$$\Delta V(x,k) \le -a\|x(k)\|^2 + l_p l_2 b^2$$

where $a = (c_3 - l_p \tilde{l}_1)$.

Then, the forward difference function $\Delta V(x,k)$ satisfies

$$\begin{aligned}\Delta V(x,k) &\le -(1-\gamma)a\|x(k)\|^2 - \gamma a\|x(k)\|^2 + l_p l_2 b^2 \\ &\le -(1-\gamma)a\|x(k)\|^2,\end{aligned}$$

for some γ such that $0<\gamma<1$ and for all $\|x(k)\| \ge \sqrt{\frac{l_p l_2 b^2}{\gamma a}}$.

It follows that $l_2 \le \frac{\gamma a}{l_p b^2}\|x(k)\|^2$ for $\|x(k)\| < \sqrt{\frac{c_1}{c_2}}r$, and a bound for l_2 is given by $l_2 \le \tilde{l}_2 < \frac{\gamma a}{l_p b^2}\frac{c_1}{c_2}r^2$. From Theorem 1, the ultimate bound of the solution of system (3.6) is given by $B = \sqrt{\frac{c_2}{c_1}}\sqrt{\frac{l_p \tilde{l}_2 b^2}{\gamma a}}$ where the solutions of the slow system satisfy

$$\|x(k)\| < \sqrt{\frac{c_2}{c_1}}\sqrt{\frac{l_p \tilde{l}_2 b^2}{\gamma a}}, \qquad \forall k \ge k_1,$$

for some finite time k_1.

To prove that the closed-loop system is locally ultimately bounded, we have the following lemma.

Lemma 2. *Consider the discrete-time nonlinear system (2.2) for which a control (3.4) is designed. Suppose that assumptions A3 and A4 hold. Then, there exist positive constants $\tilde{l}_1$ and $\tilde{l}_2$ such that, for any initial state $x(k_0)$, the solutions of the closed-loop system (3.6) are ultimately bounded.*

For $x_{ref}(k) = 0, \forall k > k_0$; the following result can be obtained.

Corollary 1: *Consider the discrete-time nonlinear system (2.2) for which a control (3.4) is designed. Suppose that assumption A3 holds. Then, there exists a positive constant $\tilde{l}_1$ such that, for any initial state $x(k_0)$, the solutions of the closed-loop system (3.6) are uniformly exponentially stable.*

4. Observer design

In this section we introduce an observer for the class of systems *(2.3)* which belongs to the class of systems with a triangular structure. This property of the nonlinearity is important because it ensures the uniform observability of the system.

An observer for the transformed system (2.3) is given by

$$\begin{aligned} z(k+1) &= A_\tau z(k) + \tau B\left[\alpha(z(k)) + \beta(z(k))u(k)\right] \\ &\quad + \tau\Delta_\theta^{-1}K\left[y(k) - \widehat{y}(k)\right] \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \Delta_\theta &= diag\left(\tfrac{1}{\theta},\cdots,\tfrac{1}{\theta^n}\right) \quad \text{for} \quad \theta \ge 1, \qquad (4.2)\\ K &= col\left(C_n^1,\cdots,C_n^n\right) \quad \text{with} \quad C_n^p = \frac{n!}{(n-p)!p!}. \end{aligned}$$

The term $\tau\Delta_\theta^{-1}K$ represents the observer gain.

Defining the estimation error as $e = z - x$, it follows that the dynamics of the estimation error is of the form

$$e(k+1) = \left(A_\tau - \tau\Delta_\theta^{-1}KC\right)e(k) + \tau B\Psi_o^\tau\left(e(k), x(k), u(k)\right) \tag{4.3}$$

where $\Psi_o^\tau(e,x,u) := \left[\alpha(e+x) - \alpha(x) + \left(\beta(e+x) - \beta(x)\right)u\right]$.

In order to make a statement on the stability of the observer we need the following hypothesis.

A5. The function Ψ_o along the trajectories of (2.3) and (4.3), driven by any admissible control input $u(k)$ satisfies

$\|B\Psi_o^\tau(e(k), x(k), u(k))\| \le l_3\|e(k)\|, \quad \forall k \ge k_0 \ge 0,$ $\forall \tau \in (0, \tau_{\max})$.

Remark 2. Notice that this assumption holds for instance if, for each compact $\mathcal{X}$, and defining

$\mathcal{U}_\tau := \{u \in R^n : u = \left[S^T\mathcal{G}_\tau(x(k))\right]^{-1} \times \left[S^T x_{ref}(k+1) + \eta S^T\left(x(k) - x_{ref}(k)\right) - S^T\mathcal{F}_\tau(x(k))\right],$ $x \in \mathcal{X}\}$ there exists $l_3 > 0$ such that $\|B\Psi_o^\tau(e,x,u)\| \le l_3\|e\|$, $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}_\tau$ for all $\tau \in (0, \tau_{\max})$ and all $k \ge k_0 \ge 0$.

Lemma 3. *Assume that the system (2.3) satisfies assumption* **A5**. *Then, there exist $\tau_{\max} > 0$ sufficiently small and $\theta_{\min} > 0$ sufficiently large such that the estimation error dynamics (4.3) is uniformly globally exponentially stable with λ_τ proportional to $\tau \in (0, \tau_{\max})$, for all $\theta > \theta_{\min}$ such that $\theta_{\min}\tau_{\max} \in (0,1)$.*

The proof of this theorem is based on the following claim.

Claim 1. *Let $A_o = I + \gamma_o(A - KC)$ where K is defined as in (4.2). Then for every $\gamma_o \in (0,1)$, the unique symmetric positive definite matrix P_o satisfying the algebraic equation,*

$$A_o^T P_o A_o - P_o = -\gamma_o P_o - \gamma_o(1-\gamma_o)^n C^T C,$$

is given by $P_o = M^T M$ where $M = \Lambda_o E_o$, $\Lambda_o = diag(1, (1-\gamma_o)^{\frac{1}{2}}, ..., (1-\gamma_o)^{\frac{n-1}{2}})$ and, letting i and j denote the rows and columns of E_o respectively, the elements of E_o, are $E_o(i,j) = (-1)^{i+j}C_{j-1}^{i-1}$ for $i \le j \le n$ and $E_o(i,j) = 0$ otherwise.

5. Application to the Synchronous Generator

In this section, we apply the previous control and observer design techniques to a synchronous generator. We consider a synchronous generator connected through purely reactive transmission lines to the rest of the network which is represented by an infinite bus, *i.e.* a machine rotating at a synchronous speed ω_s and capable of absorbing or delivering any amount of energy [6]. Such a generator can be modelled as

$$\begin{aligned} \frac{d\delta}{dt} &= \omega - \omega_s \\ M\frac{d\omega}{dt} &= T_m - P_g \\ T'_{do}\frac{dE'_q}{dt} &= -\frac{X_d}{X'_d}E'_q - \left(\frac{X'_d - X_d}{X'_d}\right)V\cos(\delta) + E_{fd} \end{aligned} \tag{5.1}$$