2. Problem setting and definitions

We denote by $\xi(k)$ the solution of the difference equation $\xi(k+1) = F_\tau(k, \xi(k))$ with initial conditions $k_0 \geq 0$ and $\xi_0 = \xi(k_0)$.

We consider the following class of continuous-time nonlinear systems

$$\Sigma_{NLC} : \begin{cases} \dot{\xi} = f(\xi) + g(\xi)u \\ y = h(\xi) \end{cases} \tag{2.1}$$

using the Euler approximation under the assumption of a sufficiently small sampling period,

$$\Sigma_{NLD} : \begin{cases} \xi(k+1) = \xi(k) + \tau\{f(\xi(k)) + g(\xi(k))u(k)\} \\ y(k) = h(\xi(k)) \end{cases} \tag{2.2}$$

where for simplicity we denote $\xi(k) = \xi(k\tau)$, for τ fixed.

In the sequel, the following definition can be used in order to design a controller and an observer.

Definition 1. Let $\Xi \subset R^n$ be a compact set. The system (2.2) is locally feedback linearizable if there exists a diffeomorphism $\Upsilon : \Xi \to \mathcal{X} \subset R^n$ such that $\mathcal{X} = \Upsilon(\Xi)$ contains the origin and defining $x = \Upsilon(\xi)$, the system (2.2) can be transformed into

$$\tilde{\Sigma}_{NLD} : \begin{cases} x(k+1) = A_\tau x(k) + \tau B\{\alpha(x(k)) + \beta(x(k))u(k)\} \\ y = Cx(k) = x_1(k) \end{cases} \tag{2.3}$$

where $A_\tau = (I_n + \tau A) = \begin{pmatrix} 1 & \tau & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \tau \\ 0 & 0 & \cdots & 1 \end{pmatrix}$,

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ and τ is the sampling period, see [4].

We will address the above mentioned control problem by designing an observer-based controller scheme for the system (2.3). More precisely, we will design an observer for which the following property can be verified:

Definition 2. **(Uniform exponential stability)** The origin of the system $\xi(k+1) = F_\tau(k, \xi(k))$ is said to be uniformly exponentially stable if there exist r, $\tau_{\max}$, $\kappa > 0$ and for each $\tau \in (0, \tau_{\max})$, $\lambda_\tau > 0$ such that,

$$\|\xi(k_0)\| \leq r \quad \Rightarrow \quad \|\xi(k)\| \leq \kappa \|\xi_o\| e^{-\lambda_\tau (k-k_o)} \quad \forall k \geq k_o . \tag{2.4}$$

If furthermore (2.4) holds for all $\xi(k_0) \in R^n$ then, the origin is said to be *uniformly globally exponentially stable.*

The property defined above is probably the most useful for discrete-time systems since it imposes a bound on the overshoots which are uniform in the initial conditions and the sampling time. Moreover, in the particular case when λ_τ is proportional to τ, this property guarantees that the *exact* discrete-time model corresponding to (2.1) (hence with a discretized control input) is (globally) asymptotically practically stable. Roughly speaking, this means that the solutions tend to an arbitrarily small ball whose size is independent of τ and can be made smaller as $\tau_{\max}$ becomes smaller.

3. Sliding-Mode Control Design

In the sequel, a control design based on sliding mode techniques is proposed. The main idea is to design an asymptotically stabilizing feedback control law assuring the sliding motion on a (n-m) dimensional space $\mathcal{M} \subset R^n$. Consider the following nonlinear discrete-time dynamics

$$\tilde{\Sigma} : \begin{cases} x(k+1) = \mathcal{F}_\tau(x(k)) + \mathcal{G}_\tau(x(k))u(k) \\ y = Cx(k) = x_1(k) \end{cases} \tag{3.1}$$

The objective of the sliding mode control strategy is to steer the states of the system into a (n-m) dimensional manifold $\mathcal{M}$ and to maintain the subsequent motion of the trajectories on $\mathcal{M}$, such that as $k \to \infty$, $x(k) \to 0$.

For this system a sliding mode control is designed by considering the following switching surface

$$\sigma(k) = \mathcal{S}^T (x(k) - x_{ref}(k)) \tag{3.2}$$

where $\mathcal{S}$ is a vector: $\mathcal{S} = col(\ \mathcal{S}_1 \ \ ,..., \ \ \mathcal{S}_n\ )$ and $x_{ref}(k+1) = x_{ref}(k)$ is a constant reference signal. We assume that $\mathcal{S}^T \mathcal{G}(x(k))$ is invertible.

Remark 1:

i) From Definition 1, the system *(2.2)* can be transformed into *(2.3)*, which can be expressed as system *(3.1)* by taking $\mathcal{F}_\tau(x(k)) = A_\tau x(k) + \tau B\alpha(x(k))$ and $\mathcal{G}_\tau(x(k)) = \tau B\beta(x(k))$.

ii) It is clear that there exist others possibilities to define the switching surface. The choice depends on the control objective.

The proposed control is designed in two steps. Firstly, the *equivalent control* $u_e(k)$ is determined when the system motion is restricted to the switching surface $\sigma(k+1) = 0$, so that the control satisfying this sliding condition is given by

$$u_e(k) = \left[\mathcal{S}^T \mathcal{G}_\tau(x(k))\right]^{-1} \left[\mathcal{S}^T \mathcal{F}_\tau(x(k)) - \mathcal{S}^T x_{ref}(k+1)\right]$$

The next step is as follows. A *regulation control* Δu is added in order to satisfy the reaching condition. A necessary and sufficient condition for assuring both sliding motion and convergence onto $\mathcal{M}$ is the discrete-time reaching condition which can be stated as

$$|\sigma(k+1)| < |\sigma(k)|$$

which must be satisfied (see [2]). For that, the switching surface can be chosen as

$$\sigma(k+1) = \eta \mathcal{S}^T (x(k) - x_{ref}(k)) = \eta\sigma(k) \tag{3.3}$$