yields:

$$\begin{aligned}\dot{\epsilon}_x &= \left\{A(u,y) - \Lambda S_\theta^{-1}\Lambda^T C^T\Sigma C - S_x^{-1}C^T\Sigma C\right\} e_x \\ &\quad +\Phi(u,y)\epsilon_\theta - \dot{\Lambda}\,\epsilon_\theta - \Lambda\,\dot{\epsilon}_\theta\,.\end{aligned}$$

Replacing the suitable expressions in the above equation, we finally get:

$$\dot{\epsilon}_x = \left\{A(u,y) - S_x^{-1}C^T\Sigma C\right\}\epsilon_x \tag{17}$$

$$\dot{\epsilon}_\theta = -S_\theta^{-1}\Lambda^T C^T\Sigma C\left(\epsilon_x + \Lambda\epsilon_\theta\right) \tag{18}$$

Now noting that under the considered excitation conditions, S_x and S_θ are positive definite matrices [4], one can choose:

$$V(\epsilon_x\,,\epsilon_\theta) = \epsilon_x^T S_x\epsilon_x + \epsilon_\theta^T S_\theta\epsilon_\theta$$

as a Lyapunov function. Then, the time derivative of V is given by:

$$\begin{aligned}\dot{V}(\epsilon_x\,,\epsilon_\theta) &= \epsilon_x^T\left\{A(u,y) - S_x^{-1}C^T\Sigma C\right\}^T S_x\epsilon_x \\ &+ \epsilon_x^T S_x\left\{A(u,y) - S_x^{-1}C^T\Sigma C\right\}\epsilon_x \\ &- (\epsilon_x + \Lambda\epsilon_\theta)^T\left\{S_\theta^{-1}\Lambda^T C^T\Sigma C\right\}^T S_\theta\epsilon_\theta \\ &- \epsilon_\theta^T S_\theta\left\{S_\theta^{-1}\Lambda^T C^T\Sigma C\right\}(\epsilon_x + \Lambda\epsilon_\theta) \\ &+ \epsilon_x^T\,\dot{S}_x\,\epsilon_x + \epsilon_\theta^T\,\dot{S}_\theta\,\epsilon_\theta\end{aligned}$$

and substituting the appropriate expressions, we obtain:

$$\begin{aligned}\dot{V}(\epsilon_x\,,\epsilon_\theta) &= -\rho_x\epsilon_x^T S_x\epsilon_x - \rho_\theta\epsilon_\theta^T S_\theta\epsilon_\theta - \epsilon_x^T C^T\Sigma C\epsilon_x \\ &\quad -\epsilon_x^T C^T\Sigma C\Lambda\epsilon_\theta - \epsilon_\theta^T\Lambda^T C^T\Sigma C\epsilon_x \\ &\quad -\epsilon_\theta^T\Lambda^T C^T\Sigma C\Lambda\epsilon_\theta\end{aligned}$$

Since $-\epsilon_x^T C^T\Sigma C\epsilon_x \;\; -\epsilon_x^T C^T\Sigma C\Lambda\epsilon_\theta \;\; -\epsilon_\theta^T\Lambda^T C^T\Sigma C\epsilon_x$ $-\epsilon_\theta^T\Lambda^T C^T\Sigma C\Lambda\epsilon_\theta$
$= -\left(\epsilon_x + \Lambda\epsilon_\theta\right)^T C^T\Sigma C\left(\epsilon_x + \Lambda\epsilon_\theta\right) <\leq 0$, it follows that:

$$\dot{V}(\epsilon_x\,,\epsilon_\theta) \leq -\rho_x\epsilon_x^T S_x\epsilon_x - \rho_\theta\epsilon_\theta^T S_\theta\epsilon_\theta$$

which finally gives:

$$\dot{V}(\epsilon_x\,,\epsilon_\theta) \leq -\rho V(\epsilon_x\,,\epsilon_\theta), \quad \text{for} \quad \rho = \min(\rho_x, \rho_\theta). \tag{19}$$

As a conclusion, ϵ_x and ϵ_θ exponentially go to zero with a rate driven by ρ, and so does e_x.

