{k}{J_l}\left[\frac{k}{J_m}(x_1 - x_3) - \frac{F_m}{J_m}x_4\right].
\end{aligned}
$$

The external control is given by $v(x) = -F\Omega_\rho x$, where the matrices F and Ω_ρ are defined as in (3.3), *i.e.* $F = \begin{pmatrix} C_4^0 & C_4^1 & C_4^2 & C_4^3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 & 4 \end{pmatrix}$ and $\Omega_\rho = diag(\rho^4, \rho^3, \rho^2, \rho)$.

Then the external control is given by

$$
\begin{aligned}
v(x) &= -F\Omega_\rho x \\
&= -(\rho^4 x_1 + 4\rho^3 x_2 + -6\rho^2 x_3 + 4\rho x_4).
\end{aligned}
$$

4.2. Observer design

According to Section 3.2, the observer is given by

$$
\begin{aligned}
z(k+1) &= A_\tau z(k) + \tau B\left\{\alpha(z(k)) + \beta(z(k))\,u(k)\right\} \\
&\quad + \tau\Delta_\theta^{-1}K\left[y(k) - \widehat{y}(k)\right]
\end{aligned}
$$

where the observer gain is

$$\tau\Delta_\theta^{-1}K = \tau\begin{pmatrix} 4\theta \\ 6\theta^2 \\ 4\theta^3 \\ \theta^4 \end{pmatrix}$$

with $\Delta_\theta^{-1} = diag(\theta, \theta^2, \theta^3, \theta^4)$;
$K = col\begin{pmatrix} C_4^1 & C_4^2 & C_4^3 & C_4^3 \end{pmatrix} = col\begin{pmatrix} 4 & 6 & 4 & 1 \end{pmatrix}$.

Therefore, the observer becomes

$$
\begin{aligned}
z_1(k+1) &= z_1(k) + \tau z_2(k) + 4\tau\theta\,(x_1(k) - z_1(k)) \\
z_2(k+1) &= z_2(k) + \tau z_3(k) + 6\tau\theta^2\,(x_1(k) - z_1(k)) \\
z_3(k+1) &= z_3(k) + \tau z_4(k) + 4\tau\theta^3\,(x_1(k) - z_1(k)) \\
z_3(k+1) &= z_3(k) + \tau\alpha(z(k)) + \tau\beta(z(k))u(k) \\
&\quad + \tau\theta^4\,(x_1(k) - z_1(k)).
\end{aligned}
$$

4.3. Simulation results

Numerical simulations were carried out to assess the closed loop responses of a flexible-joint robot using the above observer and controller algorithms was performed for the following numerical values: $m = 0.4$ Kg, $g = 9.81$ m/s^2, $l = 0.185$ m, $J_l = 0.002$ N-ms^2/rad, $J_m = 0.0059$ N-ms^2/rad, $k = 1.61$ N-m-s/rad

The initial conditions for the numerical simulation were selected as follows: $k_0 = 0$, $x(0) = col\begin{pmatrix} 0.1 & 0.2 & 0.03 & 0.04 \end{pmatrix}$ and $z(0) = col\begin{pmatrix} 0.2 & 0.3 & 0.15 & 0.25 \end{pmatrix}$. The sampling period was set to $\tau = 0.0001$. The parameter of the controller gain was set to $\rho = 30$, the parameter design of the observer was chosen as $\theta = 80$ and finally, the reference signal is $q_{r1}(t) = \frac{1}{2}\sin(4t)$. Figures 1-4 illustrate the performance of the proposed scheme.

5. Conclusions

An observer-based controller for feedback linearizable discrete-time nonlinear systems of Euler type was presented. Uniform exponential stability of the closed loop system was established. This allows to conclude on the practical asymptotic stability of the corresponding sampled-data system.

The usefulness and the performance of the proposed scheme was illustrated on the application to a flexible-joint robot. In particular, simulations show the fast convergence of the observer.

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