where $0 < \eta < 1$ is a scalar weighting value. It is clear that this choice satisfies the *reaching condition*, *i.e.*

$$\eta\,|\sigma(k)| < |\sigma(k)|.$$

Then, the regulation control Δu can be designed as follows

$$\Delta u(k) = \left[S^T \mathcal{G}_\tau(x(k))\right]^{-1}\left[\eta S^T\left(x(k) - x_{ref}(k)\right)\right].$$

Finally, the control law is given by

$$u(k) = u_e(k) + \Delta u(k) \tag{3.4}$$

The stability properties of $\sigma(k) = 0$ in (3.3) can be studied by means of the candidate Lyapunov function $V(\sigma(k)) = \sigma^T(k)\sigma(k)$. It follows that

$$\begin{aligned} V(\sigma(k+1)) - V(\sigma(k)) &= \sigma^T(k+1)\sigma(k+1) - \sigma^T(k)\sigma(k) \\ &= -(1-\eta^2)\sigma^T(k)\sigma(k) \end{aligned}$$

or equivalently $V(\sigma(k+1)) = \eta^2 V(\sigma(k))$

$$= \left(\eta^2\right)^k V(\sigma(0)).$$

Hence, $V(\sigma(k+1)) \to 0$ as $k \to \infty$.

To prove the stability of the closed-loop system under control action $u(k)$ it is necessary to introduce the notion of ultimate bound for the solutions of the unperturbed system

$$\xi(k+1) = F_\tau(\xi(k), k) \tag{3.5}$$

where $F_\tau(\xi(k), k) = \xi(k) + \tau f(\xi(k))$, which will be used to study the stability properties of a class of perturbed discrete nonlinear systems when the equilibrium point is affected by a small perturbation in some sense.

Definition 3. *The solutions of system (3.5) are said to be uniformly ultimately bounded if there exist positive constants β_1 and β_2 and for every $r \in (0, \beta_2)$ there is a constant $T = T(r)$, such that*

$$\|\xi(k_0)\| < r \Rightarrow \|\xi(k)\| < \beta_1, \quad \forall k > k_0 + T.$$

The constant β_1 is known as the ultimate bound.

Furthermore, we introduce a result of existence of *the ultimate bound* for the solution of system *(3.5)*.

Consider the following assumptions:

A1. *There exists $\mu > 0$ such that the equilibrium point $\xi = 0$ is uniformly stable on B_μ.*

A2. *There exists a continuous function $V : B_r \times Z_+ \to R$ such that*

$$c_1\|\xi(k)\|^2 \le V(\xi, k) \le c_2\|\xi(k)\|^2$$

$$\Delta V(\xi, k) \le -c_3\|\xi(k)\|^2$$

for $0 < \mu < \sqrt{\frac{c_1}{c_2}}r$, for some positive constants c_1, c_2 and c_3, for all $k > 0$ and for all $\xi \in B_r$.

Theorem 1. *Consider the system (3.5). Assume that A1 and A2 hold. There exists a class $\mathcal{KL}$ function $\varphi(.,.) = \phi(.)\rho(.)$ such that ρ is a function of class $\mathcal{K}$, ρ is a decreasing function and a finite time k_1, depending on $\xi(k_0)$ and μ, such that the solution of (3.5) satisfies*

$$\|\xi(k)\| \le \phi(\|\xi(k_0)\|)\rho(k - k_0)$$

and

$$\|\xi(k)\| \le \sqrt{\tfrac{c_2}{c_1}}\mu, \qquad \forall k \ge k_1$$

for $\|\xi(k_0)\| < \sqrt{\frac{c_1}{c_2}}r$.

Now, the system *(3.1)* under the action of the control (3.4) yields the closed-loop system

$$x(k+1) = f_\tau(x(k), 0) + p_\tau(x(k), x_{ref}(k)) \tag{3.6}$$

where

$$\begin{aligned} f_\tau(x(k), 0) &= \mathcal{F}_\tau(x(k)) \\ &+ \mathcal{G}_\tau(x(k))\left[S^T\mathcal{G}_\tau(x(k))\right]^{-1}\left[\eta S^T x(k) - S^T\mathcal{F}_\tau(x(k))\right] \end{aligned}$$

and

$$\begin{aligned} p_\tau(x(k), x_{ref}(k)) &= \mathcal{G}_\tau(x(k))\left[S^T\mathcal{G}_\tau(x(k))\right]^{-1} \times \\ &\left[S^T x_{ref}(k+1) - \eta S^T x_{ref}(k)\right] \end{aligned}$$

It is clear that the closed-loop system (3.6) can be seen as a system with a unperturbed part, represented by $f_\tau(x(k), 0)$ and a perturbed part given by $p_\tau(x(k), x_{ref}(k))$.

From the boundedness of the columns of $\mathcal{G}_\tau(x(k))$ and the non-singularity of $S^T\mathcal{G}_\tau(x(k))$, it follows that the perturbed part satisfies the following inequality

$$\|p_\tau(x(k), x_{ref}(k))\| \le l_1\|x(k)\|^2 + l_2\|x_{ref}(k)\|^2 \tag{3.7}$$

for $x(k), x_{ref}(k) \in B_r$, where l_1 and l_2 are positive constants.

Now, we consider the following assumptions about the perturbed system:

A3. *The equilibrium point of $x(k+1) = f_\tau(x(k), 0)$, is locally exponentially stable.*

A4. *The reference signal $x_{ref}(k)$ is uniformly bounded and satisfy $\|x_{ref}(k)\| \le b$, for some positive constant b.*

By a converse theorem of Lyapunov, assumption A3 assures the existence of a Lyapunov function $V(x, k)$ which satisfies

$$c_1\|x(k)\|^2 \le V(x, k) \le c_2\|x(k)\|^2 \tag{3.8}$$

$$\Delta V_1(x, k) = V(x, k+1) - V(x, k) \le -c_3\|x(k)\|^2 \tag{3.9}$$

for some positive constants c_1, c_2 and c_3.

Then, the forward difference function $\Delta V(x, k)$ along the trajectories of the closed-loop system is given by

$$\Delta V(x, k) = \Delta\, V_1(x, k) + \Delta V_2(x, k)$$

where

$$\Delta V_1(x, k) = V(f_\tau(x(k), 0), k+1) - V(x, k),$$

and

$$\begin{aligned} \Delta V_2(x, k) &= V(f_\tau(x(k), 0) + p_\tau(x(k), x_{ref}(k)), k+1) \\ &- V(f_\tau(x(k), 0), k+1). \end{aligned}$$

Furthermore, from assumption A4 and (3.7), the function $\Delta V_2(x, k)$ satisfies the following inequality

$$\begin{aligned} |\Delta V_2(x, k)| &\le l_p\|\pm f_\tau(x(k), 0) + p_\tau(x(k), x_{ref}(k))\| \\ &\le l_p l_1\|x(k)\|^2 + l_p l_2\|x_{ref}(k)\|^2 \\ &\le l_p l_1\|x(k)\|^2 + l_p l_2 b_1^2 \end{aligned}$$

Using the condition (3.9) and the above inequality, we have

$$\Delta V(x, k) \le -(c_3 - l_p l_1)\|x(k)\|^2 + l_p l_2 b^2.$$

If l_1 is sufficiently small such that $l_1 < \tilde{l}_1 < \frac{c_3}{l_p}$ is satisfied. It follows that