Discussion on observer (12)-(16):
First of all, in view of the form of the considered system (10), it is clear that extending the state vector by the vector of constant parameters θ, into $X := \begin{pmatrix} x \\ \theta \end{pmatrix}$, the state affine structure is preserved:

$$\begin{aligned}\dot{X} &= \begin{pmatrix} A(u,y) & \Phi(u,y) \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} \varphi(u,y) \\ 0 \end{pmatrix} \\ &:= F(u,y)X + G(u,y) \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} X = HX\end{aligned} \tag{20}$$

Obviously if the condition (6) is satisfied for this extended system, an observer of the form (7) can be designed for X, providing an adaptive observer for the original system.
Now our point is that observer (12)-(16) is actually *the same* as observer (7) for (20):

Proposition 2.1: *The adaptive observer design (12)-(16) for system (10) coincides with observer (7) for system (20) when $\rho_x = \rho_\theta$.*

Proof. Let S denote the solution of Riccati equation (8) for extended system (20), and consider a partition $\begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}$ corresponding to the partition in x and θ of X (namely S_1 is of same dimensions as A).
Then we can show that for the corresponding initialization, S_x, S_θ, Λ of (12)-(16) are related to S through:

$$\begin{aligned} S_x &= S_1 \\ S_\theta &= S_3 - S_2^T S_1^{-1} S_2 \\ \Lambda &= -S_1^{-1} S_2 \end{aligned} \tag{21}$$

From (8) and (20) indeed, we first have:

$$\dot{S}_1 = -\rho S_1 - A^T(u,y)S_1 - S_1 A(u,y) + C^T\Sigma C \tag{22}$$

$$\dot{S}_2 = -\rho S_2 - A^T(u,y)S_2 - S_1\Phi(u,y) \tag{23}$$

$$\dot{S}_3 = -\rho S_3 - \Phi^T(u,y)S_2 - S_2^T\Phi(u,y) \tag{24}$$

and clearly from (22), S_1 satisfies the same equation (15) as S_x (for $\rho_x = \rho$).
By using (22), (23), one can check that:

$$\frac{d}{dt}(S_1^{-1}S_2) = (A(u,y) - S_1^{-1}C^T\Sigma C)(S_1^{-1}S_2) - \Phi(u,y)$$

and thus $-S_1^{-1}S_2$ satisfies the same equation (14) as Γ.
In the same way, durect computations show that from (22)-(24), we get:

$$\begin{aligned}\tfrac{d}{dt}(S_3 - S_2^T S_1^{-1} S_2) &= -\rho(S_3 - S_2^T S_1^{-1} S_2) \\ &\quad + S_2^T S_1^{-1} C^T\Sigma C S_1^{-1} S_2\end{aligned}$$

namely, with $\Lambda = -S_2^T S_1^{-1}$, $S_3 - S_2^T S_1^{-1} S_2$ satisfies the same equation (16) as S_θ (for $\rho_\theta = \rho$).
Finally, the gain in observer (7) is given by $S^{-1}H^T\Sigma$ (with H from (20)), and from matrix manipulation, one can check that S^{-1} takes the following form:

$$S^{-1} = \begin{pmatrix} (S_1 - S_2 S_3^{-1} S_2^T)^{-1} & * \\ (S_2^T S_1^{-1} S_2 - S_3)^{-1} S_2^T S_1^{-1} & * \end{pmatrix}$$

i.e.

$$S^{-1}H^T\Sigma = \begin{pmatrix} (S_1 - S_2 S_3^{-1} S_2^T)^{-1} C^T\Sigma \\ (S_2^T S_1^{-1} S_2 - S_3)^{-1} S_2^T S_1^{-1} C^T\Sigma \end{pmatrix}$$

By using again some matrix manipulations, one can check that:

$$(S_1 - S_2 S_3^{-1} S_2^T)^{-1} C^T\Sigma \tag{25}$$

$$= S_1^{-1}(I - S_2 S_3^{-1} S_2^T S_1^{-1})^{-1} C^T\Sigma \tag{26}$$

$$= S_1^{-1}(I - S_2[S_2^T S_1^{-1} S_2 - S_3]^{-1} S_2^T S_1^{-1}) C^T\Sigma \tag{27}$$